Seminar of HA——Homology Functor

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1 Basic Defnition

Definition 1.1. *A* ***chain complex*** *in an abelian category $\mathcal{A}$ is a sequence of objects and morphisms in $\mathcal{A}$ (called* ***differentials****),*

$$(\boldsymbol{C}_\bullet, d_\bullet) = \to A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \to,$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$.

Definition 1.2. *A* ***chain map*** *$f = f_\bullet : (\boldsymbol{C}_\bullet, d_\bullet) \to (\boldsymbol{C}'_\bullet, d'_\bullet)$ is a sequence of $f_n : C_n \to C'_n$ for all $n \in \mathbb{Z}$ making the following diagram commute*

$$\begin{tikzcd}
{} \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] \arrow[dr, phantom, "\circlearrowright"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] \arrow[dr, phantom, "\circlearrowright"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & {} \\
{} \arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1} \arrow[r] & {}
\end{tikzcd}$$

$f_n d_{n+1} = d'_{n+1} f_{n+1}$

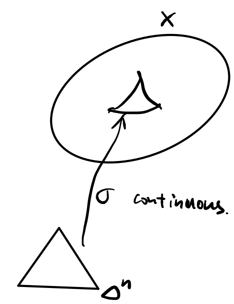
Remark 1.1. *(i) If we give two chain maps $f_\bullet : (\boldsymbol{C}_\bullet, d_\bullet) \to (\boldsymbol{C}'_\bullet, d'_\bullet)$ and $g_\bullet : (\boldsymbol{C}'_\bullet, d'_\bullet) \to (\boldsymbol{C}''_\bullet, d''_\bullet)$, and their composite is defined by $(gf)_n = g_n f_n$;*

(ii) The category of all complex in $\mathcal{A}$, denoted by **Comp**$(\mathcal{A})$*, is an abelian category;*

(iii) We define a notation $\to \boldsymbol{C}^{m+1}_\bullet \xrightarrow{f_{m+1}} \boldsymbol{C}^m_\bullet \xrightarrow{f_m} \boldsymbol{C}^{m-1}_\bullet \to$ is a large diagram as

follows:

$$
\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & C_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "d'_{n+1}"] & C_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "d_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "d''_{n+1}"] & {} \\
{} \arrow[r] & C_n \arrow[r, "i_n"] \arrow[d, "d'_n"] & C_n \arrow[r, "p_n"] \arrow[d, "d_n"] & C_n \arrow[r] \arrow[d, "d''_n"] & {} \\
{} \arrow[r] & C_{n-1} \arrow[r, "i_{n-1}"] \arrow[d] & C_{n-1} \arrow[r, "p_{n-1}"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} &
\end{tikzcd}
$$



Example 1.1. *(i) All exact sequence is a complex by adding the zeros in its head and tail.*

(ii) Singular complex. Consider the standard simplex $\Delta_q = \{\sum_{j=0}^{q} c_j \boldsymbol{e}_j : c_j \geq 0, \sum_{j=0}^{q} c_j = 1\}$ *and a topological space* X*. Consider a continuous map* $\sigma_q : \Delta_q \to X$ *and this called a singular simplex in* X*. Now we let* $\partial_q(\sigma_q) = \sum_{j=0}^{q}(-1)^j \sigma_q|[\boldsymbol{e}_0, ..., \widehat{\boldsymbol{e}}_i, ..., \boldsymbol{e}_n]$*. Then all these* σ_q *formed a basis and generate a free abelian group* $S_q(X)$ *and* $\partial_q\partial_{q+1} = 0$*.*

(iii) Complex $\to 0 \to A \to 0 \to$ *called a* ***concentrated in degree*** k *where* k*th term is* A*.*

(iv) Let U *be an open set of* $\mathbb{R}^n$*, then we have*

$$0 \longrightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^k(U) \xrightarrow{d} \Omega^{k+1}(U) \xrightarrow{d} \cdots$$

where $\Omega^k(U)$ *is a set of all* k*-forms* ω *on* U *where* $\omega : U \to \bigcup_{p \in U} A_k(T_p\mathbb{R}^n)$ *with* $p \mapsto \omega_p$*. Let* $\omega = \sum_I a_I dx^I \in \Omega^k(U)$*, then* $d\omega = \sum_I da_I dx^I$ [redacted] $= \sum_I \left(\sum_j \frac{\partial a_I}{\partial x^j} dx^j\right) \wedge dx^I \in \Omega^{k+1}(U)$*.* $dd = 0$.

(v) Consider $\partial_n : \boldsymbol{a} \mapsto \boldsymbol{A}\boldsymbol{a}$ *where* $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}$ *and* $\boldsymbol{a} \in \mathbb{R}^2$*. Let* $C_n = \mathbb{R}^2$ *with* ∂_n *formed a complex since* $\partial_n\partial_{n+1} = 0$*.* $\to \mathbb{R}^2 \xrightarrow{\partial_n} \mathbb{R}^2 \xrightarrow{\partial_n} \mathbb{R}^2 \to$

Resolutions

Definition 1.3. *A* ***projective resolution*** *of* $A \in \mathrm{Ob}(\mathcal{A})$*, where* $\mathcal{A}$ *is an abelian category, is an exact sequence* $\boldsymbol{P} : \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$ *in which each* P_n *is projective.*

Let $\boldsymbol{P}$ *is a projective resolution of* A*, then its* ***deleted projective resolution*** *is the complex* $\boldsymbol{P}_A : \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0$*. This need not be exact.*

Remark 1.2. *(i) Deleting A loses no information since we have*

$$A = \mathrm{Im}\varepsilon \cong P_0/\ker\varepsilon = P_0/\mathrm{Im}d_1 = \mathrm{coker}d_1;$$

(ii) In ${}_R\mathrm{Mod}$ or Mod_R, we can define ***free resolution*** *and* ***flat resolution****.*

A=F/K
=⟨X⟩/⟨Y⟩

Definition 1.4. *A* ***injective resolution*** *of $A \in \mathrm{Ob}(\mathcal{A})$, where $\mathcal{A}$ is an abelian category, is an exact sequence $\boldsymbol{E} : 0 \to A \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to$ in which each E_n is injective.*

Let $\boldsymbol{E}$ is a injective resolution of A, then its ***deleted injective resolution*** *is the complex $\boldsymbol{E}^A : 0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to$. This need not be exact.*

Remark 1.3. *Deleting A loses no information since $A \cong \ker d^0$.*

Remark 1.4. *In projective resolution $\boldsymbol{P} :\to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \to 0$, we define $K_0 = \ker\varepsilon$ and $K_n = \ker d_n, n \geq 1$. Then we call K_n the* ***nth syzygy*** *of $\boldsymbol{P}$.*

In injective resolution $\boldsymbol{E} : 0 \to A \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to$, we define $V_0 = \mathrm{coker}\eta$ and $V_n = \mathrm{coker}d^{n-1}, n \geq 1$. Then we call V_n the ***nth cosyzygy*** *of $\boldsymbol{E}$.*

Proposition 1.1. *Every A be a R-module has a free resolution (hence projective, flat).*

Proof. Since $A = (X|Y)$ and we can find a free module F_0 with a basis X such that we have an exact sequence $0 \to K_1 \xrightarrow{i_1} F_0 \xrightarrow{\varepsilon} A \to 0$. Then we consider K_1 and find a free module F_1, we have an exact sequence $0 \to K_2 \xrightarrow{i_2} F_1 \xrightarrow{\varepsilon'} K_1 \to 0$. Then we consider

$K_1 = \ker \varepsilon$

$$\begin{tikzcd}
 & & F_1 \arrow[rr, "d_1"] \arrow[dr, "\varepsilon_1"'] & & F_0 \arrow[r, "\varepsilon"] & A \arrow[r] & 0 \\
0 \arrow[r] & K_2 \arrow[ur] & & K_1 \arrow[ur, "i_1"'] & & &
\end{tikzcd}$$

where $\mathrm{Im}d_1 = K_1 = \ker\varepsilon$ and $\ker d_1 = K_2$. □

Proposition 1.2. *Every A be a R-module has a injective resolution.*

Proof. Every module can be imneded as a submodule of and injective module E^0, then we have $0 \to A \overset{\eta}{\hookrightarrow} E^0 \xrightarrow{\pi} V^0 \to 0$ where $\underline{V^0 = \mathrm{coker}\eta}$ and π is natural map. Then we have $V \overset{\eta'}{\hookrightarrow} E^1$. Then we consider

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, hook, "\eta"] & E^0 \arrow[rr, "d^0=\eta^1\pi"] \arrow[dr, "\pi"'] & & E^1 \arrow[dr] & & \\
 & & & V^0 \arrow[ur, hook, "\eta^1"'] & & V^1 \arrow[r] & 0
\end{tikzcd}$$

and well done. $\square$

Remark 1.5. *In an abelian category $\mathcal{A}$, if it is enough projective, it has projective resolution. If it is enough injective, it has injective resolution. The proof of these are trivial.*

Definition 1.5. *A complex $\boldsymbol{C}$ is a **positive complex** if $C_n = 0$ for all $n < 0$;*
*A complex $\boldsymbol{C}$ is a **negative complex** if $C_n = 0$ for all $n > 0$.*

2 Basic Homology

Definition 2.1. *Let $\mathcal{A}$ be an abelian category, let $(\boldsymbol{C}, d)$ is a complex in $\mathbf{Comp}(\mathcal{A})$ as below.*

$$\longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow$$

*We define C_n is n**-chains**, $Z_n(\boldsymbol{C}) = \ker d_n$ is n**-cycles** and $B_n(\boldsymbol{C}) = \mathrm{Im} d_{n+1}$ is n**-boundaries**. Then n**th homology** is $H_n(\boldsymbol{C}) = Z_n(\boldsymbol{C})/B_n(\boldsymbol{C})$.*

Remark 2.1. *In abelian category, the quotient is an equivalence class $[(f, C)]$ where $f : B \to C$ is an epic and $(f, C) \sim (f', C')$ if and only if there exists an isomorphism $g : C \to C'$ make the diagram below commute.*

$$\begin{tikzcd} B \arrow[r, two heads, "f"] \arrow[dr, two heads, "f'"'] & C \arrow[d, "g", "\cong"'] \\ & C' \end{tikzcd}$$

But now we can see $\mathcal{A}$ as a full subcategory of $\mathbf{Ab}$, then $H_n(\boldsymbol{C}) = \{z + B_n(\boldsymbol{C})\}$ and the elements of it called homology class which denoted by $cls(z)$.

Proposition 2.1. *Let $\mathcal{A}$ be an abelian category, then $H_n : \mathbf{Comp}(\mathcal{A}) \to \mathcal{A}$ is an additive functor for all $n \in \mathbb{Z}$.*

Proof. We just need to prove the case when $\mathcal{A} = \cancel{\mathbf{Ab}}$ Mod since the Mitchell's theorem. Now we need to define H_n on morphisms. If $f : (\boldsymbol{C}, d) \to (\boldsymbol{C}', d')$, then $H_n(f) : H_n(\boldsymbol{C}) \to H_n(\boldsymbol{C}')$ by $cls(z_n) \mapsto cls(f_n z_n)$. There are three steps in this property.

$\rightsquigarrow$ **Step 1. Show that $H_n(f)$ is well defined.** It suffice to show that $f_n z_n$ is a cycle and $H_n(f)$ is independent of the choice of cycle z_n. We have the following diagram.

$$\begin{tikzcd}
\arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & {} \\
\arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"] & C'_n \arrow[r, "d'_n"] & C'_{n-1} \arrow[r] & {}
\end{tikzcd}$$

Let $z \in Z_n(\boldsymbol{C})$, then $d_n z = 0$. Then $d'_n f_n z = f_{n-1} d_n z = 0$ and $f_n z_n$ is a n-cycle. Next, if $z + B_n(\boldsymbol{C}) = y + B_n(\boldsymbol{C})$, then $z - y \in B_n(\boldsymbol{C})$. Then $z - y = d_{n+1} c$ for some $c \in C_{n+1}$. Then $f_n z - f_n y = f_n d_{n+1} c = d'_{n+1} f_{n+1} c \in B_n(\boldsymbol{C}')$. Hence $cls(f_n z) = cls(f_n y)$, well done.

$\rightsquigarrow$ **Step 2. Show that H_n is a functor.** $H_n(1_{\boldsymbol{C}}) = 1$. Moreover, let f, g are chain maps and gf is well defined. Then

$$H_n(gf) : cls(z) \mapsto cls((gf)_n z) = cls(g_n f_n z) = H_n(g)(cls(f_n z)) = H_n(g)H_n(f)(cls(z)).$$

$\rightsquigarrow$ **Step 3. Show that $H_n(f)$ is additive.** Let f, g are chain maps, then

$$H_n(f+g) : cls(z) \mapsto cls(f_n z + g_n z) = cls(f_n z) + cls(g_n z) = (H_n(f) + H_n(g))cls(z).$$

Well done. $\square$

Theorem 2.1 (Long Exact Sequence)**.** *Let $\mathcal{A}$ be an abelian category. If $0 \to \boldsymbol{C}' \xrightarrow{i} \boldsymbol{C} \xrightarrow{p} \boldsymbol{C}'' \to 0$ is an exact sequence, then there is an exact sequence*

$$\begin{tikzcd}
 & {} \arrow[r] & H_{n+1}(\boldsymbol{C}) \arrow[r] & H_{n+1}(\boldsymbol{C}'') \arrow[dll, "\partial_{n+1}"] \\
 & H_n(\boldsymbol{C}') \arrow[r, "i_*"] & H_n(\boldsymbol{C}) \arrow[r, "p_*"] & H_n(\boldsymbol{C}'') \arrow[dll, "\partial_n"] \\
 & H_{n-1}(\boldsymbol{C}') \arrow[r] & H_{n-1}(\boldsymbol{C}) \arrow[r] & {}
\end{tikzcd}$$

Before we prove this important theorem, we will discuss the definition and properties of ∂_n and check whether it is well defined. ∂_n is called **connecting homomorphism**.

Proposition 2.2. *Let $\mathcal{A}$ be an abelian category. If $0 \to \boldsymbol{C}' \xrightarrow{i} \boldsymbol{C} \xrightarrow{p} \boldsymbol{C}'' \to 0$ is an exact sequence, then for each $n \in \mathbb{Z}$, there is a morphism in $\mathcal{A}$, $\partial_n : H_n(\boldsymbol{C}'') \to H_{n-1}(\boldsymbol{C}')$ by $cls(z''_n) \mapsto cls(i^{-1}_{n-1} d_n p^{-1}_n z''_n)$.*

Proof. We now look at the diagram below. needless to say.

$$
\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C'_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "d'_{n+1}"] & C_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "d_{n+1}"] & C''_{n+1} \arrow[r] \arrow[d, "d''_{n+1}"] & 0 \\
0 \arrow[r] & C'_n \arrow[r, "i_n"] \arrow[d, "d'_n"] & C_n \arrow[r, "p_n"] \arrow[d, "d_n"] \arrow[d, bend right] & C''_n \arrow[r] \arrow[d, "d''_n"] \arrow[l, bend left] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r, "i_{n-1}"] \arrow[d] & C_{n-1} \arrow[r, "p_{n-1}"] \arrow[d] \arrow[l, bend left] & C''_{n-1} \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}
$$

$\rightsquigarrow$ **We claim that $i'_{n-1}d_np_n^{-1}$ make sense.** Let $z'' \in C''_n$ and $d''_nz'' = 0$. Since p_n is surjective, there is $c \in C_n$, we called it **lifting**, with $p_nc = z''$. Let $d_nc \in C_{n-1}$. Then $p_{n-1}d_nc = d''_np_nc = d''_nz'' = 0$, so that $d_nc \in \ker p_{n-1} = \mathrm{Im} i_{n-1}$. There is a unique $c' \in C'_{n-1}$ with $i_{n-1}c' = d_nc$, for i_{n-1} is an injection. Thus $c' = i'_{n-1}d_np_n^{-1}z''$. And $\partial_n(cls(z'')) = cls(c')$.

$\rightsquigarrow$ **Independence of the choice of lifting.** If $p_n\check{c} = z''$ where $\check{c} \in C_n$. Then $c - \check{c} \in \ker p_n = \mathrm{Im} i_n$, so there is $u' \in C'_n$ with $i_nu' = c - \check{c}$. Thus $i_{n-1}d'_nu' = d_ni_nu' = d_nc - d_n\check{c}$. Hence $i_{n-1}^{-1}d_nc - i_{n-1}^{-1}d_n\check{c} = d'u' \in B'_{n-1}$, that is, $cls(i_{n-1}^{-1}d_nc) = cls(i_{n-1}^{-1}d_n\check{c})$. So this part gives a well defined map $Z''_n \to C'_{n-1}/B'_{n-1}$ by $z'' \mapsto cls(i_{n-1}^{-1}d_np_n^{-1}z'')$.

$\rightsquigarrow$ **Map $Z''_n \to C'_{n-1}/B'_{n-1}$ is a homomorphism.** Let $z'', z''_1 \in Z''_n$ with $p_nc = z'', p_nc_1 = z''_1$. Since the map is independent of the choice of lifting, we can choose $c + c_1$ as a lifting of $z'' + z''_1$.

$\rightsquigarrow$ **Element $i_{n-1}^{-1}d_np_n^{-1}z''$ is a cycle.** Let $i_{n-1}c' = d_nc$, then $0 = d_{n-1}d_nc = d_{n-1}i_{n-1}c' = i_{n-2}d'_{n-1}c'$, os $d'_{n-1}c' = 0$ since i_{n-2} is injective. So c' is a cycle. Then we have homomorphism $Z''_n \to Z'_{n-1}/B'_{n-1} = H_{n-1}(\boldsymbol{C}')$ by $z'' \mapsto cls(i_{n-1}^{-1}d_np_n^{-1}z'')$.

$\rightsquigarrow$ **Subgroup B''_n goes into B'_{n-1}.** Let $z'' = d''_{n+1}c''$ where $c'' \in C''_{n+1}$. Let $p_{n+1}u = c''$, then $p_nd_{n+1}u = d''_{n+1}p_{n+1}u = d''_{n+1}c'' = z''$. We choose $d_{n+1}u$ with $p_nd_{n+1}u = z''$, so $\partial_n(cls(z'')) = cls(i_{n-1}^{-1}d_np_n^{-1}p_nd_{n+1}u) = cls(0)$. Then well done. $\square$

Proof of Theorem 3.1. The sequence $0 \to \boldsymbol{C}' \xrightarrow{i} \boldsymbol{C} \xrightarrow{p} \boldsymbol{C}'' \to 0$ is exact, we now consider the following diagram.

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C'_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "d'_{n+1}"] & C_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "d_{n+1}"] & C''_{n+1} \arrow[r] \arrow[d, "d''_{n+1}"] & 0 \\
0 \arrow[r] & C'_n \arrow[r, "i_n"] \arrow[d, "d'_n"] & C_n \arrow[r, "p_n"] \arrow[d, "d_n"] & C''_n \arrow[r] \arrow[d, "d''_n"] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r, "i_{n-1}"] \arrow[d] & C_{n-1} \arrow[r, "p_{n-1}"] \arrow[d] & C''_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\arrow[r] & H_{n+1}(\boldsymbol{C}) \arrow[r] & H_{n+1}(\boldsymbol{C''}) \arrow[dll, "\partial_{n+1}"] \\
H_n(\boldsymbol{C'}) \arrow[r, "i_*"] & H_n(\boldsymbol{C}) \arrow[r, "p_*"] & H_n(\boldsymbol{C''}) \arrow[dll, "\partial_n"] \\
H_{n-1}(\boldsymbol{C'}) \arrow[r] & H_{n-1}(\boldsymbol{C}) \arrow[r] &
\end{tikzcd}
$$

We need to verify six inclusions as follows.

$$
\begin{tikzcd}
 & C_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "d_{n+1}"] & C''_{n+1} \arrow[d, "d''_{n+1}"] \\
C'_n \arrow[r, "i_n"] \arrow[d, "d'_n"] & C_n \arrow[r, "p_n"] \arrow[d, "d_n"] & C''_n \\
C'_{n-1} \arrow[r, "i_{n-1}"] & C_{n-1} &
\end{tikzcd}
$$

1 $\rightsquigarrow$ $\mathbf{Im}\boldsymbol{i_*} \subset \mathbf{ker}\,\boldsymbol{p_*}$. This is because $p_*i_* = (pi)_* = 0_* = 0$;

2 $\rightsquigarrow$ $\mathbf{ker}\,\boldsymbol{p_*} \subset \mathbf{Im}\boldsymbol{i_*}$. Let $p_*cls(z_n) = cls(p_nz_n) = cls(0)$, then $p_nz_n = d''_{n+1}c''_{n+1}$ where $c''_{n+1} \in C''_{n+1}$. Then there is $c''_{n+1} = p_{n+1}c_{n+1}$ where $c_{n+1} \in C_{n+1}$ since p_{n+1} is surjective. Now we have $p_nz_n = d''_{n+1}p_{n+1}c_{n+1} = p_nd_{n+1}c_{n+1}$, then $p_n(z_n - d_{n+1}c_{n+1}) = 0$. Then there is $c'_n \in C'_n$ with $i_nc'_n = z_n - d_{n+1}c_{n+1}$. Since z_n is a cycle, i_{n-1} is injective and $i_{n-1}d'_nc'_n = d_ni_nc'_n = d_nz_n - d_nd_{n+1}c_{n+1} = 0$, then $d'_nc'_n = 0$ and c'_n is a cycle. Hence $i_*cls(c'_n) = cls(i_nc'_n) = cls(z_n - d_{n+1}c_{n+1}) = cls(z_n)$.

Independence of lifting.

3 $\rightsquigarrow$ $\mathbf{Im}\boldsymbol{p_*} \subset \mathbf{ker}\,(\boldsymbol{\partial_n})$. Let $p_*cls(z_n) = cls(p_nz_n) \in \mathrm{Im}p_*$, then $\partial_ncls(p_nz_n) = cls(i_{n-1}^{-1}d_np_n^{-1}p_nz_n)$. We can let $p_n^{-1}p_nz_n = z_n$ and $i_{n-1}^{-1}d_np_n^{-1}p_nz_n = 0$.

4 $\rightsquigarrow$ $\mathbf{ker}\,(\boldsymbol{\partial_n}) \subset \mathbf{Im}\boldsymbol{p_*}$. Let $\partial_ncls(z''_n) = cls(0)$, then there exists $z'_{n-1} = i_{n-1}^{-1}d_np_n^{-1}z''_n \in B'_{n-1}$, that is, $z'_{n-1} = d'_nc'_n$ for $c'_n \in C'_n$. But $i_{n-1}z'_{n-1} = i_{n-1}d'_nc'_n = d_ni_nc'_n$ and $i_{n-1}z'_{n-1} = d_np_n^{-1}z''_n$, so $d_n(p_n^{-1}z''_n - i_nc'_n) = 0$ and $p_n^{-1}z''_n - i_nc'_n$ is a cycle. Then we have $p_*cls(p_n^{-1}z''_n - i_nc'_n) = cls(p_np_n^{-1}z''_n - p_ni_nc'_n) = cls(z''_n)$.

5 $\rightsquigarrow$ $\mathbf{Im}\boldsymbol{\partial_{n+1}} \subset \mathbf{ker}\,\boldsymbol{i_*}$. Since $i_nz'_n = d_{n+1}p_{n+1}^{-1}z''_{n+1} \in B_{n-1}$, then we have $i_*\partial_{n+1}cls(z''_{n+1}) = cls(i_nz'_n) = 0$.

6 $\rightsquigarrow$ $\mathbf{ker}\,\boldsymbol{i_*} \subset \mathbf{Im}\boldsymbol{\partial_{n+1}}$. Let $i_*cls(z'_n) = cls(i_nz'_n) = cls(0)$, then $i_nz'_n = d_{n+1}c_{n+1}$ for some $c_{n+1} \in C_{n+1}$. Then $d''_{n+1}p_{n+1}c_{n+1} = p_nd_{n+1}c_{n+1} = p_ni_nz'_n = 0$. So $p_{n+1}c_{n+1}$ is a cycle. Then $\partial_{n+1}cls(p_{n+1}c_{n+1}) = cls(i_n^{-1}d_{n+1}p_{n+1}^{-1}p_{n+1}c_{n+1}) = cls(i_n^{-1}d_{n+1}c_{n+1}) = cls(i_n^{-1}i_nz'_n) = cls(z'_n)$. $\square$

Corollary 2.1 (Snake Lemma)**.** *Let $\mathcal{A}$ be an abelian category. We have the following diagram.*

$$
\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' \arrow[r] & 0
\end{tikzcd}
$$

Then we have $0 \to \ker f' \to \ker f \to \ker f'' \to \mathrm{coker} f' \to \mathrm{coker} f \to \mathrm{coker} f'' \to 0$.

Proof. This is trivial. Why snake? Look at the diagram below!

$$
\begin{tikzcd}
0 \arrow[r] & \ker f' \arrow[r, "g_0"] \arrow[d, "i'"] & \ker f \arrow[r, "h_0"] \arrow[d, "i"] & \ker f'' \arrow[d, "i''"] \arrow[dddll, dotted] & \\
0 \arrow[r] & M' \arrow[r, "g_1"] \arrow[d, "f'"] & M \arrow[r, "h_1"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "g_2"] \arrow[d, "j'"] & N \arrow[r, "h_2"] \arrow[d, "j"] & N'' \arrow[r] \arrow[d, "j''"] & 0 \\
 & \mathrm{coker} f' \arrow[r, "g_3"] & \mathrm{coker} f \arrow[r, "h_3"] & \mathrm{coker} f'' \arrow[r] & 0
\end{tikzcd}
$$

This is a snake. □

Remark 2.2. *Except this, we have the real snake lemma,consider the commutative diagram with exact rows.*

$$
\begin{tikzcd}
 & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r, "p"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i"] & B' \arrow[r] & C' &
\end{tikzcd}
$$

Then after define $\Delta : \ker\gamma \to \mathrm{coker}\alpha$ *by* $z \mapsto i^{-1}\beta p^{-1} z + \mathrm{Im}\alpha$, *which is well defined, we have* $\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\Delta} \mathrm{coker}\alpha \xrightarrow{i'} \mathrm{coker}\beta \to \mathrm{coker}\gamma$, *where* $i' : a' + \mathrm{Im}\alpha \mapsto ia' + \mathrm{Im}\beta$.

Theorem 2.2 (Naturality of ∂)**.** *Let $\mathcal{A}$ be an abelian category. Given a diagram with exact rows.*

$$
\begin{tikzcd}
0 \arrow[r] & \mathbf{C}' \arrow[r, "i"] \arrow[d, "f"] & \mathbf{C} \arrow[r, "p"] \arrow[d, "g"] & \mathbf{C}'' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \mathbf{A}' \arrow[r, "j"] & \mathbf{A} \arrow[r, "q"] & \mathbf{A}'' \arrow[r] & 0
\end{tikzcd}
$$

Then we have the following diagram with exact rows.

$$
\begin{tikzcd}
{} \arrow[r] & H_n(\mathbf{C}') \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(\mathbf{C}) \arrow[r, "p_*"] \arrow[d, "g_*"] & H_n(\mathbf{C}'') \arrow[r, "\partial"] \arrow[d, "h_*"] & H_{n-1}(\mathbf{C}') \arrow[r] \arrow[d, "f_*"] & {} \\
{} \arrow[r] & H_n(\mathbf{A}') \arrow[r, "j_*"] & H_n(\mathbf{A}) \arrow[r, "q_*"] & H_n(\mathbf{A}'') \arrow[r, "\partial'"] & H_{n-1}(\mathbf{A}') \arrow[r] & {}
\end{tikzcd}
$$

Proof. Trivial by diagram chase. □

Example 2.1. *We know that if F is flat, then $\mathrm{Tor}_n^R(F, M) = \{0\}$ for all $n \geq 1$ and $M \in {}_R\mathbf{Mod}$. Conversely, if $\mathrm{Tor}_1^R(F, M) = \{0\}$ for all $M \in {}_R\mathbf{Mod}$, then F is flat. Now we consider a* ***proposition: if $0 \to A \to B \to C \to 0$ is an exact sequence of modules, and C is flat, then A is flat iff B is flat.*** *Now for any module X,* *we have* $\mathrm{Tor}_2^R(C,X) \to \mathrm{Tor}_1^R(A,X) \to \mathrm{Tor}_1^R(B,X) \to \mathrm{Tor}_1^R(C,X)$, *since* $\mathrm{Tor}_2^R(C,X) = \mathrm{Tor}_1^R(C,X) = 0$, *then* $\mathrm{Tor}_1^R(A,X) \cong \mathrm{Tor}_1^R(B,X)$, *well done.*

How to prove the long exact sequence of left derived functor? First by Horseshoe Lemma, we have $0 \to \boldsymbol{P}'_{A'} \to \widetilde{\boldsymbol{P}}_A \to \boldsymbol{P}''_{A''} \to 0$. *Since additive functor preserve split short exact sequence, then we have exact* $0 \to T\boldsymbol{P}'_{A'} \to T\widetilde{\boldsymbol{P}}_A \to T\boldsymbol{P}''_{A''} \to 0$. *Then there a long exact sequence. Next just need to consider the relation between* $\widetilde{\boldsymbol{P}}_A$ *and original* $\boldsymbol{P}_A$.

Example 2.2. *Let X be a space and A be a nonempty closed subspace that is a deformation retract of some neighborhood in X, then we have*

$$\begin{tikzcd}
 & {} \arrow[r] & \widetilde{H}_{n+1}(X) \arrow[r] & \widetilde{H}_{n+1}(X/A) \arrow[dll, "\partial"] \\
 & \widetilde{H}_n(A) \arrow[r, "i_*"] & \widetilde{H}_n(X) \arrow[r, "j_*"] & \widetilde{H}_n(X/A) \arrow[dll, "\partial"] \\
 & \widetilde{H}_{n-1}(A) \arrow[r] & \widetilde{H}_{n-1}(X) \arrow[r] & {}
\end{tikzcd}$$

Then consider $X = D^n$ and $A = S^{n-1}$, then $D^n/S^{n-1} \cong S^n$. Use this and the knowledge in algebraic topology inductively, we have $\widetilde{H}_n(S^n) \cong \mathbb{Z}$ and $\widetilde{H}_i(S^n) = 0, i \neq n$.

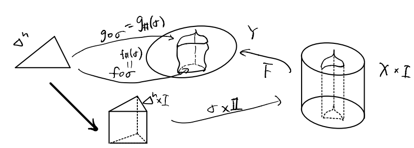
Definition 2.2. *Let $\boldsymbol{C}, \boldsymbol{D}$ be complexes, let $p \in \mathbb{Z}$. A* ***map of degree*** *p, denoted by $s : \boldsymbol{C} \to \boldsymbol{D}$, is a sequence $s = (s_n)$ with $s_n : C_n \to D_{n+p}$ for all n.*

Definition 2.3. *Chain maps $f, g : (\boldsymbol{C}, d) \to (\boldsymbol{C}', d')$ are* ***homotopic****, denoted by $f \simeq g$, if, for all n, there is a map $s = (s_n) : \boldsymbol{C} \to \boldsymbol{C}'$ of degree $+1$ with the following diagram and $f_n - g_n = d'_{n+1}s_n + s_{n-1}d_n$.*

$S_n : C_n \to C'_{n+1}$

Homotopy $F : X \times I \to Y$.

$$\begin{tikzcd}
 & {} \arrow[r] & C_n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & C_{n-1} \arrow[r] \arrow[dl, "s_{n-1}"] & {} \\
{} \arrow[r] & C'_{n+1} \arrow[r, "d'_{n+1}"'] & C'_n \arrow[r] & {} &
\end{tikzcd}$$



$\partial P - P\partial = g_\# - f_\#$.

prism operator $\Rightarrow P = \sum (-1)^i F \circ (\sigma \times \mathbb{I}) | [v_0, \dots, v_i, w_i, \dots, w_n]$.

$P : C_n(X) \longrightarrow C_{n+1}(Y)$.

If $f \simeq 0$, we called f is null-homotopic. 收缩同伦

A complex $(\boldsymbol{D}, e)$ has a ***contracting homotopy*** *if its identity $1_{\boldsymbol{D}}$ is null-homotopic.*

Theorem 2.3. *Homotopic chain maps induce the same morphism in homology. That is, if $f, g : (\boldsymbol{C}, d) \to (\boldsymbol{C}', d')$ are chain maps with $f \simeq g$, then $f_{*n} = g_{*n} : H_n(\boldsymbol{C}) \to H_n(\boldsymbol{C}')$ for all n.*

Proof. Let z is a n-cycle, then $f_n z - g_n z = d'_{n+1}s_n z - s_{n-1}\underbrace{d_n z}_{=0} = d'_{n+1}s_n z \in B_n(\boldsymbol{C}')$, so $f_{*n} = g_{*n}$, well done. $\square$

Proposition. So if a complex C having a contracting homotopy, then $H_n(C) = 0$ for all n.

[Pf.]

[Pf.] Since $1_C \simeq 0$, this is trivial by Thm 2.5.

3 More Theorems

Theorem 3.1. *If $(\boldsymbol{C}^i)_{i \in I}, (\varphi^i_j)_{i \leq j}$ is a direct system of complexes over a directed index set, then for all $n \geq 0$, we have*

$$H_n(\varinjlim \boldsymbol{C}^i) \cong \varinjlim H_n(\boldsymbol{C}^i).$$

Theorem 3.2 (Barratt-Whitehead)**.** *Consider the diagram with exact rows.*

$$\begin{tikzcd}
\arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "f_n"] & B_n \arrow[r, "p_n"] \arrow[d, "g_n"] & C_n \arrow[r, "\partial_n"] \arrow[d, "h_n", "\cong"'] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & {} \\
\arrow[r] & A'_n \arrow[r, "j_n"] & B'_n \arrow[r, "q_n"] & C'_n \arrow[r] & A'_{n-1} \arrow[r] & {}
\end{tikzcd}$$

If each h_n is an isomorphism, then we have

$$\begin{aligned}
&\longrightarrow A'_{n+1} \oplus B_{n+1} \longrightarrow B'_{n+1} \\
&\hookrightarrow A_n \xrightarrow{(f_n, i_n)} A'_n \oplus B_n \xrightarrow{j_n - g_n} B'_n \\
&\xrightarrow{\partial_n h_n^{-1} q_n} A_{n-1} \longrightarrow A'_{n-1} \oplus B_{n-1} \longrightarrow
\end{aligned}$$

where $(f_n, i_n) : a_n \mapsto (f_n a_n, i_n a_n)$ and $j_n - g_n : (a'_n, b_n) \mapsto j_n a'_n - g_n b_n$.

Theorem 3.3 (Mayer-Vietoris)**.** *Given a commutative diagram with exact rows.*

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{C}' \arrow[r, "i"] \arrow[d, "f"] & \mathbf{C} \arrow[r, "p"] \arrow[d, "g"] & \mathbf{C}'' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \mathbf{A}' \arrow[r, "j"] & \mathbf{A} \arrow[r, "q"] & \mathbf{A}'' \arrow[r] & 0
\end{tikzcd}$$

If every h_* in the following diagram

$$\begin{tikzcd}
\arrow[r] & H_n(\mathbf{C}') \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(\mathbf{C}) \arrow[r, "p_*"] \arrow[d, "g_*"] & H_n(\mathbf{C}'') \arrow[r, "\partial"] \arrow[d, "h_*", "\cong"'] & H_{n-1}(\mathbf{C}') \arrow[r] \arrow[d, "f_*"] & {} \\
\arrow[r] & H_n(\mathbf{A}') \arrow[r, "j_*"] & H_n(\mathbf{A}) \arrow[r, "q_*"] & H_n(\mathbf{A}'') \arrow[r, "\partial'"] & H_{n-1}(\mathbf{A}') \arrow[r] & {}
\end{tikzcd}$$

is an isomorphism, then we have an exact sequence.

$$
\begin{aligned}
&\longrightarrow H_{n+1}(\mathbf{A}') \oplus H_{n+1}(\mathbf{C}) \longrightarrow H_{n+1}(\mathbf{A}) \\
&\hookrightarrow H_n(\mathbf{C}') \longrightarrow H_n(\mathbf{A}') \oplus H_n(\mathbf{C}) \longrightarrow H_n(\mathbf{A}) \\
&\hookrightarrow H_{n-1}(\mathbf{C}') \longrightarrow H_{n-1}(\mathbf{A}') \oplus H_{n-1}(\mathbf{C}) \longrightarrow
\end{aligned}
$$