§11. Sphere Bundles.

By the Leray–Hirsch thm, if $\exists$ closed global n-form on E restrict to fibers $\Rightarrow$ generated cohom of fiber, then $H^*(E) \cong H^*(M) \otimes H^*(S^n)$

So we need to know when this global form exists!

$\boxed{1. \text{ Orientability.}}$

$E \xrightarrow{S^n} M$. Orientable

Choose generator $[\sigma_x] \in H^n(E_x)$

$\Rightarrow$ [local compatibility]:

① $\forall x, \exists x \in U$, & $[\sigma_U] \in H^n(E|_U)$ s.t. $\forall y \in U$, $[\sigma_U]$ restrict to y is $[\sigma_y] \in H^n[E_y]$.

$\Leftrightarrow$ ② $\Leftrightarrow$ cover $\{U_\alpha\}$ of M & generators $[\sigma_\alpha] \in H^n(E|_{U_\alpha})$ s.t. $[\sigma_\alpha] = [\sigma_\beta] \in H^n(E|_{U_\alpha \cap U_\beta})$.

$\int \sigma = 1 \Rightarrow$ two choices $[\sigma_\alpha] = \pm[\sigma_\beta]$ on $U_\alpha \cap U_\beta$.

$\Rightarrow S^0$-bundle over M

we have ~~...~~ $E \to M$ orientable $\Leftrightarrow S^0$-bundle has 2-connected components.

Remark: $[\sigma_\alpha] = [\sigma_\beta]$ over $U_\alpha \cap U_\beta$ but doesn't mean $\Rightarrow$ global cohomology class! since $\sigma_\alpha - \sigma_\beta$ = exact. May be $\Rightarrow$ not a global form!

$\Rightarrow E \Rightarrow$ rank $n+1$ vect-bundle $\gg O(n+1)$

$S(E)$ (unit sphere bundle)

fibers $\Rightarrow$ all unit vectors in E_x.

Remark: Fix orientation on S^n

$\int_{S^n} g^*\sigma = \int_{g(S^n)} \sigma = \int_{S^n} \sigma = 1$. if $g \in SO$

$\Rightarrow \forall g \in O(n+1)$, $[g^*\sigma] = [\sigma]$ $\Leftrightarrow$ g positive

▷ **Prop.** Vect-bundle E orientable $\Leftrightarrow S(E)$ orientable.

Proof. ($\Rightarrow$) Trivialization $\{U_\alpha, \phi_\alpha\}$ of E

$g_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1} \in SO(n+1)$

Fix generator σ on S^n.

$\rho_\alpha : U_\alpha \times S^n \to S^n$.

Let $\pi : S(E) \to M$.

$$\begin{array}{ccc} S(E)|_{U_\alpha} & \xrightarrow{\pi} & M \\ \downarrow \phi_\alpha & & \\ U_\alpha \times S^n & & \\ \downarrow \rho_\alpha & & \\ S^n & & \end{array}$$

Define $[\sigma_\alpha] \in H^n(S(E)|_{U_\alpha})$

$[\sigma_\alpha] = \phi_\alpha^* \rho_\alpha^* [\sigma]$

~~Let~~ Denote $[\sigma_\alpha]|_x$ & $\phi_\alpha|_x$ for $[\sigma_\alpha]|_{\pi^{-1}(x)}$ & $\phi_\alpha|_{\pi^{-1}(x)}$.

$[\sigma_\alpha]|_x = (\phi_\alpha|_x)^*[\sigma]$

$\Rightarrow [\sigma_\beta]|_x = [\sigma_\alpha]|_x \Leftrightarrow [\sigma] = g_{\alpha\beta}(x)^*[\sigma]$ on $x \in U_\alpha \cap U_\beta$

$\Rightarrow [\sigma_\beta] = [\sigma_\alpha]$ on $U_\alpha \cap U_\beta$.

($\Leftarrow$) $\{U_\alpha, [\sigma_\alpha]\}$ orientation on $S(E)$.

(S^n, σ) oriented sphere in $\mathbb{R}^{n+1}$.

Take $\phi_\alpha : S(E)|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times S^n$.

s.t. ϕ_α preserve metric, $\phi_\alpha^* \rho_\alpha^* [\sigma] = [\sigma_\alpha]$.

$\Rightarrow \forall x \in U_\alpha \cap U_\beta$, $[\sigma] = g_{\alpha\beta}(x)^*[\sigma]$

$\Rightarrow g_{\alpha\beta} \in SO(n+1)$ □

($SO(1) = \{1\} \Rightarrow$ line bundle orientable $\Leftrightarrow$ trivial! $\Rightarrow$ stable classifying)

▷ **Prop.** Vect-bundle E orientable $\Leftrightarrow \det(E)$ orientable

Proof. $(U_\alpha, g_{\alpha\beta})$ on $E \Rightarrow (U_\alpha, \det(g_{\alpha\beta}))$ on $\det(E)$

$g_{\alpha\beta} \in SO(n+1) \Leftrightarrow \det(g_{\alpha\beta}) = 1 \in SO(1)$.

□

As the orien-bundle

$\Rightarrow E$ orientable $\Leftrightarrow S(\det E)$ disconnected.
(see $\det E$ line bundle)

▷ **Prop.** M simply-connected
$\Rightarrow$ ~~vect bun~~ VB $E \to M$ orientable!

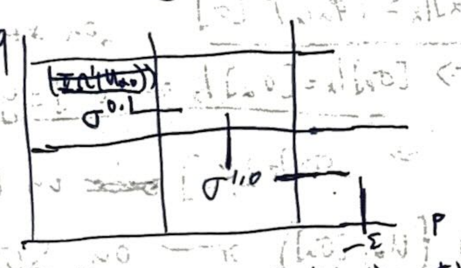
Proof. M is itself universal cover of M
$\Rightarrow S(\det E)$ disconnected ✓ □

▷ **Cor.** M simply-connected
$\Rightarrow M$ orientable.
(use tangent bundle)

2. The Euler Class of an Oriented Sphere Bundle.

① ▷ First consider $\pi: E \xrightarrow{S^1} M$ with $\mathrm{Diff}(S^1)$.
Good cover $\{U_\alpha\}$, choose generators $[\sigma_\alpha] \in H^1(E|_{U_\alpha})$

$\{\sigma_\alpha\}$ is $\sigma^{0,1}$ in $C^*(\pi^{-1}\mathcal{U}, \Omega^*)$:



Since $\pi^*: H^*_{DR}(E) \cong H_D(C^*(\pi^{-1}\mathcal{U}, \Omega^*))$,
we will extend $\sigma^{0,1}$ to a D-cocycle.

Since $d\sigma^{0,1}=0$, we need $(\delta\sigma^{0,1})_{\alpha\beta} = \sigma_\beta - \sigma_\alpha$
exact. i.e. $[\sigma_\alpha]=[\sigma_\beta]$, $\forall \alpha,\beta$ on $U_\alpha \cap U_\beta$

Since E orientable, we choose $\{\sigma_\alpha\}$ to its orientation! So this is right.

$\exists \sigma^{1,0} \in C^1(\pi^{-1}\mathcal{U}, \Omega^0)$ s.t. $d\sigma^{1,0} = \delta\sigma^{0,1}$!

[So, $\sigma^{0,1}+\sigma^{1,0}$ is D-cocycle $\Leftrightarrow \delta\sigma^{1,0}=0$.]

Since $d\delta\sigma^{1,0} = \delta d\sigma^{1,0} = \delta\delta\sigma^{0,1}=0$, then
$\exists -\varepsilon \in C^2(\pi^{-1}\mathcal{U}, \mathbb{R}) \xrightarrow[\text{combinatorics}]{\text{same}} C^2(\mathcal{U}, \mathbb{R})$

s.t. $\delta\sigma^{1,0}$ comes from $-\varepsilon$. So $\delta\varepsilon=0$.

$\Rightarrow [\varepsilon] \in H^2(\mathcal{U}, \mathbb{R}) \cong H^2_{DR}(M)$.

Let $e(E) = [\varepsilon] \in H^2_{DR}(M)$

Euler class. [Affirm pre th?] is coincident to before].

② ▷ General. $\pi: E \xrightarrow{S^n} M$ $(n \geq 1)$.

Let E orientable, then
$\exists \underset{\text{closed}}{\sigma^{0,n}} \in C^0(\pi^{-1}\mathcal{U}, \Omega^n)$ with
$\underbrace{\delta\sigma^{0,n} = d\sigma^{1,n-1} = -D''\sigma^{1,n-1}}_{\text{(compatible)}}$.

Since every closed $(n-1)$-form on $E|_{U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}}$
is exact (since $H^k(S^n)=0$ for $0<k<n$).

Then we can extend to $\sigma^{n,0} \in C^n(\pi^{-1}\mathcal{U}, \Omega^0)$.

$\Rightarrow$ As a D-cocycle, we have
$$D\sigma = D\Big(\sum_{p+q=n} \sigma^{p,q}\Big) = \delta\sigma^{n,0}.$$

Since $d\delta\sigma^{n,0} = \delta d\sigma^{n,0} = \pm\delta\delta\sigma^{n-1,1} = 0$,
we have $D\sigma = \delta\sigma^{n,0} = i(-\varepsilon)$
for $\varepsilon \in C^{n+1}(\pi^{-1}\mathcal{U}, \mathbb{R}) = C^{n+1}(\mathcal{U}, \mathbb{R})$.

$\delta\varepsilon = 0 \Rightarrow [\varepsilon] = e(E) \in H^{n+1}(\mathcal{U}, \mathbb{R})$
$\cong H^{n+1}_{DR}(M)$.

[Define S^0-bundle $\Rightarrow e=0$]

RMK1: If we choose the opposite orientation, the Euler class is $-e(M)$!

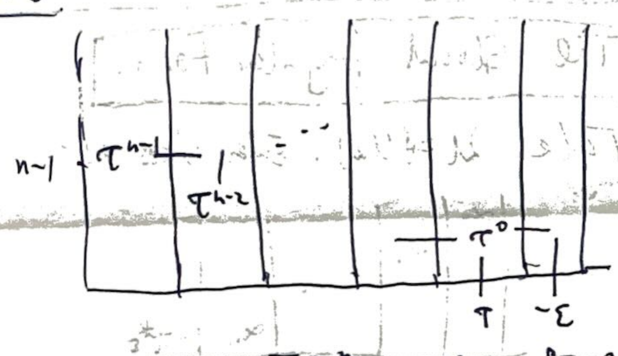
RMK2: If E oriented vector bundle, then $E^0 = E\setminus\{0\}$ has the same homotopy type of an oriented sphere bundle. [If rank $E = n \Rightarrow S^{n-1}$-bundle]

$\Longleftarrow$ Endow E with Riemann metric $\Rightarrow S(E) \simeq E^0$. $\Rightarrow S(E)$ — Euler.

Show that $e(E)$ is well defined

▷ Prop. Given an orientation $\{[\sigma_\alpha]\}$, the Euler class is independent of choice of $\sigma^{j,n-j}$, $j=0,\dots,n$.

Proof



Take another $\bar\sigma^{0,n}$ ⇒ represent $\{[\sigma_\alpha]\}$.

$\Rightarrow \bar\sigma^{0,n} - \sigma^{0,n} = d\tau^{n-1}$ for $\tau^{n-1} \in C^0(\pi^{-1}\mathcal{U}, \Omega^{n-1})$

Since $d(\delta\tau^{n-1}) = \delta d\tau^{n-1}$

$= \delta(\bar\sigma^{0,n} - \sigma^{0,n}) = d(\bar\sigma^{1,n-1} - \sigma^{1,n-1})$,

then $\delta\tau^{n-1} - (\bar\sigma^{1,n-1} - \sigma^{1,n-1}) = d\tau^{n-2}$

for some $\tau^{n-2} \in C^1(\pi^{-1}\mathcal{U}, \Omega^{n-2})$.

Repeat this process, we have

$\Rightarrow \delta\tau^0 - (\bar\sigma^{n,0} - \sigma^{n,0}) = i\tau$,

$\tau \in C^n(\pi^{-1}\mathcal{U}, \mathbb{R})$. Then

$\delta\delta\tau^0 - (\delta\bar\sigma^{n,0} - \delta\sigma^{n,0}) = \bar\varepsilon - \varepsilon = \delta\tau$

$\Rightarrow [\bar\varepsilon] = [\varepsilon] \in H^{n+1}(\mathcal{U}, \mathbb{R})$

$\cong H^{n+1}_{DR}(M)$. □

▷ Prop. Euler class $e(E)$ is independent of choice of the good cover!

Proof. Given two good covers $\mathcal{U}$, $\mathcal{W}$, we find a common refinement (good cover) $\mathcal{B}$. We have:

Let $[\varepsilon_{\mathcal{U}}] \in H^{n+1}(\mathcal{U}, \mathbb{R})$ with

$[\varepsilon_{\mathcal{B}}] \in H^{n+1}(\mathcal{B}, \mathbb{R})$ (Euler class)

$$\begin{tikzcd} H^{n+1}(\mathcal{U},\mathbb{R}) \arrow[rr, "r^*"] \arrow[dr, "\alpha"', "\cong"] & & H^{n+1}(\mathcal{B},\mathbb{R}) \arrow[dl, "\beta", "\cong"'] \\ & H^{n+1}_{DR}(M) & \end{tikzcd}$$

We claim that $\alpha[\varepsilon_{\mathcal{U}}] = \beta[\varepsilon_{\mathcal{B}}]$.

Choose $\{\sigma^{0,n}\}$ on $\pi^{-1}\mathcal{B}$ as restriction of $\{\sigma^{0,n}\}$ on $\pi^{-1}\mathcal{U}$, then $\alpha[\varepsilon_{\mathcal{U}}] = \beta[\varepsilon_{\mathcal{B}}]$ (by the previous prop).

~~Then~~ Similarly, do this for $\mathcal{W}$ & $\mathcal{B}$. □

RMK. When $e(E)=0$, then choose $\sigma^{n,0} \Rightarrow D\sigma = 0$.

$\Rightarrow$ D-cocycle $\sigma \Rightarrow$ global form restricts to d-cohom $\sigma^{0,n}$.

So: [∃ global form res— fiber ⇒ generator iff

① E orientable;

② $e(E) = [0]$.]

(Another's idea as follows:

① the global form nowhere vanish s.t. (fiber generator) ⇒ orientable. ✓

② $D\sigma = 0 \iff \delta\sigma^{n,0} = 0 \iff e(E) = 0$. ✓)

If $E = M\times S^n$, then we have $\sigma^{0,n}$, then $(\delta\sigma^{0,n})_{\alpha\beta} = \sigma_\beta - \sigma_\alpha = \underline{\underline{0}} \Rightarrow \sigma^{1,n-1} \simeq 0$

$\Rightarrow e(E) = 0$. Actually, we have:

Prop. $E \Rightarrow$ oriented sphere bundle has a global section, then $e(E) = 0$.

Proof. Let $s: M \to E \in \Gamma(M, E)$, then $\pi\circ s = 1 \Rightarrow s^*\pi^* = 1$. Since $D\sigma = -\pi^*\varepsilon$

$\Rightarrow -\varepsilon = Ds^*\sigma = \delta s^*\sigma$

$\Rightarrow e(E) = [0] \in H^*(\mathcal{U}, \mathbb{R}) = H^*(M)$. □

~~Alex~~ **RMK**. In the case of circle-bundle, we will show that the construction above of Euler class coincident to before.

$\{\theta_\alpha\}$ angular coordinate over U_α.

$\left[\frac{d\theta_\alpha}{2\pi}\right] \Rightarrow$ generator of $H^1(E|_{U_\alpha})$.

Furthermore, $\frac{d\theta_\beta}{2\pi} - \frac{d\theta_\alpha}{2\pi} = \pi^* \frac{d\phi_{\alpha\beta}}{2\pi} = \pi^*\xi_\beta - \pi^*\xi_\alpha$ for some 1-form ξ_α on U_α.

As follows:

$C^*(\pi^{-1}\mathcal{U}, \Omega^*)$

$\Omega^*(E)$			
	$\frac{d\theta_\alpha}{2\pi}$	$\frac{\pi^* d\phi_{\alpha\beta}}{2\pi}$	
		$\frac{\pi^* d\phi_{\alpha\beta}}{2\pi}$	$-\pi^*\varepsilon$
			$-\pi^*\varepsilon$

$C^*(\pi^{-1}\mathcal{U}, \mathbb{R})$

As $\check{H}^2(\mathcal{U}, \mathbb{R}) \xrightarrow{\simeq} H^2_{DR}(M)$

$\varepsilon \longmapsto (-D''K)^2\varepsilon \in \prod_\alpha \Omega^2(U_\alpha)$

$\xi_\beta - \xi_\alpha = \frac{1}{2\pi} d\phi_{\alpha\beta} \Rightarrow \delta\xi = \frac{1}{2\pi} d\phi$

$\Rightarrow \xi = \delta K\xi + \frac{1}{2\pi} K d\phi$. Take ξ as $\frac{1}{2\pi}Kd\phi$

Since $\frac{\delta\phi}{2\pi} = -\varepsilon \Rightarrow -K\varepsilon = \frac{K\delta\phi}{2\pi}$

$= \frac{\phi}{2\pi} - \frac{\delta K\phi}{2\pi}$

$= \frac{\phi}{2\pi}$ (modulo δ-coboundary).

$\Rightarrow (-D''K)^2\varepsilon = -dKdK\varepsilon$

$= dKd\left(\frac{\phi}{2\pi} - \frac{\delta K\phi}{2\pi}\right)$

$= dK\frac{d\phi}{2\pi} - \frac{dKd\delta(K\phi)}{2\pi}$

$= d\xi + dKd\delta\tau$

$dKd\delta\tau = dK\delta d\tau$

$= d(1-\delta K)d\tau$

$= -\delta dKd\tau$

$Kd\tau \in \Omega^1(M) \Rightarrow dKd\tau \Rightarrow$ exact (global)

$\delta dKd\tau \in \prod_\alpha \Omega^2(U_\alpha) \Rightarrow \delta dKd\tau \simeq 0$

$\Rightarrow (-D''K)^2\varepsilon = d\xi$ ✓ □

$(d\phi = -\pi^* d\xi = -\pi^* e)$.

3. The Global Angular Form.

Take $\mathcal{U} = \{U_\alpha\}$. Euler class as:

α_0	α_1	$\cdots$		
			α_n	$-\pi^*\varepsilon$

where $\alpha_0 \in C^0(\pi^{-1}\mathcal{U}, \Omega^n) \Rightarrow$ orientation of E

$\begin{cases} \delta\alpha_i = -D''\alpha_{i+1}, & i = 0, \dots, n-1 \\ \delta\alpha_n = -\pi^*\varepsilon. \end{cases}$

$\Rightarrow D(\alpha_0 + \dots + \alpha_n) = -\pi^*\varepsilon \quad (\alpha_i = \sigma^{i, n-i})$.

P.O.U $\Rightarrow \{\rho_\alpha\}$ of $\mathcal{U} \Rightarrow \{\pi^*\rho_\alpha\} \Rightarrow$ of $\pi^{-1}\mathcal{U}$

Define homotopy operator K on both $C^*(\mathcal{U}, \Omega^*)$ & $C^*(\pi^{-1}\mathcal{U}, \Omega^*)$ as $(K\omega)_{\alpha_0\cdots\alpha_{p-1}} = \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_0\cdots\alpha_{p-1}}$

Both as $\delta K + K\delta = 1$.

Since $(K\pi^*\omega)_{\alpha_0\cdots\alpha_{p-1}} = \sum_\alpha (\pi^*\rho_\alpha)(\pi^*\omega)_{\alpha\alpha_0\cdots\alpha_{p-1}}$

$= \pi^* \sum_\alpha \rho_\alpha \omega_{\alpha\alpha_0\cdots\alpha_{p-1}} = (\pi^* K\omega)$

$\Rightarrow K\pi^* = \pi^* K$.

- Let section $s: M \to E$, claim: $Ks^* = s^*K$.

Actually, we have:

$$(K s^* \omega)_{\alpha_0 \cdots \alpha_{p-1}}$$
$$= \sum_\alpha \rho_\alpha (s^*\omega)_{\alpha\alpha_0 \cdots \alpha_{p-1}}$$
$$= \sum_\alpha \rho_\alpha s^* \omega_{\alpha\alpha_0 \cdots \alpha_{p-1}} \qquad (s^*\pi^* = id)$$
$$= \sum_\alpha s^* \pi^* \rho_\alpha s^* \omega_{\alpha\alpha_0 \cdots \alpha_{p-1}}$$
$$= s^* \sum_\alpha \pi^* \rho_\alpha \omega_{\alpha\alpha_0 \cdots \alpha_{p-1}}$$
$$= (s^* K \omega)_{\alpha_0 \cdots \alpha_{p-1}}$$
$$\Rightarrow \underline{s^* K = K s^*}.$$

Use collating formula,

$$\psi = \sum_{i=0}^{n} (-1)^i (D'' K)^i \alpha_i + (-1)^{n+1} K (D''K)^n (-\pi^* \varepsilon)$$

$\Rightarrow$ global form on E. Moreover,

① $d\psi = (-1)^{n+1} dK (D''K)^n (-\pi^* \varepsilon)$

$= -\pi^* (-1)^{n+1} (D''K)^{n+1} \varepsilon \qquad (\pi^* K = K \pi^*)$

$= \underline{-\pi^* e}$ by the iso betw Čech & de Rham!

② ψ res to fiber, we have

(i). $\sum_{i=0}^{n} (-1)^i (D''K)^i \alpha_{i\,\text{res}} \sim \alpha_0 \pmod d$. $\longrightarrow$ generator

(ii). $(-1)^{n+1} K (D''K)^n (-\pi^* \varepsilon) = -\pi^* \left((-1)^{n+1} K (D''K)^n \varepsilon \right)$

res fun is $\underline{0}$ ~~(for S^n, $\varepsilon \in H^{n+1}$)~~.

$\longrightarrow -\pi^* \left((-1)^{n+1} K (D''K)^n \varepsilon \right) \Big|_{E_x}$

$= -\pi^* \left((-1)^{n+1} K (D''K)^n \varepsilon \big|_x \right)$

$= -\pi^* \left((-1)^{n+1} \sum_\alpha \rho_\alpha(x) (D''K)^n \varepsilon \right) \sim 0 \pmod d$

~~$\Rightarrow \psi|_{\text{fiber}} = \text{generator}$!~~

Σ So $\underline{\psi} \Rightarrow$ global angular form on Sphere bundles!

RMK. Use ψ, we can show if $\exists$ global section $s: M \to E$, then $e \simeq 0$.

Proof. $s^* d\psi = d s^* \psi = -s^* \pi^* e = -e$

$\Rightarrow e$ is exact $\Rightarrow [e] = 0$. $\square$

Something about orientation & double cover.

Claim. For a double cover $p: Y \to X$ where X is connected. The following statement is equivalent:

(a). $(p: Y \to X) \cong (\tau: X \amalg X \to X)$;

(b) p has a continuous section;

(c) Y is not connected.

Proof. (a) $\Rightarrow$ (b) consider natural inclusion $\underline{X \to X \amalg X}$ $\Rightarrow$ continuous section!

(b) $\Rightarrow$ (c) $\forall x \in X$, $\exists U_x \subseteq X$ of x s.t. $p^{-1}U_x = V_x \amalg V_x'$ $\Rightarrow$ locally trivialization

So $p|_{p^{-1}U_x}: V_x \amalg V_x' \to U_x$ has two exactly continuous sections.

If p has a continuous section $s: X \to Y$, then $s|_{U_x}$ is $p|_{V_x}$ & $p|_{V_x'}$,

So $s(U_x) = V_x$, or $s(U_x) = V_x'$ depend on whether $s(x) \in V_x$ or V_x'.

WLOG let $s(U_x) = V_x \Rightarrow s(X) = \bigcup_{x \in X} s(U_x) = \bigcup_{x \in X} V_x$

$Y - s(X) = \bigcup_{x \in X} V_x' \Rightarrow Y = s(X) \amalg (Y - s(X))$. ✓

(c) $\Rightarrow$ (a) $Y = X_1 \amalg X_2$. $X_i \to X$ one-sheeted $\Rightarrow X_i \cong X$. ✓ □