BASIC HODGE THEORY ON COMPLEX MANIFOLDS

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Abstract

In this article, we will introduce the Hodge Theory on compact Hermitian and Kähler manifolds. First of all, we will talk about the basic definitions of some operators and cohomology to make sure we can discuss our theory smoothly. Then we write many properties about Hermitian manifolds which every one should know it. Then two versions of Hodge decomposition theorems are discussed, but we only give the proof of the second version since the first version will use some PDEs. Then we focus on the applications of Hodge decomposition of Kähler manifolds such as hard Lefschetz theorem and Hodge index theorem. Finally, we will introduce some basic things about Hodge structure and introduce the meaning of Hodge conjecture. So, as you see, this article just contains some most basic Hodge theory because the writer also has to learn many things.

Contents

1 Introductions

Hodge theory, named after W.V.D.Hodge, has its origin in works by Abel, Jacobi, Gauss, Legendre and Weierstrass among many others on the periods of integrals of rational one-forms. In 1931, Hodge assimilated de Rham's theorem and defined the Hodge star operator. It would allow him to define harmonic forms and so fine the de Rham theory. Hodge's major contribution was in the conception of harmonic integrals and their relevance to algebraic geometry.

The relative theory appeared in the late 1960's with the work of Griffiths. He found that higher weights generalization of the ordinary Jacobian, the intermediate Jacobian, need not be polarized. He generalized Abel-Jacobi maps in this set-up and used these to explain the difference of cycles and divisors. The important insight that any algebraic variety has a generalized notion of Hodge structure was worked out in Hodge II,III. In the relative setting, if the family acquires singularities, the Hodge structure on the cohomology of a fiber may degenerate when the base point goes to the singular locus, leading to the so-called limit mixed Hodge structure. Morihiko Saito introduced the theory of Mixed Hodge Modules around 1985, which unifies many theories: algebraic D-modules and perverse sheaves.

So this is a brief history of Hodge theory, extracted from [12]. In our field, we just discuss the most basic Hodge theory. Including the basic properties of Hermitian manifolds and Kähler manifolds, then we will introduce the Hodge decomposition of Hermitian manifolds. This theorem is amazing and it can used to prove the Hodge decomposition in Kähler case. Then it has many applications, these applications help us to discover the structures of cohomology of the compact Kähler manifolds. With these structures, we can distinguish the manifold is Kähler or not. Like the famous Hopf surface, with these analysis, we can find that it is not a Kähler manifold. In this part, we will divide it into two parts, including lower dimension and higher dimension cohomology. Finally, we will introduce a little bit of Hodge structures. Its goal is to abstract the Hodge structure in Kähler case into more general case, as any R-modules. Use this we introduce the meaning of Hodge Conjecture, one of the most famous open problem in algebraic geometry.

As you see, this article is just the summary of the most basic Hodge theory and this is also a basic part of complex geometry. The further materials as [3] is about the Hodge theory on complex algebraic geometry, including many things we not introduced, as the Frölicher spectral sequence. And [12], a specialized book about Hodge structures, including many applications of mixed Hodge structures.

2 Preliminaries

In this section, we will introduce some basic definitions about some cohomology and some differential operators and linear operators to make sure we can introduce Hodge theory smoothly.

We will omit the basic definition about complex manifolds, complex structure and almost complex structure. It's worth mentioning that the complex manifolds have natural almost complex structure, but converse statement is wrong. But if X endowed by an integrable almost complex structure, then it must induced by a complex structure. This due to *Newlander-Nirenberg*, see [1] for a easy proof in the case of analytic manifolds.

Definition 2.1. *Let X be a manifold endowed by an integrable almost complex structure, then the $(p,q)-$*Dolbeault cohomology *is the vector space*

$$H^{p,q}(X) = \frac{\ker(\overline{\partial} : \mathscr{A}^{p,q}(X) \to \mathscr{A}^{p,q+1}(X))}{\operatorname{Im}(\overline{\partial} : \mathscr{A}^{p,q-1}(X) \to \mathscr{A}^{p,q}(X))}.$$

Actually[1], it's easy to see that $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$. This is very important.

Definition 2.2. *Let X be a complex manifold, then the* Bott-Chern cohomology *is*

$$H_{BC}^{p,q}(X) = \frac{\{\alpha \in \mathscr{A}^{p,q} : d\alpha = 0\}}{\partial\overline{\partial}\mathscr{A}^{p-1,q-1}(X)}.$$

Now we will endow the complex manifolds with a metric.

Definition 2.3. *Let X endowed by a Riemannian metric g and g is compatible with the natural almost complex structure, that is, for $x \in X$, we have $g_x(I_x(\alpha), I_x(\beta)) = g_x(\alpha, \beta)$, then we denote X be a* Hermitian manifold.

Remark 2.4. *This can induce a real $(1,1)-$form ω, called* Fundamental form, *by $\omega = g(I(\cdot), \cdot)$. Actually, we can use the almost complex structure and the fundamental form to determine a unique Hermitian matric $g(\cdot,\cdot) = \omega(\cdot, I(\cdot))$. Locally, we can write the fundamental form as $\omega = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^n h_{ij} dz_i \wedge d\overline{z_j}$ where (h_{ij}) is a positive definite Hermitian matrix.*

Now we begin to introduce some important differential and linear operators which we will use them to construct the whole theory.

[1]We omit the definition of (p,q)-forms where $\mathscr{A}^{p,q}(X)$ is global section of sheaf $\mathscr{A}_X^{p,q}$, the sheaf of sections of $\bigwedge^{p,q} X = \bigwedge^p (T^{1,0}X)^* \otimes_{\mathbb{C}} \bigwedge^q (T^{0,1}X)^*$.We also omit the definition of ∂ and $\overline{\partial}$, that is, $\partial = \Pi^{p+1,q} d, \overline{\partial} = \Pi^{p,q+1} d$. Locally, in some chart we can see that $\partial(f dz_{i_1} \wedge \cdots dz_{i_k} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_l}) = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i \wedge dz_{i_1} \wedge \cdots dz_{i_k} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_l}$, $\overline{\partial}$ is similar. This is just some linear algebra and you can see these in any complex geometry books.

Definition 2.5. *(i)* LEFSCHETZ OPERATOR*:* $L: \bigwedge_{\mathbb{C}}^k X \to \bigwedge_{\mathbb{C}}^{k+2} X, \alpha \mapsto \alpha \wedge \omega$*;*
(ii) HODGE $*-$OPERATOR*:* $*: \bigwedge_{\mathbb{C}}^k X \to \bigwedge_{\mathbb{C}}^{2n-k} X$ *with* $\alpha \wedge *\beta = g(\alpha, \beta) \cdot \mathrm{Vol}$[2]*;*
(iii) DUAL LEFSCHETZ OPERATOR*:* $\Lambda = *^{-1} \circ L \circ * : \bigwedge_{\mathbb{C}}^k X \to \bigwedge_{\mathbb{C}}^{k-2} X$*;*
(iv) We denote $H = \sum_{k=0}^{2n} (k-n)\Pi^k$, $\boldsymbol{I} = \sum_{p,q=0}^{n} (\sqrt{-1})^{p-q} \Pi^{p,q}$*;*
(v) For a m-dimensional oriented Riemannian manifold (M, g)*, the adjoint operator is* $d^* = (-1)^{m(k+1)+1} * \circ d \circ * : \mathscr{A}^k(M) \to \mathscr{A}^{k-1}(M)$*. The* LAPLACE OPERATOR $\Delta_d = dd^* + d^*d$*. But in the complex case, we have* $d^* = - * \circ d \circ *$*;*
(vi) we have $d^* = \partial^* + \overline{\partial}^*, (\partial^*)^2 = (\overline{\partial}^*)^2 = 0$*. The Laplace operator correspond to* $\partial, \overline{\partial}$ *is* $\Delta_\partial = \partial^*\partial + \partial\partial^* : \mathscr{A}^{p,q}(X) \to \mathscr{A}^{p,q}(X), \Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^* : \mathscr{A}^{p,q}(X) \to \mathscr{A}^{p,q}(X)$.

3 Hodge Theory on Hermitian Manifolds

Definition 3.1. *Let* (X, g) *be a Hermitian manifolds, we equipped* $\mathscr{A}_{\mathbb{C}}^*(X)$ *with a scalar product as*

$$(\alpha, \beta) = \int_X g(\alpha, \beta) * 1 = \int_X \alpha \wedge *\overline{\beta}.$$

Then we have some ADJOINT PROPERTY about this.

Proposition 3.2. *The operators* $\partial, \overline{\partial}$ *and* $\partial^*, \overline{\partial}^*$ *are adjoint about the scalar product* $(,)$.

Proof. For $\alpha \in \mathscr{A}^{p-1,q}(X), \beta \in \mathscr{A}^{p,q}(X)$, we have

$$\int_X \partial\alpha \wedge *\overline{\beta} = \int_X \partial(\alpha \wedge *\overline{\beta}) - (-1)^{p+q-1} \int_X \alpha \wedge \partial(*\overline{\beta}).$$

We find that $\int_X \partial(\alpha \wedge *\overline{\beta}) = 0$ by Stokes theorem since $\alpha \wedge *\overline{\beta}$ has bidegree $(n-1, n)$, so

$$(\partial\alpha, \beta) = -(-1)^{p+q-1} \int_X \alpha \wedge \partial(*\overline{\beta}) = \varepsilon \int_X g_{\mathbb{C}}(\alpha, -\partial^*\beta) * 1 = (\alpha, \partial^*\beta),$$

well done. □

Remark 3.3. *We could also verify that* $(\alpha, L\beta) = (\Lambda\alpha, \beta)$.

Proposition 3.4. *Let* (X, g) *be a compact Hermitian manifold. Then the following decompositions are orthogonal with respect to* $(,)$*:*

(i) DEGREE DECOMPOSITION*:* $\mathscr{A}_{\mathbb{C}}^*(X) = \bigoplus_k \mathscr{A}_{\mathbb{C}}^k(X)$*;*

(ii) BIDEGREE DECOMPOSITION*:* $\mathscr{A}_{\mathbb{C}}^k(X) = \bigoplus_{p+q=k} \mathscr{A}^{p,q}(X)$*;*

(iii) LEFSCHETZ DECOMPOSITION*:* $\mathscr{A}_{\mathbb{C}}^k(X) = \bigoplus_{i \geq 0} L^i P_{\mathbb{C}}^{k-2i}(X)$*.(*$P^j = \ker(\Lambda) \subset \mathscr{A}^k$*)*

[2] Let $e_1, ..., e_d$ be an orthonormal of vector space V and $\{i_1, ..., i_k, j_1, ..., j_{d-k}\} = \{1, ..., d\}$, then $*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \mathrm{sgn}(i_1, ..., i_k, j_1, ..., j_{d-k}) e_{j_1} \wedge \cdots \wedge e_{j_{d-k}}$.

Proof. Actually, these are purely linear algebra. See chapter 1 in [2]. □

Now we have a nice decomposition and we will prove this by the propositions above. But before doing this, we will prove the following useful lemma first.

Lemma 3.5. *Let (X, g) be a compact Hermitian manifold, then α is $\overline{\partial}-$harmonic if and only if $\overline{\partial}\alpha = \overline{\partial}^*\alpha = 0$. So is ∂.*

Proof. We find that

$$(\Delta_{\overline{\partial}}\alpha, \alpha) = (\overline{\partial}\overline{\partial}^*\alpha, \alpha) + (\overline{\partial}^*\overline{\partial}\alpha, \alpha) = \|\overline{\partial}^*\alpha\|^2 + \|\overline{\partial}\alpha\|^2,$$

and well done where the last step follows from Proposition 3.2. □

Corollary 3.6. *Let (X, g) be a Hermitian manifold, then*

$$\mathscr{H}^k_{\partial}(X, g) = \bigoplus_{p+q=k} \mathscr{H}^{p,q}_{\partial}(X, g), \mathscr{H}^k_{\overline{\partial}}(X, g) = \bigoplus_{p+q=k} \mathscr{H}^{p,q}_{\overline{\partial}}(X, g).$$

Proof. We can write $\alpha = \sum_{p,q} \alpha^{p,q}$, use Propisition 3.4 and the bidegree analysis, this is easy to see. □

Theorem 3.7 (Hodge Decomposition)**.** *Let (X, g) be a compact Hermitian manifold, then we have the following natural orthogonal decomposition*

$$\mathscr{A}^{p,q}(X) = \partial\mathscr{A}^{p-1,q}(X) \oplus \mathscr{H}^{p,q}_{\partial}(X, g) \oplus \partial^*\mathscr{A}^{p+1,q}(X),$$

$$\mathscr{A}^{p,q}(X) = \overline{\partial}\mathscr{A}^{p-1,q}(X) \oplus \mathscr{H}^{p,q}_{\overline{\partial}}(X, g) \oplus \overline{\partial}^*\mathscr{A}^{p+1,q}(X);$$

Moreover, we have

$$\ker\partial = \partial\mathscr{A}^{p-1,q}(X) \oplus \mathscr{H}^{p,q}_{\partial}(X, g), \ker\overline{\partial} = \overline{\partial}\mathscr{A}^{p-1,q}(X) \oplus \mathscr{H}^{p,q}_{\overline{\partial}}(X, g).$$

Proof. Omitted for the first part. This proof depend on some knowledge of PDEs. Now we prove the second part by assuming the first is proved. We just prove the statement about $\overline{\partial}$ and ∂ is the same.

First we claim that $\overline{\partial}\overline{\partial}^*\beta = 0$ if and only if $\partial^*\beta = 0$. Actually, we have $(\overline{\partial}\overline{\partial}^*\beta, \beta) = \|\overline{\partial}^*\beta\|^2$, then the claim is right. Use the claim, we can find that

$$\overline{\partial}\mathscr{A}^{p-1,q}(X) \oplus \mathscr{H}^{p,q}_{\overline{\partial}}(X, g) \subset \ker\overline{\partial},$$

so just need to prove that $\overline{\partial}^*\mathscr{A}^{p+1,q}(X) \cap \ker\overline{\partial} = 0$. Let α in it with $\alpha = \overline{\partial}^*\gamma$, then $\overline{\partial}\overline{\partial}^*\gamma = 0$. Use the claim we have $\alpha = \overline{\partial}^*\gamma = 0$, well done. □

Corollary 3.8. *Let (X, g) be a compact Hermitian manifold, then the canonical map $\mathscr{H}^{p,q}_{\overline{\partial}}(X, g) \to H^{p,q}(X)$ is an isomorphism.*

Proof. This map defined by $\alpha \mapsto [\alpha]$ is well defined by Lemma 3.5. Moreover, we use the previous theorem we have $\ker \overline{\partial} = \overline{\partial}\mathscr{A}^{p-1,q}(X) \oplus \mathscr{H}^{p,q}_{\overline{\partial}}(X,g)$, this is exactly what we need. □

Proposition 3.9. *Let (X,g) be a compact Hermitian manifold and $[\alpha] \in H^{p,q}(X)$. Then the harmonic representative of $[\alpha]$ (existence is the previous corollary) is the unique $\overline{\partial}-$closed form with the minimal norm $\|\alpha\|$.*

Proof. For any representation $\alpha + \overline{\partial}\beta$ of $[\alpha]$ where $\alpha \in \mathscr{H}^{p,q}(X,g)$, we use the orthogonal Hodge decomposition find that $\alpha \perp \overline{\partial}\beta$, then well done. □

4 Hodge Theory on Kähler Manifolds

4.1 Main Results

Definition 4.1. *We call a Hermitian manifold (X,g) a* Kähler manifold *if its fundamental form ω is $d-$closed. In this case, we say g is its* Kähler metric *and ω is its* Kähler form[3].

Actually, a Kähler manifold is so good and it has much better properties than the usual Hermitian manifolds. First we will prove the most important properties about Kähler manifolds which play a vital role. Now the dreams begin.

Theorem 4.2 (Kähler Identities)**.** *Let (X,g) be a Kähler manifold, then the following identities hold ture.*

(i) $[\overline{\partial}, L] = [\partial, L] = 0$ and $[\overline{\partial}^*, \Lambda] = [\partial^*, \Lambda] = 0$;

(ii) $[\overline{\partial}^*, L] = i\partial, [\partial^*, L] = -i\overline{\partial}$ and $[\Lambda, \overline{\partial}] = -i\partial^*, [\Lambda, \partial] = i\overline{\partial}^*$;

(iii) $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$ and Δ_d commutes with $*, \partial, \overline{\partial}, \partial^*, \overline{\partial}^*, L, \Lambda$.

Proof. We only prove (i) and (iii). You can see the proof of (ii) in [2, 3, 5].

For (i), we have $[\overline{\partial}, L]\alpha = \overline{\partial}(\omega \wedge \alpha) - \omega \wedge \overline{\partial}\alpha = \overline{\partial}\omega \wedge \alpha = 0$ since $d-$closed implies $\overline{\partial}-$closed. And $[\partial, L] = 0$ is similar. Moreover, let $\alpha \in \mathscr{A}^k(X)$, then

$$[\overline{\partial}^*, \Lambda]\alpha = - * \partial * *^{-1} L * \alpha + *^{-1} L * * \partial * \alpha$$

$$= - * \partial L * \alpha - (-1)^k *^{-1} L \partial * \alpha = - * [\partial, L] * \alpha = 0.$$

[3]We can prove that any Kähler manifold (X,g) has a natural Symplectic Structure since the Kähler form is a natural symplectic form, that is, ω is closed form and nowhere degenerate. So you can put Kähler geoemetry into symplectic geometry.

Another as $[\partial^*, \Lambda] = \overline{[\overline{\partial}^*, \overline{\Lambda}]} = \overline{[\overline{\partial}^*, \Lambda]} = 0$, well done.

For (iii), it's easy to see $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$ by (ii). Then

$$\begin{aligned}\Delta_\partial &= \partial\partial^* + \partial^*\partial = i[\Lambda, \overline{\partial}]\partial + i\partial[\Lambda, \overline{\partial}] \\ &= i(\Lambda\overline{\partial}\partial - \overline{\partial}\Lambda\partial + \partial\Lambda\overline{\partial} - \partial\overline{\partial}\Lambda) \\ &= i(\Lambda\overline{\partial}\partial - (\overline{\partial}[\Lambda, \partial] + \overline{\partial}\partial\Lambda) + ([\partial, \Lambda]\overline{\partial} + \Lambda\partial\overline{\partial}) - \partial\overline{\partial}\Lambda) \\ &= i(\Lambda\overline{\partial}\partial - i\overline{\partial}\overline{\partial}^* - \overline{\partial}\partial\Lambda - i\overline{\partial}^*\overline{\partial} + \Lambda\partial\overline{\partial} - \partial\overline{\partial}\Lambda) = \Delta_{\overline{\partial}}.\end{aligned}$$

The claim $\Delta_d = 2\Delta_{\overline{\partial}}$ is trivial. Well done. □

Remark 4.3. *This is an important theorem that describe some special information about Kähler manifolds. Actually, we focus on the third statement of the theorem, that is, $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$ tell us that $\mathscr{H}^{p,q}_\partial(X, g) = \mathscr{H}^{p,q}_{\overline{\partial}}(X, g) = \mathscr{H}^{p+q}_d(X, g) \cap \mathscr{A}^{p,q}(X)$. So from now on, we denote $\mathscr{H}^{p,q}(X, g) := \mathscr{H}^{p,q}_\partial(X, g) = \mathscr{H}^{p,q}_{\overline{\partial}}(X, g)$ in the case of Kähler.*

Now we will introduce an important and interesting lemma which we use it frequently.

Lemma 4.4 ($\partial\overline{\partial}$ Lemma)**.** *Let (X, g) be a compact Kähler manifold, then for all $d-$closed (p, q) form α, the following statements are equivalence:*

(i) $\alpha = d\beta, \beta \in \mathscr{A}^{p+q-1}_{\mathbb{C}}(X)$;

(ii) $\alpha = \partial\beta, \beta \in \mathscr{A}^{p-1,q}(X)$;

(iii) $\alpha = \overline{\partial}\beta, \beta \in \mathscr{A}^{p,q-1}(X)$;

(iv) $\alpha = \partial\overline{\partial}\beta, \beta \in \mathscr{A}^{p-1,q-1}(X)$;

(v) $\alpha \perp \mathscr{H}^{p,q}(X, g)$.

Proof. Actually, use the Hodge decompositon theorem of Hermitian manifolds, we know that (v) implies (i)-(iv) and (iv) implies (i)-(iii). So we just need to prove that (v) implies (iv) as follows.

Let α is a $d-$closed $(p, q)-$form and orthogonal to $\mathscr{H}^{p,q}(X, g)$. Use Hodge decomposition with respect to ∂, we have $\alpha = \partial\gamma$ and use with respect to $\overline{\partial}$, we have $\gamma = \overline{\partial}\beta + \overline{\partial}^*\beta' + \beta''$ where β'' is harmonic. Then $\alpha = \partial\overline{\partial}\beta + \partial\overline{\partial}^*\beta' + \partial\beta'' = \partial\overline{\partial}\beta + \partial\overline{\partial}^*\beta'$ by Lemma 3.5. Now we claim that $\partial\overline{\partial}^*\beta' = 0$. Use the proof of the third statement of Theorem 4.2, we have $\partial\overline{\partial}^*\beta' = -\overline{\partial}^*\partial\beta'$. Then since $\overline{\partial}\alpha = 0$, we have $\overline{\partial}\overline{\partial}^*\partial\beta' = 0$. But we find that $(\overline{\partial}\overline{\partial}^*\partial\beta', \partial\beta') = \|\overline{\partial}^*\partial\beta'\|^2$, so $\overline{\partial}^*\partial\beta' = 0$, well done. □

Corollary 4.5. *If X be a complex manifold, then we have the natural map $H^{p,q}_{BC}(X) \to H^{p,q}(X)$. Additively, if (X, g) is compact Kähler manifold, then this natural is an isomorphism.*

Proof. Recall the definition of the Bott-Chern cohomology $H^{p,q}_{BC}(X) = \ker d/\mathrm{Im}\partial\overline{\partial}$, then we have $\ker d \subset \ker\overline{\partial}$ and $\mathrm{Im}\partial\overline{\partial} \subset \mathrm{Im}\overline{\partial}$, so we have the natural map $H^{p,q}_{BC}(X) \to H^{p,q}(X)$.

In the case of compact Kähler, we have $\mathrm{Im}\partial\overline{\partial} = \mathrm{Im}\partial$ by $\partial\overline{\partial}$ lemma. Similarly, we know that $d-$closed if and only if both $\partial-$ and $\overline{\partial}-$closed, then use $\partial\overline{\partial}$ lemma we have $\mathrm{Im}\partial = \mathrm{Im}\overline{\partial}$, moreover, by Hodge decomposition we find that

$$\ker\partial = \mathrm{Im}\partial \oplus \mathscr{H}^{p,q}(X,g) = \mathrm{Im}\overline{\partial} \oplus \mathscr{H}^{p,q}(X,g) = \ker\overline{\partial},$$

well done. □

Theorem 4.6 (Hodge Decomposition)**.** *If (X,g) be a compact Kähler manifold, then we have the decomposition*

$$H^k_{\mathbb{C}}(X) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

Moreover, this decomposition is not depend on the Kähler structure.

Proof. We have $H^k_{\mathbb{C}}(X) \cong \mathscr{H}^k(X,g) \cong \bigoplus_{p+q=k}\mathscr{H}^{p,q}(X,g) = \bigoplus_{p+q=k} H^{p,q}(X)$.

Moreover, we let g' is another Kähler structure, then $\mathscr{H}^{p,q}(X,g) \cong H^{p,q}(X) \cong \mathscr{H}^{p,q}(X,g')$. Let $\alpha \in \mathscr{H}^{p,q}(X,g)$ correspond to $\alpha' \in \mathscr{H}^{p,q}(X,g')$, we need to show that $[\alpha] = [\alpha'] \in H^k(X,\mathbb{C})$.We find that $\alpha = \alpha' + \overline{\partial}\gamma$. Use the Hodge decomposition with respect to d and $d\overline{\partial}\gamma = 0$ and $\overline{\partial}\gamma \perp \mathscr{H}^k(X,g)$, we have $\overline{\partial}\gamma \in \mathrm{Im}d$, so well done. □

Remark 4.7. *So in the case of compact Kähler, we can use this to see the structure of $\mathbb{C}$-coefficient de Rham cohomology and break it into Dolbeault cohomology.*

Corollary 4.8. *If (X,g) be a compact Kähler manifold, then*

$$H^k(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{BC}(X).$$

Proof. Immediately from the Hodge decomposition of the Kähler manifold. □

4.2 Applications

Now we try to discuss the cohomology of compact Kähler manifolds using what we just learned. First, Let talk about the relation between the Picard group (All holomorphic line bundles on X formed a group by tensor and dual) of a compact Kähler manifold X and the cohomology group $H^{1,1}(X)$.

Now we introduce the most famous exact sequence $0 \to \mathbb{Z} \to \mathscr{O} \to \mathscr{O}^* \to 0$. The long exact sequence of it yields the maps as follows

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathrm{Pic}(X) = H^1(X,\mathscr{O}^*) & \longrightarrow & H^2(X,\mathbb{Z}) & \longrightarrow \cdots \\
& \downarrow & \swarrow & & \\
& H^2(X,\mathbb{C}) & \xrightarrow[\text{Hodge}]{\cong} & H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X) &
\end{array}$$

Moreover, we find that $H^{0,2}(X) \cong H^2(X, \mathscr{O}_X)$. So, for any $\alpha \in H^2(X, C)$, we now have two ways to associate to α a class of degree $(0, 2)$. First, using the canonical projection $H^2(X, \mathbb{C}) \to H^{0,2}(X)$. Second, using the map $H^2(X, \mathbb{C}) \to H^2(X, \mathscr{O}_X)$ induced by the inclusion map $\mathbb{C} \hookrightarrow \mathscr{O}_X$. Two our surprise, the two ways coincide as follows.

Lemma 4.9. *Let (X, g) be a compact Kähler manifold, then the previous maps coincide.*

Proof. We have the following acyclic resolutions

$$\begin{array}{ccccccccc}
\mathbb{C} & \longrightarrow & \mathscr{A}^0(X) & \xrightarrow{d} & \mathscr{A}^1(X) & \xrightarrow{d} & \mathscr{A}^2(X) & \xrightarrow{d} & \cdots \\
\big\downarrow & & \big\| & & \big\downarrow \Pi^{0,1} & & \big\downarrow \Pi^{0,2} & & \\
\mathscr{O}_X & \longrightarrow & \mathscr{A}^{0,0}(X) & \xrightarrow{\bar\partial} & \mathscr{A}^{0,1}(X) & \xrightarrow{\bar\partial} & \mathscr{A}^{0,2}(X) & \xrightarrow{\bar\partial} & \cdots
\end{array}$$

This is commute by easily verifying. Use the harmonic forms, well done. □

Now we back to the Picard group. As the following diagram shows that whole composition is trivial.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & H^2(X, \mathbb{Z}) & \longrightarrow & H^2(X, \mathscr{O}_X) & \longrightarrow & \cdots \\
 & & & & \big\downarrow & \nearrow & \big\| & & \\
 & & & & H^2(X, \mathbb{C}) & \xrightarrow{\supset} & H^{0,2}(X) & &
\end{array}$$

So $\mathrm{Im}(\mathrm{Pic}(X) \to H^2(X, \mathbb{C})) \subset \ker(H^2(X, \mathbb{C}) \to H^{0,2}(X)) \cap \mathrm{Im}(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}))$. Using the fact that it is invariant under complex conjugation, we find that

$$\mathrm{Im}(\mathrm{Pic}(X) \to H^2(X, \mathbb{C})) \subset H^{1,1}(X, \mathbb{Z}) := \mathrm{Im}(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})) \cap H^{1,1}(X).$$

More surprisingly, we have

Theorem 4.10 (Lefschetz Theorem on (1,1)-classes)**.** *Let (X, g) be a compact Kähler manifold, then $\mathrm{Pic}(X) \to H^{1,1}(X, \mathbb{Z})$ is* SURJECTIVE.

Proof. We take $\alpha = \rho(\hat\alpha) \in H^{1,1}(X, \mathbb{Z})$, then use bidegree decomposition, we have $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$ with $\alpha^{2,0} = \overline{\alpha^{0,2}}$. So $\alpha^{0,2} = 0$. Conversely, if $\beta \in \mathrm{Im}(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}))$, then $\beta^{0,2} = 0$ implies $\beta \in H^{1,1}(X, \mathbb{Z})$.

Actually, use the previous lemma, we find that $\alpha \mapsto \alpha^{0,2}$ similar as it induced by $\mathbb{C} \hookrightarrow \mathscr{O}_X$. So just to check that $\hat\alpha \in \ker(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}) \to H^2(X, \mathscr{O}_X))$. Consider the following diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & H^2(X, \mathbb{Z}) & \longrightarrow & H^2(X, \mathscr{O}_X) & \longrightarrow & \cdots \\
 & & & \searrow & \big\downarrow \rho & \nearrow & & & \\
 & & & & H^2(X, \mathbb{C}) & & & &
\end{array}$$

This is easy to see. □

Remark 4.11. *(i) We call the image of* $\mathrm{Pic}(X) \to H^2(X, \mathbb{R})$ *the* NERON-SEVERI GROUP $\mathrm{NS}(X)$. *By the case of Kähler, use the previous theorem, we know that* $\mathrm{NS}(X) = H^{1,1}(X, \mathbb{Z})$*;*

(ii) We define the PICARD NUMBER $\rho(X)$ *as the rank of the image of* $\mathrm{Pic}(X) \to H^2(X, \mathbb{R})$. *So in the case of Kähler, we have* $\rho(X) = \mathrm{rank}(H^{1,1}(X, \mathbb{Z})) = \mathrm{rank}(\mathrm{NS}(X))$.

Definition 4.12. *Let* X *be a complex manifold, then* JACOBIAN *as* $\ker(\mathrm{Pic}(X) \to H^2(X, \mathbb{Z}))$*, denoted by* $\mathrm{Pic}^0(X)$. *So* $\mathrm{Pic}^0(X) = H^1(X, \mathscr{O}_X)/H^1(X, \mathbb{Z})$.

Now we discuss a corollary that reflect some special things about the compact Kähler manifolds in order to finish the discussion of lower dimensional cohomology and go to the area of higher dimensional cohomology.

Corollary 4.13. *If* (X, g) *be a compact Kähler manifold, then* $\mathrm{Pic}^0(X)$ *is a complex torus of dimension* $b_1(X)$.

Proof. Use Hodge decomposition we have

$$(H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus \overline{H^{1,0}(X)},$$

then $H^{0,1}(X) \to H^1(X, \mathbb{C}) \to H^{0,1}(X)$ is injective with discrete image formed a lattice, well done. □

EXAMPLE. Let X be a Hopf surface, that is, $\mathbb{Z}$ act on $\mathbb{C}^2 \backslash \{0\}$ by $(z_1, z_2) \mapsto (\lambda^k z_1, \lambda^k z_2)$ and $X = (\mathbb{C}^2 \backslash \{0\})/\mathbb{Z}$ where $0 < \lambda < 1$. Then we know that X diffeomorphic to $S^1 \times S^3$. Now we claim that it is not a complex torus, then conclude that X is not Kähler.

First, we use the KÜNNETH FORMULA as

$$H^1(X, \mathbb{Z}) \cong H_1(S^1 \times S^3, \mathbb{Z}) \cong (H_1(S^1) \otimes_{\mathbb{Z}} H_0(S^3)) \oplus (H_0(S^1) \otimes_{\mathbb{Z}} H_1(S^3)) \cong \mathbb{Z}.$$

Moreover, we have $H^1(X, \mathscr{O}_X) \cong H^{0,1}(X) \cong \mathbb{C}$ by TODD-HIRZEBBRUCH FORMULA in page 172 in [13]. So well done.

In order to discuss the higher dimensional cohomology, we first point out a fact we have never discuss. If X is a compact Kähler manifold, easy to verify that the Lefschetz and the dual of it as $L : H^{p,q}(X) \to H^{p+1,q+1}(X)$ and $\Lambda : H^{p,q}(X) \to H^{p-1,q-1}(X)$. In this part, we will find some symmetric relation in whole Hodge numbers $h^{p,q}$. First we statement some conclusion without proof.

Definition 4.14. *If* X *is a compact Kähler manifold, then* PRIMITIVE CONOMOLOGY *is* $H^k(X, \mathbb{R})_p = \ker \Lambda$*, so is* $H^{p,q}(X)_p$.

Theorem 4.15 (HARD LEFSCHETZ THEOREM)**.** *If* X *is* n*-dimensional compact Kähler manifold, then for all* $k \le n$*, we have*

$$L^{n-k} : H^k(X, \mathbb{R}) \cong H^{2n-k}(X, R).$$

Moreover $H^k(X,\mathbb{R}) \cong \bigoplus_{i\geq 0} L^i H^{k-2i}(X,\mathbb{R})_p$ *and* $H^k(X,\mathbb{R})_p \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X)_p$.

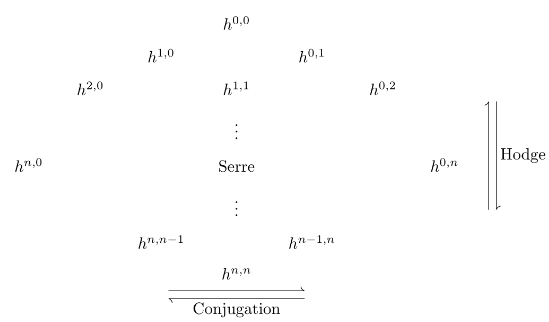
Proposition 4.16. *If* X *be a compact Hermitian manifold.*

*(i)(*HODGE DUALITY*)The Hodge* $*-$*operator induces* $* : \mathscr{H}^{p,q}_{\overline{\partial}}(X,g) \cong \mathscr{H}^{n-q,n-p}_{\partial}(X,g)$*;*

*(ii)(*SERRE DUALITY*)One has* $\mathscr{H}^{p,q}_{\overline{\partial}}(X,g) \cong \mathscr{H}^{n-p,n-q}_{\partial}(X,g)^*$.

So in the case of compact Kähler, we have $* : H^{p,q}(X) \cong H^{n-q,n-p}(X)$ as Hodge duality. And $H^{p,q}(X) \cong H^{n-p,n-q}(X)^*$ as Serre duality. Moreover, use Hodge theorem, we have $H^{p,q}(X) = \overline{H^{q,p}(X)}$. So these things yields many symmetric relation in whole Hodge numbers $h^{p,q}$. Now we will draw a diagram to summarize these, called HODGE DIAMOND as follows.

Recall that $h^{p,q} = \dim H^{p,q}(X)$. Then



Theorem 4.17 (HODGE-RIEMANN BILINEAR RELATION)**.** *Let* (X,g) *be a* $n-$*dimensional compact Kähler manifold and* $[\omega]$ *is its Kähler form, then for* $0 \neq \alpha \in H^{p,q}(X)_p$*, we have*

$$i^{p-q}(-1)^{\frac{(p+q)(p+q-1)}{2}} \int_X \alpha \wedge \overline{\alpha} \wedge [\omega]^{n-p-q} > 0.$$

Proof. This is pure linear algebra. See any book, such as [2, 3, 5, 6]. □

Proposition 4.18 (HODGE INDEX THEOREM)**.** *Let* (X,g) *be a compact Kähler surface, then the index of*

$$H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) \to \mathbb{R}, (\alpha,\beta) \mapsto \int_X \alpha \wedge \beta$$

is $(2h^{2,0}+1, h^{1,1}-1)$.

Proof. We find that $H^2(X,\mathbb{R}) = \big((H^{2,0}(X) \oplus H^{0,2}(X)) \cap H^2(X,\mathbb{R})\big) \oplus H^{1,1}(X)$, take α is an elements in $H^{2,0}(X) \oplus H^{0,2}(X)$. We find that all class in it is primitive, let $\alpha = \alpha^{2,0} + \alpha^{0,2}$, then

$$\int_X \alpha^2 = 2\int_X \alpha^{2,0} \wedge \alpha^{0,2} = 2\int_X \alpha^{2,0} \wedge \overline{\alpha^{2,0}} > 0,$$

Then we only need to consider $H^{1,1}(X)$, use Lefschetz decomposition, we have

$$H^{1,1}(X) = H^{1,1}(X) \oplus LH^0(X) = H^{1,1}(X) \oplus [\omega]\mathbb{R},$$

Use Hodge $*$ duality and the orthogonality of decomposition, we have $\int_X \omega \wedge \alpha = 0$. And $\int_X \omega^2 > 0$, with Hodge-Riemann bilinear relation, we have $\int_X \alpha^2 < 0$, well done. $\square$

This result can be generalized. We omit it here. See [2, 3].

5 Basic Hodge Structures

Definition 5.1. *A* Hodge structure of weight *k on a $\mathbb{Z}$-module $V_{\mathbb{Z}}$ of finite type is a direct decomposition*

$$V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}, V^{p,q} = \overline{V^{q,p}}.$$

Definition 5.2. *Let $V_{\mathbb{Z}}, W_{\mathbb{Z}}$ are two Hodge structure of weight k, then a morphism of it is a homomorphism of $\mathbb{Z}$-modules with its complexification $f_{\mathbb{C}}$ send $V^{p,q}$ to $W^{p,q}$.*

Definition 5.3. *A* real Hodge structure *on V is a decomposition*

$$V_{\mathbb{C}} = V \otimes \mathbb{C} = \bigoplus_{p,q\in\mathbb{Z}} V^{p,q}, V^{p,q} = \overline{V^{q,p}}.$$

We call the polynomial $P_{hn}(V) = \sum_{p,q\in\mathbb{Z}} h^{p,q}(V)u^p v^q$ is associated Hodge number polynomial.

Actually, if a real Hodge structure V is of form $V = V_R \otimes_R \mathbb{R}$ where R is a subring of $\mathbb{R}$ and V_R is an R-module of finite type, then we say that V_R carries an R-Hodge structure.

If $V^{(k)}$ is the real vector space underlying $\bigoplus_{p+q=k} V^{p,q}$, we say that it is the weight k part of V. So if $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ and $V = V^{(k)}$, then it is a Hodge structure of weight k.

Example.(i) So in the case of compact Kähler (X, g), the de Rham cohomology $H^k_{DR}(X)$ has real Hodge structure of weight k by Hodge decomposition where the direct sum of Dolbeault cohomology.

(ii) We define the Hodge structures of Tate on R as $R(k) = R \otimes_{\mathbb{Z}} \mathbb{Z}(k) = R \otimes_{\mathbb{Z}} [2\pi i]^k\mathbb{Z}$ with $\mathbb{Z}(m) \otimes \mathbb{C} = [\mathbb{Z}(m) \otimes \mathbb{C}]^{-m,-m}$. Actually, if we have any Hodge structure $V_{\mathbb{Z}}$ of weight k, then we have the Tate twist $V(m)$ is a Hodge structure of weight $k - 2m$. It has $V \otimes (2\pi i)^m$ as underlying $\mathbb{Z}$-module, while $V(m)^{p,q} = V^{p-m,q-m}$.

Next, we focus on the Hodge structures in cohomology and homology.

Recall that the fundamental class $\mathrm{cl}(Y) \in H^{2c}(X)$ of a codimension c subvariety Y of a compact complex manifold X is that for all integral cohomology class $\alpha \in H^{2n-2c}(X)$, we have $\int_X \alpha \wedge \mathrm{cl}(Y) = \int_Y \alpha|_Y$.

Proposition 5.4. *The fundamental cohomology class* $\mathrm{cl}(Y) \in H^{2c}(X)$ *of a codimension* c *subvariety* Y *of a compact complex manifold* X *has pure type* (c, c)*. In particular, if* X *is connected, the twist trace map is an isomorphism*

$$\mathrm{tr}_X : H^{2n}(X) \cong \mathbb{Z}(-n), [\alpha] \mapsto (2\pi i)^n \int_X \alpha.$$

Recall that the trace map is induced by Poincáre duality when $k = 0$. In this case, we have $\mathrm{tr} : H^{2n}(X) \cong H_0(X) = \mathbb{Z}$. This proposition is easy to see. Now we will introduce the famous conjecture, Hodge Conjecture.

An element $\sum n_i Y_i$ of the free group $Z_k(X)$ on k-dimensional subvarieties of X called an algebraic k-cycle. We defines the cycle class map $\mathrm{cl} : Z^c(X) \to H^{2c}(X)$ by $\sum n_i Y_i \mapsto \sum n_i \mathrm{cl}(Y_i)$. The image of it is called an **algebraic class**. Note that if X is projective with $c = 1$, then the algebraic classes are exactly integral $(1, 1)$-class by Theorem 4.10. Going over to rational classes, we let

$$H^{2c}_{Hdg}(X) = H^{c,c}(X) \cap \mathrm{Im}(H^{2c}(X, \mathbb{Q}) \to H^{2c}(X, \mathbb{C})).$$

★ **Hodge Conjecture** ★ Let X be a smooth projective variety. Every (c, c)-class with rational coefficients is algebraic. That is, every class in $H^{2c}_{Hdg}(X)$ is a rational combination of fundamental cohomology classes of subvarieties of X.

6 Postscript

As you see, this is a brief introduction about basic Hodge theory. After reading this, you can smoothly read [3]. You can also read [12] after you are familiar with spectral sequence, representation theory and derived category. Since I haven't read this (I will do it in the next term), I didn't write about this in the article. This is the first time for me to write this kind of formal article, the wrong place also hopes the criticism corrects.

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