§6. Local Theory & Preliminaries

$\rho: M\times\mathbb{R}\to M$

Def 6.1. $\rho \Rightarrow$ isotopy if each $\rho_t: M\to M$ is a diffeomorphism and $\rho_0 = id_M$.

RMK. Actually, we obtain a family of vector fields v_t s.t. $\forall p\in M$, $q=\rho_t^{-1}(p)$,

$$\begin{cases} v_t(p) = \frac{d}{ds}\rho_s(q)\big|_{s=t} \\ \rho_t(q)=p \end{cases} \qquad \left(\frac{d}{ds}\rho_s(q)\big|_{s=t} = v_t(\rho_t(q))\right)$$

Conversely, if M compact ($\Rightarrow$ compact support), by ODEs:

$$v_t \Longrightarrow \rho_t.$$

Def 6.2. If $v_t\equiv v$, the associated isotopy $\Rightarrow$ flow of v ($\exp tv$).

$$\Rightarrow \begin{cases} \frac{d}{dt}(\exp tv)(p) = v(\exp tv(p)) \\ \exp tv|_{t=0} = id_M\end{cases} \qquad \left[v_t \text{ time-indep} \Leftrightarrow \rho_{t+s}=\rho_t\circ\rho_s\right]$$

Def 6.3. Lie derivative of vector field v:

$$\mathcal{L}_v: \Omega^k(M)\to\Omega^k(M)$$
$$\omega\longmapsto \frac{d}{dt}(\exp tv)^*\omega\big|_{t=0}$$

RMK. In general bundle $F(M)\xrightarrow{\pi}M$, and diffeomorphism $f: M\to M$.

$$\Rightarrow \begin{tikzcd} F(M) \arrow[r, "F(f)"] \arrow[d, "\pi"] \arrow[dr, phantom, "\circlearrowright"] & F(M) \arrow[d, "\pi"] \\ M \arrow[r, "f"] & M \end{tikzcd}$$

let $s\in\Gamma(F(M))$ & isotopy ρ_t,

$$\Rightarrow \rho_t^* s = F(\rho_t^{-1})\circ s\circ F(\rho_t).$$

D. Lie derivative by v_t:

$$\mathcal{L}_{v_t}:\Omega^k(M)\to\Omega^k(M)$$
$$\omega\mapsto\frac{d}{ds}(\rho_s\circ\rho_t^{-1})^*\omega\big|_{s=t}.$$

Exercise 1. For vector fields v,

$$\Rightarrow \mathcal{L}_v\omega = \frac{d}{dt}(\exp tv)^*\omega\big|_{t=0}$$

Show: $\mathcal{L}_v\omega = \iota_v d\omega + d\iota_v\omega$.

proof. Let $\omega\in\Omega^k(M)$,

$$\begin{aligned} v(\omega(Y_1,\dots,Y_k)) &= \frac{d}{dt}\Big|_{t=0}(\exp tv)^*(\omega(Y_1,\dots,Y_k)) \\ &= \frac{d}{dt}\Big|_{t=0}((\exp tv)^*\omega)((\exp tv)^*Y_1,\dots,(\exp tv)^*Y_k) \\ &= (\mathcal{L}_v\omega)(Y_1,\dots,Y_k)+\sum_{i=1}^k \omega(Y_1,\dots,[X,Y_i],\dots,Y_k). \end{aligned}$$

$$\Rightarrow (\mathcal{L}_v\omega)(Y_1,\dots,Y_k) = v(\omega(Y_1,\dots,Y_k)) - \sum_{i=1}^k \omega(Y_1,\dots,[X,Y_i],\dots,Y_k)$$

On the other hand

$$\begin{aligned} d\omega(X_0,\dots,X_k) &= \sum_{i=0}^k (-1)^i X_i(\omega(X_0,\dots,\hat X_i,\dots,X_k)) \\ &+ \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,\hat X_i,\dots,\hat X_j,\dots,X_k) \end{aligned}$$

$\Rightarrow$ ✓ □

Exercise 2. Show: $\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{v_t}\omega$.

proof.
$$\frac{d}{dt}\rho_t^*\omega = \lim_{s\to t}\frac{\rho_s^*\omega-\rho_t^*\omega}{s-t} = \rho_t^*\lim_{s\to t}\frac{(\rho_t^{-1})^*\rho_s^*\omega-\omega}{s-t} = \rho_t^*\mathcal{L}_{v_t}\omega.$$

Prop 6.4. For family ω_t of $\Omega^d(M)$

$$\Rightarrow \frac{d}{dt}\rho_t^*\omega_t = \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right).$$

proof. Let $f(x,y)=\rho_x^*\omega_y$.

$$\begin{aligned} \Rightarrow \frac{d}{dt}f(t,t) &= \frac{d}{dx}f(x,t)\Big|_{x=t} + \frac{d}{dy}f(t,y)\Big|_{y=t} \\ &= \frac{d}{dx}\rho_x^*\omega_t\Big|_{x=t} + \frac{d}{dy}\rho_t^*\omega_y\Big|_{y=t} \\ &= \rho_t^*\mathcal{L}_{v_t}\omega + \rho_t^*\frac{d\omega_t}{dt} \\ &= \rho_t^*\left(\mathcal{L}_{v_t}\omega+\frac{d\omega_t}{dt}\right). \end{aligned}$$ □

RMK. $i: X \hookrightarrow M$, $\dim X = k < n = \dim M$

$N_x X = T_x M / T_x X$

$\Rightarrow NX = \{(x,v) \mid x \in X, v \in N_x X\}$.

$i_0 : X \hookrightarrow NX$, $x \mapsto (x,0)$ as submanifold.

$\to$ Neighborhood $\mathcal{U}_0$ of X in NX convex if $\mathcal{U}_0 \cap N_x X$ convex, $\forall x \in X$.

Thm 6.5. $\exists$ convex neighborhood $\mathcal{U}_0$ of X in NX, a neighborhood $\mathcal{U}$ of X in M & diffeomorphism $\varphi: \mathcal{U}_0 \to \mathcal{U}$ s.t.

$$\begin{tikzcd} NX \supseteq \mathcal{U}_0 \arrow[rr, "\varphi", "\cong"'] & & \mathcal{U} \subseteq M \\ & X \arrow[ul, "i_0"] \arrow[ur, "i"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

RMK. Tubular neighborhood fibration:

$$\begin{tikzcd} NX \supseteq \mathcal{U}_0 \arrow[r, "\varphi", "\cong"'] \arrow[d, "\pi_0"] & \mathcal{U} \subseteq M \arrow[d, "\pi"] \\ X \arrow[r, "="] & X \end{tikzcd}$$

by $\pi = \pi_0 \circ \varphi^{-1} : \mathcal{U} \to X$.

Lemma 6.6. Consider $\rho_t : \mathcal{U}_0 \to \mathcal{U}_0$, $(x,v) \mapsto (x, tv)$.

with $\rho_0 = i_0 \circ \pi_0$ & $\rho_1 = id$.

Define $Q : \Omega^k(\mathcal{U}_0) \to \Omega^{k-1}(\mathcal{U}_0)$

$$\omega \mapsto \int_0^1 \rho_t^*(\iota_{v_t}\omega)\,dt,$$

then $\Rightarrow Id - (i_0 \circ \pi_0)^* = dQ + Qd$.

proof. $\forall \omega \in \Omega^k(\mathcal{U}_0)$, we have

$$dQ\omega + Qd\omega = d\int_0^1 \rho_t^*(\iota_{v_t}\omega)\,dt + \int_0^1 \rho_t^*(\iota_{v_t}d\omega)\,dt$$

$$= \int_0^1 \rho_t^*(\mathcal{L}_{v_t}\omega)\,dt = \int_0^1 \frac{d}{dt}\rho_t^*\omega\,dt$$

$$= \rho_1^*\omega - \rho_0^*\omega. \quad \square$$

Thm 6.7. ~~[illegible]~~ $\forall$ tubular neighborhood $\mathcal{U}$ of X in M, then

if closed d-form ω on $\mathcal{U}$ s.t. $i^*\omega = 0$, then $\exists \mu \in \Omega^{d-1}(\mathcal{U})$ s.t. $\omega = d\mu$ and $\mu_x = 0$, $\forall x \in X$.

proof. Via $\varphi : \mathcal{U}_0 \xrightarrow{\cong} \mathcal{U}$, we just need to work on $\mathcal{U}_0$.

Define $\rho_t : \mathcal{U}_0 \to \mathcal{U}_0$, $(x,v) \mapsto (x, tv)$.

By Lemma 6.6 $\Rightarrow Id - (i_0 \circ \pi_0)^* = dQ + Qd$

$\Rightarrow \omega = dQ\omega$. $\mu = Q\omega$.

As $\rho_t(x) = x$, $\forall x \in X \Rightarrow \mu_x = 0$, $\forall x \in X$.

$v_t = 0$ at all x. $\square$

§7. Moser Theorems.

Now let M ⇒ $2n$-dim with two symplectic forms ω_0 & ω_1.

<u>Def. 7.1</u> We say (M,ω_0) & (M,ω_1) are.

① <u>symplectomorphic</u> if $\exists$ diffeo $\varphi: M \to M$ s.t. $\varphi^*\omega_1 = \omega_0$;

② <u>Strongly isotopic</u> if $\exists$ isotopy $\rho_t: M \to M$ s.t. $\rho_0 = id$, $\rho_1^*\omega_1 = \omega_0$.

③ <u>Deformation-equivalent</u>. $\exists$ family of symplec forms ω_t is smooth resp. to t joint ~~with~~ ω_0 & ω_1.

④ <u>Isotopic</u>. $\exists$ <u>deformation-equiv</u> $[\omega_t]$ s.t. $\frac{d}{dt}[\omega_t] = [\frac{d}{dt}\omega_t] = 0$.

<u>RMK</u>. We have:

$$\begin{array}{ccc} \text{Strongly isotopic} & \Longrightarrow & \text{Symplectomorph} \\ ①\Big\Downarrow & \Big\Uparrow \text{Moser thm.} + \text{compact manifd.} & \\ \text{isotopic} & \Longrightarrow & \text{deform-equiv} \end{array}$$

① : Similarly, $Q_t\omega = \int_0^t \rho_s^*(\iota_{v_s}\omega)\, ds$

$\Rightarrow id^*\omega_1 - \rho_t^*\omega_1$

$= dQ_t\omega_1 - Q_t d\omega_1 = dQ_t\omega_1$

$\Rightarrow [\omega_1] = [\rho_t^*\omega_1]$ as $\omega_t := \rho_t^*\omega_1$. ✓

$\boxed{\text{Moser's Trick}}$

(or forms are all compact-supp)

$\boxed{\text{Main idea}}$ In a compact manifold M & two forms α_0, α_1. We want to find a diffeo $\varphi: M \to M$ s.t. $\varphi^*\alpha_1 = \alpha_0$.

The <u>main idea</u>: We want to find

$\Rightarrow$ isotopy $\rho_t: M \to M$ s.t. ~~$\rho_t^*\omega_t = \omega_0$~~ $\rho_t^*\alpha_t = \alpha_0$.

$$\Rightarrow \frac{d}{dt}(\rho_t^*\alpha_t) = 0 = \rho_t^*\left(\mathcal{L}_{v_t}\alpha_t + \frac{d\alpha_t}{dt}\right)$$

Just need to find v_t s.t.

$$\mathcal{L}_{v_t}\alpha_t + \frac{d\alpha_t}{dt} = 0$$

i.e. $d\iota_{v_t}\alpha_t + \iota_{v_t}d\alpha_t + \frac{d\alpha_t}{dt} = 0$. !

▷ If moreover, $d\alpha_t = 0$, $\forall t$,

then ⟺ $d\iota_{v_t}\alpha_t + \frac{d\alpha_t}{dt} = 0$.

▷ More usefully, we sometimes will

Let $\alpha_t = (1-t)\alpha_0 + t\alpha_1$, with $[\alpha_0] = [\alpha_1]$ $\alpha_1 - \alpha_0 = d\beta$

then ⟺ $d(\iota_{v_t}\alpha_t) + \alpha_1 - \alpha_0 = 0$,

⟺ $d(\iota_{v_t}\alpha_t + \beta) = 0$.

just need to find v_t s.t.

$\iota_{v_t}\alpha_t + \beta = 0$.

<u>Thm 7.2</u>. [Moser thm 1]. M compact.

$[\omega_0] = [\omega_1]$ & $\omega_t = (1-t)\omega_0 + t\omega_1$ symplectic.

then $\exists$ isotopy ρ s.t. $\rho_t^*\omega_t = \omega_0$.

<u>Proof</u>. This is exactly the Moser's trick.
(use ω_t nondegeneracy) □

<u>Thm 7.3</u>. [Moser thm 2]. M compact.

Let ω_t ⇒ smooth family of closed 2-forms joint ω_0 & ω_1 with:

① $[\omega_t]$ independ of t.

② ω_t is nondegenerate.

Then $\exists$ isotopy s.t. $\rho_t^*\omega_t = \omega_0$.

proof. ① $\Rightarrow \exists \mu_t$ s.t. $\frac{d\omega_t}{dt} = d\mu_t$.

② $\Rightarrow$ same argument

$\Rightarrow \exists v_t$ s.t. $\iota_{v_t}\omega_t + \mu_t = 0$.

(Moser equn).

As M compact $\Rightarrow \rho_t \longleftrightarrow v_t$

$$\Rightarrow \frac{d}{dt}(\rho_t^*\omega_t) = \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right)$$
$$= \rho_t^*(d\iota_{v_t}\omega_t + d\mu_t) = 0.$$
$$\Rightarrow \rho_t^*\omega_t = \rho_0^*\omega_0 = \omega_0, \quad 0 \le t \le 1. \quad \square$$

☆ **Thm 7.4** [Relative Moser thm].

$M \Rightarrow$ symp^c, $X \subseteq M$ s.t. X compact. with $i: X \hookrightarrow X$. Let ω_0 & ω_1 are symp on M, s.t.

$\omega_0|_p = \omega_1|_p$, $\forall p \in X$.

Then $\exists$ neighborhoods $\mathcal{U}_0$ & $\mathcal{U}_1$ of X in M and diffeorphism $\varphi: \mathcal{U}_0 \to \mathcal{U}_1$ s.t. $\varphi^*\omega_1 = \omega_0$ and

$$\begin{tikzcd} \mathcal{U}_0 \arrow[rr, "\varphi"] & & \mathcal{U}_1 \\ & X \arrow[ul, "i"] \arrow[ur, "i"'] & \end{tikzcd} \quad \Leftrightarrow \varphi|_X = id_X$$

proof. ~~Indentify $\mathcal{U}_0 \Rightarrow \varepsilon \subseteq NX$~~, ~~we~~ We have ω_1, ω_0 closed with $(\omega_1 - \omega_0)|_X = 0$.

~~2nd~~ Choose tubular neighborhood $\mathcal{U}_0$.

Use thm 6.7.

$\Rightarrow \exists \mu \in \Omega^1(\mathcal{U}_0)$ s.t. $\omega_1 - \omega_0 = d\mu$ with $\mu|_X = 0$.

Consider $\omega_t = (1-t)\omega_0 + t\omega_1$

$= \omega_0 + t\,d\mu$

As $(d\mu)|_X = 0 \Rightarrow$ by shrinking $\mathcal{U}_0$ we may assume that ω_t non-degenerated. i.e. ω_t are symplectic forms with $0 \le t \le 1$.

So ~~we~~ by solving $\iota_{v_t}\omega_t = -\mu$ to get v_t with $v_t|_X = 0$.

As X compact, we can again shrink $\mathcal{U}_0$ s.t. all of $\iota_{v_t}\omega_t = -\mu$ are have compact support. $\Rightarrow$ find ρ_t associated v_t. As ~~$\rho_t|_X$~~ $v_t|_X = 0 \Rightarrow \rho_t|_X = 0$. $\square$

Coro. ~~7.6~~ 7.5 [Darboux thm]

$(M, \omega) \Rightarrow$ symplectic, $\forall p \in M$.

Then $\exists$ coordinate system $(\mathcal{U}, x_1 \cdots x_n, y_1 \cdots y_n)$ centered at p on $\mathcal{U}$ s.t.

$$\Rightarrow \omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

proof. Use Thm 7.4 with $X = \{p\}$:

First, choose symplectic basis on T_pM to construct coordinates (x_i', y_i') on some $\mathcal{U}'$ s.t. $\omega_p = \sum dx_i' \wedge dy_i'|_p$.

Let $\omega_0 = \omega$, $\omega_1 = \sum dx_i' \wedge dy_i'$ on $\mathcal{U}'$, use Thm 7.4. $\exists \mathcal{U}_0$ & $\mathcal{U}_1$ or p with diffeo $\varphi: \mathcal{U}_0 \to \mathcal{U}_1$ s.t.

$\varphi(p) = p$, $\varphi^*\omega_1 = \omega_0$.

Since $\varphi^*\omega_1 = \sum d(x_i' \circ \varphi) \wedge d(y_i' \circ \varphi)$ and we let $x_i = x_i' \circ \varphi$, $y_i = y_i' \circ \varphi$.

$\Rightarrow \omega_0 = \sum dx_i \wedge dy_i$. ✓ $\square$