

## §6. Local Thm & Preliminaries

$$p: M \times \mathbb{R} \rightarrow M$$

Def 6.1.  $p \Rightarrow$  isotopy if each  $p_t: M \rightarrow M$  is a diffeomorphism and  $p_0 = \text{id}_M$ .

RMK. Actually, we obtain a family of vector fields  $v_t$  s.t.  $\forall p \in M$ ,

$$\begin{cases} v_t(p) = \frac{d}{ds} p_s(q) \Big|_{s=t} \\ p_t(q) = p \end{cases} \quad \left( \begin{array}{l} q = p_t^{-1}(p) \\ \frac{d}{ds} p_s(q) \Big|_{s=t} = v_t(p_t(q)) \end{array} \right)$$

Conversely, if  $M$  compact, <sup>compact support</sup> by ODEs:

$$\textcircled{D} v_t \Rightarrow p_t.$$

Def 6.2. If  $v_t \equiv v$ , the associated isotopy

$\Rightarrow$  flow of  $v$  ( $\text{exp } tv$ ).

$$\Rightarrow \begin{cases} \frac{d}{dt} (\text{exp } tv)(p) = v(\text{exp } tv(p)) \\ \text{exp } tv|_{t=0} = \text{id}_M \end{cases} \quad \left[ \begin{array}{l} v_t \text{ tie-id} \\ \Leftrightarrow p_{t+s} = p_t \circ p_s \end{array} \right]$$

Def 6.3. Lie derivative of vector field  $v$ :

$$L_v: \Omega^k(M) \rightarrow \Omega^k(M)$$

$$w \mapsto \frac{d}{dt} (\text{exp } tv)^* w \Big|_{t=0}$$

RMK. In general bundle  $F(M) \xrightarrow{\pi} M$ , ~~and~~ diffeomorphism  $f: M \rightarrow M$ .

$$\begin{array}{ccc} F(M) & \xrightarrow{F(f)} & F(M) \\ \downarrow \pi & \Downarrow & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array}$$

let  $s \in \Gamma(F(M))$  & isotopy  $p_t$ ,

$$\Rightarrow p_t^* s = F(p_t^*) \circ s \circ F(p_t).$$

Def. Lie derivative by  $v_t$ :

$$L_{v_t}: \Omega^k(M) \rightarrow \Omega^k(M)$$

$$w \mapsto \frac{d}{ds} (p_s \circ p_t^{-1})^* w \Big|_{s=t}.$$

Exercise 1. For vector fields  $v$ ,

$$\Rightarrow L_v w = \frac{d}{dt} (\text{exp } tv)^* w \Big|_{t=0}$$

Show:  $L_v w = v \cdot dw + dw \cdot v$ .

Proof. Let  $w \in \Omega^k(M)$ ,

$$\begin{aligned} & v(w(\gamma_0 \rightarrow \gamma_k)) \\ &= \frac{d}{dt} \Big|_{t=0} (\text{exp } tv)^* (w(\gamma_0 \rightarrow \gamma_k)) \\ &= \frac{d}{dt} \Big|_{t=0} (\text{exp } tv)^* w(\text{exp } tv \gamma_0 \rightarrow \text{exp } tv \gamma_k) \\ &= (L_v w)(\gamma_0 \rightarrow \gamma_k) + \sum_{i=1}^k w(\gamma_0 \rightarrow [X, \gamma_i] \rightarrow \gamma_k) \end{aligned}$$

$$\Rightarrow (L_v w)(\gamma_0 \rightarrow \gamma_k) = v(w(\gamma_0 \rightarrow \gamma_k)) - \sum_{i=1}^k w(\gamma_0 \rightarrow [X, \gamma_i] \rightarrow \gamma_k)$$

On the other hand

$$\begin{aligned} & dw(X_0, \dots, X_k) \\ &= \sum_{i=0}^k (-1)^i X_i(w(X_0, \dots, X_{i-1}, \dots, X_k)) \\ &+ \sum_{i < j} (-1)^{i+j} w([X_i, X_j], X_0, \dots, X_{i-1}, X_{i+1}, \dots, X_{j-1}, X_{j+1}, \dots, X_k) \end{aligned}$$

Exercise 2. Show:  $\frac{d}{dt} p_t^* w = p_t^* L_{v_t} w$ .

$$\begin{aligned} \text{Proof. } \frac{d}{dt} p_t^* w &= \lim_{s \rightarrow t} \frac{p_s^* w - p_t^* w}{s-t} \\ &= p_t^* \lim_{s \rightarrow t} \frac{(p_t^{-1})^* p_s^* w - w}{s-t} = p_t^* L_{v_t} w. \end{aligned}$$

Prop. 6.4. For family  $w_t$  of  $\Omega^d(M)$

$$\Rightarrow \frac{d}{dt} p_t^* w_t = p_t^* (L_{v_t} w_t + \frac{dw_t}{dt}).$$

Proof. Let  $f(x, y) = p_x^* w_y$ .

$$\Rightarrow \frac{d}{dt} f(t, t) = \frac{d}{dx} f(x, t) \Big|_{x=t} + \frac{d}{dy} f(t, y) \Big|_{y=t}$$

$$= \frac{d}{dx} p_x^* w_t \Big|_{x=t} + \frac{d}{dy} p_t^* w_y \Big|_{y=t}$$

$$= p_t^* L_{v_t} w + p_t^* \frac{dw_t}{dt}$$

$$= p_t^* (L_{v_t} w + \frac{dw_t}{dt}). \quad \square$$

RMK.  $i: X \hookrightarrow M$ ,  $\dim X = k < n = \dim M$

$$N_x X = T_x M / T_x X$$

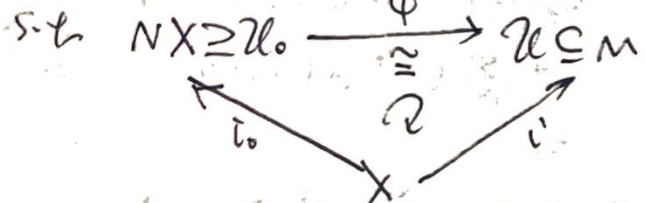
$$\Rightarrow NX = \{(x, v) \mid x \in X, v \in N_x X\}$$

$i_0: X \hookrightarrow NX$  as submanifold.  
 $x \mapsto (x, 0)$

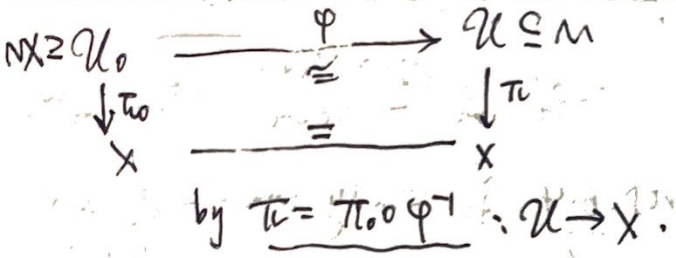
$\rightarrow$  Neighborhood  $\mathcal{U}_0$  of  $X$  in  $NX$  convex  
 if  $\mathcal{U}_0 \cap N_x X$  convex,  $\forall x \in X$ .

Thm 6.5.  $\exists$  convex neigh  $\mathcal{U}_0$  of  $X$

in  $NX$ , a neighborhood  $\mathcal{U}$  of  $X$  in  $M$  & diffeomorphism  $\varphi: \mathcal{U}_0 \rightarrow \mathcal{U}$



RMK. Tubular neighborhood fibration:



Lemma 6.6. Consider  $p_t: \mathcal{U}_0 \rightarrow \mathcal{U}$   
 $(x, v) \mapsto (x, tv)$ .

with  $p_0 = i_0 \circ \pi_0$  &  $p_1 = id$ .

Define  $Q: \Omega^k(\mathcal{U}_0) \rightarrow \Omega^{k-1}(\mathcal{U}_0)$

$$w \mapsto \int_0^1 p_t^*(L_{tv} w) dt,$$

then  $\Rightarrow Id - (i_0 \circ \pi_0)^* = dQ + Qd$ .

proof.  $\forall w \in \Omega^k(\mathcal{U}_0)$ , we have

$$\begin{aligned} dQw + Qd w &= d \int_0^1 p_t^*(L_{tv} w) dt \\ &\quad + \int_0^1 p_t^*(L_{tv} d w) dt \\ &= \int_0^1 p_t^*(L_{tv} w) dt = \int_0^1 \frac{d}{dt} p_t^* w dt \\ &= p_1^* w - p_0^* w. \quad \square \end{aligned}$$

Thm 6.7. ~~Let~~  $\forall$  tubular neighborhood

$\mathcal{U}$  of  $X$  in  $M$ , then

if closed  $d$ -form  $w$  on  $\mathcal{U}$  s.t.  $i_0^* w = 0$ ,  
 then  $\exists \mu \in \Omega^{d-1}(\mathcal{U})$  s.t.  $w = d\mu$   
 and  $\mu_x = 0, \forall x \in X$ .

proof. Via  $\varphi: \mathcal{U}_0 \xrightarrow{\cong} \mathcal{U}$ , we just  
 need to work on  $\mathcal{U}_0$ .

Define  $p_t: \mathcal{U}_0 \rightarrow \mathcal{U}_0$   
 $(x, v) \mapsto (x, tv)$

By lemma 6.6  $\Rightarrow Id - (i_0 \circ \pi_0)^* = dQ + Qd$

$$\Rightarrow w = dQw. \quad \mu = Qw.$$

As  $p_t(x) = x, \forall x \in X \Rightarrow \mu_x = 0, \forall x \in X$ .  
 $\mu = 0$  at all  $x$ .  $\square$

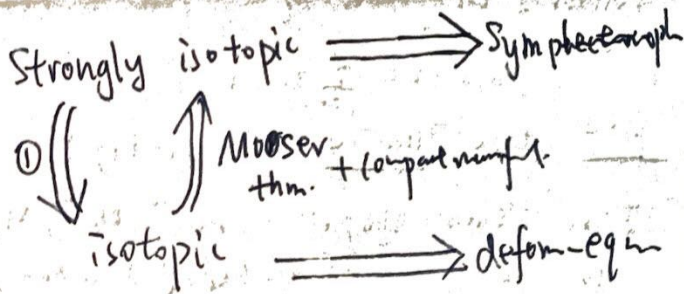
# §7. Moser Theorems.

Now let  $M \Rightarrow 2n$ -dim with two symplectic forms  $\omega_0$  &  $\omega_1$ .

Def. 7.1 We say  $(M, \omega_0)$  &  $(M, \omega_1)$  are:

- ① symplectomorphic if  $\exists$  diffeo  $\phi: M \rightarrow M$  s.t.  $\phi^* \omega_1 = \omega_0$ ;
- ② Strongly isotopic if  $\exists$  isotopy  $P_t: M \rightarrow M$  s.t.  $P_0 = \text{id}$ ,  $P_1^* \omega_1 = \omega_0$ .
- ③ Deformation-equivalent.  $\exists$  family of symplectic forms  $\omega_t$  is smooth resp. to  $t$  joint with  $\omega_0$  &  $\omega_1$ .
- ④ Isotopic.  $\exists$  deformation-equival.  $[\omega_t]$  s.t.  $\frac{d}{dt}[\omega_t] = \left[ \frac{d}{dt} \omega_t \right] = 0$ .

RMK. We have:



①: Similarly,  $Q_t \omega = \int_0^t \rho_s^* (L_{v_s} \omega) ds$

$\implies d^* \omega_1 = \rho_t^* \omega_1$

$= d Q_t \omega_1 = Q_t d \omega_1 = d Q_t \omega_1$

$\implies [\omega_1] = [\rho_t^* \omega_1]$  as  $\omega_t = \rho_t^* \omega_1$ .

## Moser's Trick

(or forms on all compact supp)

Main idea In a compact manifold  $M$  & two forms  $\omega_0, \omega_1$ . We want to find a diffeo  $\phi: M \rightarrow M$  s.t.  $\phi^* \omega_1 = \omega_0$ .

The main idea: We want to find an isotopy  $P_t: M \rightarrow M$  s.t.  $P_0^* \omega_1 = \omega_0$  &  $P_1^* \omega_1 = \omega_0$ .

$$\implies \frac{d}{dt} (P_t^* \omega_t) = 0 = P_t^* (L_{v_t} \omega_t + \frac{d\omega_t}{dt})$$

Just need to find  $v_t$  s.t.

$$L_{v_t} \omega_t + \frac{d\omega_t}{dt} = 0$$

$$\text{i.e. } d(L_{v_t} \alpha_t + L_{v_t} d\alpha_t + \frac{d\alpha_t}{dt}) = 0!$$

$\triangleright$  If moreover,  $d\alpha_t = 0$ , then

$$\text{then } \Leftrightarrow d(L_{v_t} \alpha_t + \frac{d\alpha_t}{dt}) = 0.$$

$\triangleright$  More usefully, we sometimes will let  $\alpha_t = (1-t)\alpha_0 + t\alpha_1$ , with  $[\alpha_0] = [\alpha_1]$

$$\text{then } \Leftrightarrow d(L_{v_t} \alpha_t) + \alpha_1 - \alpha_0 = 0,$$

$$\Leftrightarrow d(L_{v_t} \alpha_t + \beta) = 0.$$

just need to find  $v_t$  s.t.

$$L_{v_t} \alpha_t + \beta = 0$$

Thm 7.2. [Moser thm 1].  $M$  compact.

$[\omega_0] = [\omega_1]$  &  $\omega_t = (1-t)\omega_0 + t\omega_1$ , symplectic.

then  $\exists$  isotopy  $\rho$  s.t.  $\rho_t^* \omega_t = \omega_0$ .

Proof. This is exactly the Moser's trick. (use  $\omega_t$  nondegen.)  $\square$

Thm 7.3. [Moser thm 2].  $M$  compact.

Let  $\omega_t \Rightarrow$  smooth family of closed 2-forms joint  $\omega_0$  &  $\omega_1$  with:

- ①  $[\omega_t]$  indep. of  $t$ .
- ②  $\omega_t$  is nondegenat.

Then  $\exists$  isotopy s.t.  $\rho_t^* \omega_t = \omega_0$ .

proof ①  $\Rightarrow \exists \mu_t$  s.t.  $\frac{d\omega_t}{dt} = d\mu_t$ .

②  $\Rightarrow$  same argument

$\Rightarrow \exists \nu_t$  s.t.  $\iota_{\nu_t} \omega_t + \mu_t = 0$ .  
(Moser eqn).

As  $M$  compact  $\Rightarrow \rho_t \leftrightarrow \nu_t$

$$\Rightarrow \frac{d}{dt} (\rho_t^* \omega_t) = \rho_t^* (\iota_{\nu_t} \omega_t + \frac{d\omega_t}{dt})$$

$$= \rho_t^* (d\iota_{\nu_t} \omega_t + d\mu_t) = 0$$

$$\Rightarrow \rho_t^* \omega_t = \rho_0^* \omega_0 = \omega_0, \quad 0 \leq t \leq 1. \quad \square$$

Thm 7.4 [Relative Moser thm].

$M \Rightarrow \text{symp}^L$ ,  $X \in M$  s.t.  $X$  compact.

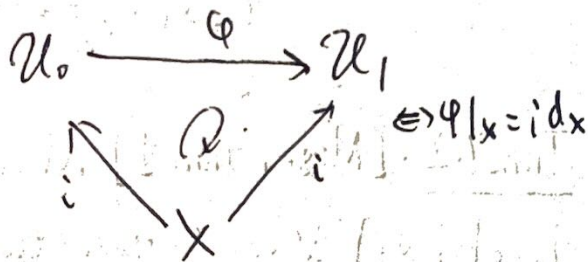
with  $i: X \hookrightarrow M$ . Let  $\omega_0$  &  $\omega_1$  are symplectic on  $M$ , s.t.

$$\omega_0|_p = \omega_1|_p, \quad \forall p \in X.$$

Then  $\exists$  neighborhoods  $\mathcal{U}_0$  &  $\mathcal{U}_1$  of  $X$  in  $M$

and diffeomorphism  $\varphi: \mathcal{U}_0 \rightarrow \mathcal{U}_1$ ,

$$\text{s.t. } \varphi^* \omega_1 = \omega_0 \text{ and}$$



proof. Identify  $\mathcal{U}_0 \rightarrow \mathcal{U}_1$ .

We have  $\omega_1|_p = \omega_0|_p$  closed

with  $(\omega_1 - \omega_0)|_X = 0$ .

Choose tubular neighborhood  $\mathcal{U}_0$ .

Use thm 6.7.

$$\Rightarrow \exists \mu \in \mathcal{R}^1(\mathcal{U}_0) \text{ s.t. } \omega_1 - \omega_0 = d\mu$$

with  $\mu|_X = 0$ .

$$\text{Consider } \omega_t = (1-t)\omega_0 + t\omega_1$$

$$= \omega_0 + t d\mu$$

As  $(d\mu)|_X = 0 \Rightarrow$  by shrinking  $\mathcal{U}_0$

we may assume that  $\omega_t$  non-degenerate. i.e.  $\omega_t$  are symplectic forms with  $\partial \mathcal{U}_0$ .

So by solving  $\iota_{\nu_t} \omega_t = -\mu$  to get  $\nu_t$  with  $\nu_t|_X = 0$ .

As  $X$  compact, we can again shrink  $\mathcal{U}_0$  s.t. all of  $\omega_t$  are non-degenerate and have compact support.  $\Rightarrow$  find  $\rho_t$  associated  $\nu_t$ . As  $\rho_t|_X = 0 \Rightarrow \rho_t|_X = 0$ .  $\square$

Coro. 7.5 [Darboux thm]

$(M, \omega) \Rightarrow \text{symplectic}$ ,  $p \in M$ .

Then  $\exists$  coordinate system  $(\mathcal{U}, x_i \rightarrow x_n, y_i \rightarrow y_n)$  centered at  $p$  on  $\mathcal{U}$

$$\Rightarrow \omega = \sum_{i=1}^n dx_i \wedge dy_i$$

proof. Use Thm 7.4 with  $X = \{p\}$ :

First, choose symplectic basis on  $T_p M$  to construct coordinates  $(x'_i, y'_i)$  on some  $\mathcal{U}'$

$$\text{s.t. } \omega_p = \sum dx'_i \wedge dy'_i|_p$$

Let  $\omega_0 = \omega$ ,  $\omega_1 = \sum dx'_i \wedge dy'_i$  on  $\mathcal{U}'$ ,

use Thm 7.4.  $\exists \mathcal{U}_0$  &  $\mathcal{U}_1$  or  $p$

with diffeo  $\varphi: \mathcal{U}_0 \rightarrow \mathcal{U}_1$  s.t.

$$\varphi(p) = p, \quad \varphi^* \omega_1 = \omega_0$$

Since  $\varphi^* \omega_1 = \sum d(x'_i \circ \varphi) \wedge d(y'_i \circ \varphi)$

and we let  $x_i = x'_i \circ \varphi$ ,  $y_i = y'_i \circ \varphi$ .

$$\Rightarrow \omega_0 = \sum dx_i \wedge dy_i. \quad \checkmark \quad \square$$