Notes on Algebraic Geometry by U.Görtz and T.Wedhorn

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Abstract

This is a note about the book [UT1] which aims to fix some gaps in it. So we will cite the book [UT1] without statement.

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1 Page 217, Subsection (8.6)

Consider the scheme S and quasi-coherent sheaf $\mathscr{E}$, then we define a functor $\mathrm{Grass}^e(\mathscr{E}) : (Sch/S)^{opp} \to (Sets)$ as

$$\mathrm{Grass}^e(\mathscr{E})(T) = \{\mathscr{U} \subset h^*(\mathscr{E}) : h^*(\mathscr{E})/\mathscr{U} \text{ is locally free of rank } e\},$$

where $h : T \to S$ be a S-scheme.

Easy to see that $\mathrm{Grass}^e(\mathscr{E})(T) \to \mathrm{Grass}^e(\mathscr{E})(T'), \mathscr{U} \mapsto f^*(\mathscr{U})$ induced by $f : T' \to T$ is well defined.

Lemma 1. *Surjections of quasi-coherent $\mathscr{O}_S$-modules $v : \mathscr{E}_1 \to \mathscr{E}_2$ induce $i_v : \mathrm{Grass}^e(\mathscr{E}_2) \to \mathrm{Grass}^e(\mathscr{E}_1)$.*

Proof. Consider $h : T \to S$, then define $i_v(T) : \mathscr{U} \mapsto \ker(h^*(\mathscr{E}_1) \to h^*(\mathscr{E}_2)/\mathscr{U})$. □

Theorem 1. *Morphism $i_v : \mathrm{Grass}^e(\mathscr{E}_2) \to \mathrm{Grass}^e(\mathscr{E}_1)$ is representable and is a closed immersion.*

Proof. Let $h : X \to S$ and take any $f : X \to \mathrm{Grass}^e(\mathscr{E}_1)$. Via the Yoneda lemma, it's corresponding to $\mathscr{V}_X \in \mathrm{Grass}^e(\mathscr{E}_1)(X)$. Consider:

$$\begin{array}{ccc} \mathrm{Grass}^e(\mathscr{E}_2) \times_{\mathrm{Grass}^e(\mathscr{E}_1)} X & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Grass}^e(\mathscr{E}_2) & \xrightarrow{\quad i_v \quad} & \mathrm{Grass}^e(\mathscr{E}_1) \end{array}$$

Via the inverse isomorphism in Yoneda lemma it's easy to see that $f(T)(w) = w^*(\mathscr{V}_X)$. So for $g : T \to S$ we have

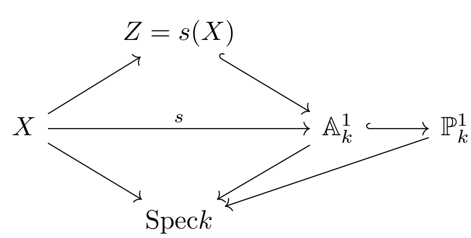
$$\mathrm{Grass}^e(\mathscr{E}_2) \times_{\mathrm{Grass}^e(\mathscr{E}_1)} X(T) = \{w \in X(T) : w^*(\mathscr{V}_X) \in \mathrm{Im}(i_v(T))\},$$

so $w^*(\mathscr{V}_X) \in \mathrm{Im}(i_v(T))$ if and only if $g^*(\mathscr{E}_1)/w^*(\mathscr{V}_X) \to g^*(\mathscr{E}_2)/g^*(v)(w^*(\mathscr{V}_X)) \to g^*(\mathscr{E}_2)/\mathscr{U}$ induced by $g^*(v)$ is an isomorphism where these are locally free of rank e and $g^*(\mathscr{E}_1)/w^*(\mathscr{V}_X) \to g^*(\mathscr{E}_2)/g^*(v)(w^*(\mathscr{V}_X))$ is surjective, then if and only if $g^*(\mathscr{E}_1)/w^*(\mathscr{V}_X) \to g^*(\mathscr{E}_2)/g^*(v)(w^*(\mathscr{V}_X))$ is an isomorphism, which is equivalent to $\ker g^*(v) \subset w^*(\mathscr{V}_X)$, that is, $\ker(g^*(v)) \to g^*(\mathscr{E}_1)/w^*(\mathscr{V}_X)$ is zero. So we use Proposition 8.4 to $\ker(g^*(v)) \to g^*(\mathscr{E}_1)/w^*(\mathscr{V}_X)$ and well done. □

2 Page 350, Subsection (12.14)

Proposition 1 (Proposition 12.66)**.** *Let k be a field and let X be a proper geometrically connected and geometrically reduced k-scheme. Then $\Gamma(X, \mathscr{O}_X) = k$.*

Proof. By the condition we may assume k is an algebraically closed field. Take $s \in \Gamma(X, \mathscr{O}_X)$, we consider $s \in \mathrm{Hom}_k(k[T], \Gamma(X, \mathscr{O}_X)) = \mathrm{Hom}_k(X, \mathbb{A}^1_k)$. Consider the following diagram.



As X is proper, we know that $X \to \mathbb{A}^1_k$ and $X \to \mathbb{P}^1_k$ are also proper. The image $Z = s(X)$ is closed in $\mathbb{A}^1_k$. Consider Z as a closed reduced subscheme of $\mathbb{A}^1_k$. Since $X \to \mathbb{P}^1_k$ is proper, Z is a proper closed in $\mathbb{P}^1_k$, so Z is a finite set in $\mathbb{P}^1_k$. As X is connected, Z is connected. So $Z = \mathrm{Spec}k'$ for some finite extension k'/k. As k is an algebraically closed field we have $k' = k$. So $\Gamma(X, \mathscr{O}_X) = k$. □

Thus we use Corollary 12.64 and the previous Proposition, we find that:

Theorem 2. *Let k be a field and let X be a proper k-scheme. Then $\Gamma(X, \mathscr{F})$ is a finite-dimensional k-vector space for every coherent $\mathscr{O}_X$-module $\mathscr{F}$.*

3 Page 383, Subsection (13.7)

In this subsection many details and definitions are omitted, so we aim to add some content about the related proj.

Let S be a scheme and $\mathscr{A}$ be a graded quasi-coherent $\mathscr{O}_S$-algebra. Then for any affine open $U \subset S$ we have graded $\Gamma(U, \mathscr{O}_S)$-algebra $\Gamma(U, \mathscr{A})$. So we have a separated U-scheme $\mathrm{Proj}\Gamma(U, \mathscr{A})$.

Take affine open sets $U \subset V \subset S$, we have the following pullback diagram

$$\begin{array}{ccc} \mathrm{Proj}\Gamma(U,\mathscr{A}) & \longrightarrow & \mathrm{Proj}\Gamma(V,\mathscr{A}) \\ \downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi_V} \\ U & \longrightarrow & V \end{array}$$

Then we can glue it into a scheme $\underline{Proj}_S\mathscr{A} \to S$ as the following pullback diagram

$$\begin{array}{ccc} \mathrm{Proj}\Gamma(U,\mathscr{A}) & \xrightarrow{\ \eta_U\ } & \underline{Proj}_S\mathscr{A} \\ \downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi} \\ U & \longrightarrow & S \end{array}$$

where ψ_U be an open immersion and $\pi^{-1}(U) = \mathrm{Im}\eta_U \cong \mathrm{Proj}\Gamma(U, \mathscr{A})$.

Next we will find out the functor $\widetilde{-} : \mathfrak{GrQcoMod}(\mathscr{A}) \to \mathfrak{QcoMod}(\underline{Proj}_S\mathscr{A})$ where $\mathfrak{GrQcoMod}(\mathscr{A})$ be the category of graded quasi-coherent $\mathscr{A}$-modules and $\mathfrak{QcoMod}(\underline{Proj}_S\mathscr{A})$ be the category of quasi-coherent $\mathscr{O}_{\underline{Proj}_S\mathscr{A}}$-modules.

Theorem 3. *Let $\mathscr{M}$ be a graded quasi-coherent $\mathscr{A}$-module, then there exists a canonical quasi-coherent $\mathscr{O}_{\underline{Proj}_S\mathscr{A}}$-module $\widetilde{\mathscr{M}}$ such that for any affine open $U \subset S$ we have*

$$\eta_U^*\widetilde{\mathscr{M}} \cong \Gamma(U, \mathscr{M})^\sim, (\eta'_U)_*\Gamma(U, \mathscr{M})^\sim \cong \widetilde{\mathscr{M}}|_{\mathrm{Im}\eta_U}$$

where $\eta_U : \mathrm{Proj}\Gamma(U, \mathscr{A}) \to \underline{Proj}_S(\mathscr{A})$ be an open immersion and $\eta'_U : \mathrm{Proj}\Gamma(U, \mathscr{A}) \to \mathrm{Im}\eta_U$ be the isomorphism.

If $u : \mathscr{M} \to \mathscr{N}$ be a morphism of graded quasi-coherent $\mathscr{A}$-modules, then there is a canonical morphism $\widetilde{u} : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{N}}$. So we defines a exact functor $\widetilde{-} : \mathfrak{GrQcoMod}(\mathscr{A}) \to \mathfrak{QcoMod}(\underline{Proj}_S\mathscr{A})$.

Proof. We omit the proof and one can see [DM1] for the detailed information. The exactness of the functor follows from that $\widetilde{-} : \Gamma(U, \mathscr{A}) - \mathfrak{GrQcoMod} \to \mathfrak{QcoMod}(\mathrm{Proj}\Gamma(U, \mathscr{A}))$ is exact. □

By this theorem, we can prove that $\underline{Proj}_{S'}(g^*\mathscr{A}) \cong \underline{Proj}_S\mathscr{A} \times_S S'$ where $g : S' \to S$. Similar as $(g^*\mathscr{M})^\sim \cong g'^*(\widetilde{\mathscr{M}})$ where $g' : \underline{Proj}_{S'}(g^*\mathscr{A}) \to \underline{Proj}_S(\mathscr{A})$.

Difinition 1. *Let $X = \underline{Proj}_S \mathscr{A}$, then we denote $\mathscr{O}_X(n) := \widetilde{\mathscr{A}(n)}$. For a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we let $\mathscr{F}(n) := \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X$.*

Proposition 2. *If $\mathscr{M}, \mathscr{N}$ be graded quasi-coherent $\mathscr{A}$-modules, then we have*

$$\widetilde{\mathscr{M}} \otimes_{\mathscr{O}_{\underline{Proj}_S \mathscr{A}}} \widetilde{\mathscr{N}} \cong (\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N})^{\sim}.$$

Proof. Also use the previous theorem and this is easy to see. □

Then we have $\mathscr{F}(m) \otimes_{\mathscr{O}_{\underline{Proj}_S \mathscr{A}}} \mathscr{F}(n) \cong \mathscr{F}(m+n)$ for all quasi-coherent $\mathscr{O}_{\underline{Proj}_S \mathscr{A}}$-module $\mathscr{F}$. Similarly we have $(\mathscr{M}(n))^{\sim} \cong \widetilde{\mathscr{M}}(n)$ for all graded quasi-coherent $\mathscr{A}$-module $\mathscr{M}$.

Difinition 2. *From now we let $\mathscr{A}$ generated by $\mathscr{A}_1$ as $\mathscr{A}_0$-algebra. Let $X = \underline{Proj}_S \mathscr{A}$ and $\pi : X \to S$. First we define $\alpha_n : \mathscr{M}_n \to \pi_*(\mathscr{M}(n))^{\sim}$ for graded quasi-coherent $\mathscr{A}$-module $\mathscr{M}$. Since for affine U we have $\mathscr{M}_n(U) \to \Gamma(\mathrm{Proj}\Gamma(U, \mathscr{A}), \Gamma(U, \mathscr{M}(n))^{\sim}) \cong \Gamma(\pi^{-1}U, \widetilde{\mathscr{M}(n)})$ before as $m \mapsto m/1$ at affine local, then by gluing we have the morphism $\alpha_n : \mathscr{M}_n \to \pi_*(\mathscr{M}(n))^{\sim}$.*

Then for quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ we let

$$\Gamma_*(\mathscr{F}) = \bigoplus_n \pi_* \mathscr{F}(n).$$

Moreover, if $f : \mathscr{F} \to \mathscr{G}$, we have $\bigoplus_n \pi_*(f(n)) : \Gamma_*(\mathscr{F}) \to \Gamma_*(\mathscr{G})$. So we defined a functor $\Gamma_*(-) : \mathfrak{QcoMod}(X) \to \mathfrak{GrQcoMod}(\mathscr{A})$.

Theorem 4. *Let $\mathscr{A}$ is finitely generated as $\mathscr{O}_S$-algebra, then we have an adjoint pair $(\widetilde{-}, \Gamma_*(-))$:*

$$\mathfrak{GrQcoMod}(\mathscr{A}) \underset{\Gamma_*(-)}{\overset{\widetilde{-}}{\rightleftarrows}} \mathfrak{QcoMod}(X)$$

Proof. This is a little bit complicated and we omitted the proof of it. □

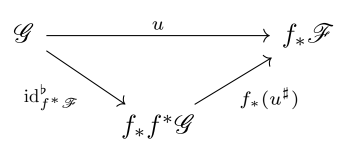
4 Page 397, Subsection (13.12)

For $f : X \to Y$, the bijection

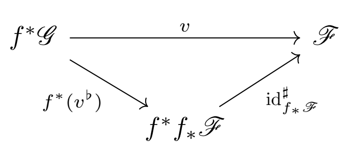
$$\mathrm{Hom}_{\mathscr{O}_X}(f^*\mathscr{G}, \mathscr{F}) \cong \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{G}, f_*\mathscr{F})$$

gives us the following decompositions.

For $u : \mathscr{G} \to f_*\mathscr{F}$ we have



and for $v : f^*\mathscr{G} \to \mathscr{F}$ we have



References

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