Some Notes for Algebraic Spaces and Stacks by Martin Olsson

Xiaolong Liu

Abstract

I take some notes about the book *Algebraic Spaces and Stacks* written by Prof. Martin Olsson [5], aiming to study some basic theory of algebraic stacks. We will also read some pages of [2], which is a wonderful website of algebraic geometry.

Contents

1 Some Notes

♣(Page 30, Theorem 1.5.7) Read [1].
♣(Page 54, 2.3.14) The aim is that we will introduce: Consider ringed topos (C,T,Λ), if $F\in \mathrm{Mod}_\Lambda$ and let $\underline{\mathscr{H}^i}(F)\in \mathrm{PMod}_\Lambda$ sending $X\in C$ to $H^i(C/X,F)$. Then why $\underline{\mathscr{H}^i}(-)$ is the i-th derived functor of inclusion functor $\mathrm{Mod}_\Lambda\hookrightarrow \mathrm{PMod}_\Lambda$?

Actually this is nearly trivial when we show some fundamental results.

First, giving a ringed topos (C,T,Λ) and let $X\in C$, we need to know something about the localization of ringed sites (follows from St 03DH). Consider forgetful functor $j_X: C/X\to C$, which trivially induce a morphism between topoi $j_X=(j_X^*,j_{X,*}):(C,T)\to(C/X,T/X)$. Let $\Lambda_X=j_{X,*}\Lambda$ such that making $(C/X,T/X,\Lambda_X)$ to be a ringed topos (Note that there are some difference between this book and the stacks project, actually f^*,f_* in this book coorespond to the $u_!,u^{-1}$ in the stacks project).

Second, as we defined $H^i(C,-)=R^i\Gamma(C,-):=R^i\mathrm{Hom}(\Lambda,-)$, we again defined for $X\in C$, $H^i(X,-)=R^i\Gamma(X,-)$.

Lemma.(St 03F3) (1) If I is an injective Λ-module, then $I|_U=j_{X,*}I$ is an injective Λ_X-module;

(2) For any $F\in\mathrm{Mod}_\Lambda$, we have $H^p(X,F)=H^p(C/X,F)$.

Proof of Lemma. Trivial as j_X^* exact. □

Proposition.(St 06YK) Consider ringed topos (C,T,Λ), if $F\in\mathrm{Mod}_\Lambda$ and let $\underline{\mathscr{H}^i}(F)\in\mathrm{PMod}_\Lambda$ sending $X\in C$ to $H^i(C/X,F)$. Then $\underline{\mathscr{H}^i}(-)$ is the i-th derived functor of inclusion functor $i:\mathrm{Mod}_\Lambda\hookrightarrow\mathrm{PMod}_\Lambda$.

Proof. Easy to see that i is left exact, choose injective resolution $F\to I^*$. So $R^pi=H^p(I^*)$. Hence the section of $R^pi(F)$ over $X\in C$ is given by

$$\frac{\ker(I^n(X)\to I^{n+1}(X))}{\mathrm{Im}(I^{n-1}(X)\to I^n(X))},$$

which is just $H^p(X,F)=H^p(C/X,F)$. Well done. □

♣(Page 54, Proposition 2.3.15)
Lemma.(St 01FW) Let ringed site (C, Λ). Let $F \in \mathrm{Mod}_\Lambda$ and let $X \in C$. Let $p > 0$ and we take $\xi \in H^p(C/X, F)$. Then there exists a covering $\{X_i \to X\}$ such that $\xi|_{U_i} = 0$ for all i.

Proof of Lemma. Easy to see, just as St 01FW. □

So now in the proof of the Proposition 2.3.15, α is zero in $\check{H}^0(\mathscr{X}, \underline{\mathscr{H}^{i_0}}(F))$. But it is not zero in $H^{i_0}(C/X, F)$! And we find that if $s + t = i_0, 0 < s < i_0$, then $\check{H}^s(\mathscr{X}, \underline{\mathscr{H}^t}(F)) = 0$. Together with $\check{H}^{i_0}(\mathscr{X}, \underline{\mathscr{H}^0}(F)) = \check{H}^{i_0}(\mathscr{X}, F)$ and the spectral sequence

$$E_2^{s,t} = \check{H}^s(\mathscr{X}, \underline{\mathscr{H}^t}(F)) \Rightarrow H^{s+t}(C/X, F),$$

we get that α is not zero in $\check{H}^{i_0}(\mathscr{X}, F)$, well done.
♣(Page 122, 5.1.15)
♣(Page 129, 5.4.3) Let $f : X \to Y$ be morphism of algebraic spaces which is representable by schemes and P be a property of morphisms of schemes which is stable in the étale topology. We claim that f have property P iff there is an étale cover $V \to Y$ such that $V \times_Y X \to V$ has property P. Actually for any scheme T with $T \to Y$, we consider

$$\begin{array}{ccccc} V \times_Y X & \longleftarrow & T \times_Y V \times_Y X & \longrightarrow & T \times_Y X \\ \downarrow f & & \downarrow g & & \downarrow h \\ V & \longleftarrow & T \times_Y V & \longrightarrow & T \end{array}$$

where both squares are fibre product squares. Now as $f : V \times_Y X \to V$ has property P, then so is g. Then as P is stable (locally on the base), so h has property P.
♣(Page 135, 5.5.2, see [4]) After assuming U and X are quasi-compact, we will rewrite the proof of this theorem, because the proof in this book is not so clear and confusing.

Let's start. We first need a lemma:
Lemma 1.(St 03H6) Let S be a scheme. Let X be an algebraic space over S. Then X is quasi-compact if and only if there exists an étale surjective morphism $U \to X$ with U an affine scheme.

Proof of Lemma 1. This is from a fact that for any algebraic space X, one can choose a surjective étale morphism $U \to X$, where $U = \coprod_i U_i$ where U_i are affine schemes by trivial reasons. □

So now we choose a étale surjective morphism $\pi : X_0 \to X$ where X_0 affine by lemma. By St 06NF, we get π **is separated**, as we can have a $\pi : X_0 \to X_1 \to X$ where X_1 is an open subscheme of X and $X_0 \to X_1$ is étale surjective.

Next, as π is étale, then it is formally étale and locally of finite presentation. We need to show π is quasi-compact. Actually consider the fundamental diagram:

$$\begin{array}{ccccc} X_0 & \xrightarrow[\mathrm{id}\times\pi]{} & X_0 \times_S X & \xrightarrow[\mathrm{pr}_2]{} & X \\ \pi\downarrow & & \downarrow(\pi,\mathrm{id}) & & \\ X & \xrightarrow[\Delta_{X/S}]{} & X \times_S X & & \end{array}$$

where the composition of the top horizontal arrow is π and the the square below is cartesian. As $\Delta_{X/S}$ is quasi-compact and by the transitivity of fibre products, we find that $X_0 \to X_0 \times_S X$ is quasi-compact. And pr_2 is quasi-compact as we all passing S into $\mathrm{Spec}(\mathbb{Z})$. Hence π is quasi-compact. So we get π **is quasi-finite**. (This paragraph is just the standard method in basic scheme theory) Now we use the weak version of Zariski main theorem (St 02LR) to some base-change, we get π **is quasi-affine**.

As we already know $q : X \to \overline{X}$ is a sheaf-theoric injective, we need to show q is a sheaf-theoric surjective. Just need to show that $\pi \circ q$ is a sheaf-theoric surjective.

[To be continued...]

♣(Page 140, LEMMA 6.2.9, see [3]) Actually the diagram in the previous of the lemma is **wrong**. It use the same notation as [3] but have the different meanings. Now we use the meanings in [5]. So the diagram should be

$$\begin{array}{ccccc} A_0 & \xrightarrow{\delta_0} & A_1 & \overset{\delta_2'}{\underset{\delta_1'}{\rightrightarrows}} & A_2 \\ & & \uparrow{\scriptstyle \delta_1} & & \uparrow{\scriptstyle \delta_0'} \\ & & A_0 & \overset{\delta_1}{\underset{\delta_0}{\rightrightarrows}} & A_1 \end{array} \qquad \begin{array}{ccccc} X_1 \times_{s,X_0,t} X_1 & \overset{m}{\underset{\mathrm{pr}_2}{\rightrightarrows}} & X_1 & \xrightarrow{s} & X_0 \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle t} & & \\ X_1 & \overset{t}{\underset{s}{\rightrightarrows}} & X_0 & & \end{array}$$

which is comes from the definitions of groupoids in EXERCISE 3.E. So we can split the diagram into three parts (we only do this at the first diagram)

$$\begin{array}{ccc} A_1 & \xrightarrow{\delta_1'} & A_2 \\ \uparrow{\scriptstyle \delta_1} & & \uparrow{\scriptstyle \delta_0'} \\ A_0 & \xrightarrow{\delta_0} & A_1 \end{array} \qquad \begin{array}{ccc} A_1 & \xrightarrow{\delta_2'} & A_2 \\ \uparrow{\scriptstyle \delta_0} & & \uparrow{\scriptstyle \delta_1'} \\ A_0 & \xrightarrow{\delta_0} & A_1 \end{array} \qquad \begin{array}{ccc} A_1 & \xrightarrow{\delta_2'} & A_2 \\ \uparrow{\scriptstyle \delta_1} & & \uparrow{\scriptstyle \delta_0'} \\ A_0 & \xrightarrow{\delta_1} & A_1 \end{array}$$

which are all cocartesians now. So by the base change of the first and the third diagram, we get

$$\delta_0(P_{\delta_1}(T, \delta_0(a))) = P_{\delta_0'}(T, \delta_1'\delta_0(a)),$$

$$\delta_1(P_{\delta_1}(T, \delta_0(a))) = P_{\delta_0'}(T, \delta_2'\delta_0(a))$$

as the characteristic polynomials (actually it is norms here) commutes commutes with arbitrary base change. See St 0BD2. By the second diagram, we get $\delta_1'\delta_0 = \delta_2'\delta_0$. Hence $P_{\delta_1}(T, \delta_0(a))$ has all coefficients in A. By Cayley-Hamilton theorem, we have

$$\begin{aligned} &\delta_0(a)^n - \delta_1(\sigma_1)\delta_0(a)^{n-1} + \cdots + (-1)^n\delta_1(\sigma_n) \\ &= \delta_0(a^n - \sigma_1 a^{n-1} + \cdots + (-1)^n\sigma_n) = 0. \end{aligned}$$

As there exists $\tau : A_1 \to A_0$ such that $\tau \circ \delta_0 = \mathrm{id}$ since the definition of groupoids, we get δ_0 (and δ_1, similarly) is injective. Hence well done.

♣(Page 142, 6.2.13)

♣(Page 142, COROLLARY 6.2.14)

♣(Page 147, PROPOSITION 6.3.4)

♣(Page 149, THEOREM 6.4.1) Assume that we reduced to the case that U is quasi-compact.

♣(Page 158, THEOREM 7.2.10)

♣(Page 161, EXAMPLE 7.2.15)

♣(Page 162, THEOREM 7.3.1)

♣(Page 163, LEMMA 7.4.2)

♣(Page 166, 7.5.3)

♣(Page 169, DEFINITION 8.1.1) Here we need to think of a S-scheme U and an algebraic space X over S (even for a presheaf over (Sch/S) by the same construction) as two fibered categories over groupoids. First we consider U. This is easy, we think of U as the natural functor $(Sch/U)_{et} \to (Sch/S)_{et}$.

For algebraic space X over S, we think of X as a category S_X with object class is $\{(U,x) : U \in (Sch/S), x \in X(U)\}$ and morphisms as

$$\mathrm{Hom}_{S_X}((U,x),(V,y)) = \{f \in \mathrm{Hom}_{Sch/S}(U,V) : f^*y = x\},$$

hence we get $S_X \to (Sch/S), (U,x) \mapsto U$ fibered in sets (so in groupoids) over (Sch/S) (this need some argument and we omitted here, see this at St 0049).

These two things are compactible by PROPOSITION 3.2.8.

♣(Page 169, LEMMA 8.1.3) The core of this proof is EXERCISE 5.G which is easy to prove by using étale equivalence relation. We will rewrite this proof follows from St 0300.

First $V \times_{\mathscr{Y}} \mathscr{X}$ is fibred in setoids, so by some category-theoric method (St 0045) we get that it is equivalent to S_F for some presheaf F over $(Sch/S)_{et}$. By some formal argument we get $V \times_{\mathscr{Y}} \mathscr{X} \to V$ is representable by algebraic spaces and correspond to $F \to G$ by some category-theoric method (St 04SC). Hence $F \to G$ is representable by algebraic spaces by 2-Yoneda lemma. Use EXERCISE 5.G and well done.

♣(Page 171, EXAMPLE 8.1.12) Note that this definition of quotient stack is actually the same of the original one, by St 04WM.

Here the final step of proving $[X/G]$ is an algebraic stack need some corrections as follows.

$$\begin{array}{ccc} & & X \\ & & \big\downarrow{\scriptstyle (G_X,\rho)} \\ T & \xrightarrow{(\mathscr{P},\pi)} & [X/G] \end{array} \qquad\qquad \begin{array}{ccc} & & S_X \\ & & \big\downarrow{\scriptstyle q} \\ (Sch/T)_{et} & \xrightarrow[p]{} & [X/G] \end{array}$$

The diagram on the left is actually the right side one where $q : (V,x) \mapsto (V, x^*G_X, x^*\rho)$ and $p : (f : U \to T) \mapsto (U, f^*\mathscr{P}, f^*\pi)$ (Use some kind of Yoneda lemma).

Now $\mathrm{Ob}(S_{\mathscr{P}}) = \{(U,x) : U \in (Sch/T), x \in \mathscr{P}(U)\}$ (As section be trivialization), so we find that the fiber product of that diagram is $S_{\mathscr{P}}$.

[To be continued...]

♣(Page 176, LEMMA 8.2.4) We actually have further more results of this.

► LEMMA. Let P be a property of schemes which is local in the smooth topology, Let $\mathscr{X}$ be an algebraic stack. The following are equivalent:

(1) $\mathscr{X}$ has property P;

(2) for every scheme X and every smooth morphism $X \to \mathscr{X}$ the scheme X has property P;

(3) for some algebraic space X and some surjective smooth morphism $X \to \mathscr{X}$ the algebraic space X has property P;

(4) for every algebraic space X and every smooth morphism$X \to \mathscr{X}$ the algebraic space X has property P.

Proof. Actually (2) implies (1), (4) implies (2) and (1) implies (3) are all trivial, we only need to show (3) implies (4). Let surjective smooth morphism $X \to \mathscr{X}$ and smooth morphism$Y \to \mathscr{X}$ with algebraic spaces X, Y.

Consider surjective étale morphisms $X' \to X$ and $Y' \to Y$ with schemes X', Y'. As $\mathscr{X}$ be an algebraic stack (hence its diagonal is representable), we get $X' \times_{\mathscr{X}} Y'$ be an algebraic space.

Consider a surjective étale morphisms $W \to X' \times_{\mathscr{X}} Y'$ we get the diagram as follows

$$\begin{array}{ccccccc} W & \longrightarrow & X' \times_{\mathscr{X}} Y' & \longrightarrow & Y' & \longrightarrow & Y \\ & & \downarrow & & & & \downarrow \\ & & X' & \longrightarrow & X & \longrightarrow & \mathscr{X} \end{array}$$

Hence this is easy to see. □

♣(Page 178, Definition 8.2.12) Here we defined for any morphism $f : \mathscr{X} \to \mathscr{Y}$ between algebraic stacks the separated and quasi-separated property of f (here f **may not representable**!). But these are have some different points.

Why? Because the diagonal of morphisms of algebraic spaces are separated (St 03HK), so is the morphisms of algebraic stacks which is **representable** by algebraic spaces. BUT the morphisms of algebraic stacks (**may not representable**) need not be separated!

But now we show that there are no conflicts with the already existing notions if f is **representable** by algebraic spaces.

► Proposition 1. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent:
(1) f is separated in sence of Definition 8.2.9;
(2) $\Delta_{\mathscr{X}/\mathscr{Y}}$ is a closed immersion;
(3) $\Delta_{\mathscr{X}/\mathscr{Y}}$ is proper;
(4) $\Delta_{\mathscr{X}/\mathscr{Y}}$ is universally closed.

Proof. See St 04YS for details. □

► Proposition 2. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks representable by algebraic spaces. Then the following are equivalent:
(1) f is quasi-separated in sence of Definition 8.2.9;
(2) $\Delta_{\mathscr{X}/\mathscr{Y}}$ is quasi-compact (hence qcqs);
(3) $\Delta_{\mathscr{X}/\mathscr{Y}}$ is of finite type.

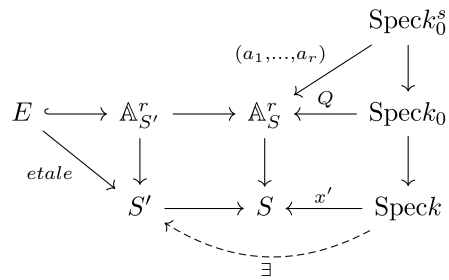
Proof. See St 04YT for details. □

♣(Page 179, 8.3.4) Why we can check on algebraic closed fields? Since we let $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is locally of finite presentation, so $\underline{Isom}(x, y) \to U$ is locally of finite presentation. Hence, we can check formally unramified for $\underline{Isom}(x, y) \to U$ on geometric points, where we have the fiber product:

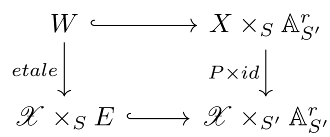
$$\begin{array}{ccc} \underline{Isom}(x,y) & \longrightarrow & U \\ \downarrow & & \downarrow{\scriptstyle (x,y)} \\ \mathscr{X} & \xrightarrow{\Delta} & \mathscr{X} \times_S \mathscr{X} \end{array}$$

When we prove this converse-direction, if $\underline{Aut}_x$ is formally unramified and let $f : \mathrm{Spec}(\Omega) \to U$, then $P_\Omega = \underline{Isom}(f^*u_1, f^*u_2)$. If P_Ω is empty, well done. If not, we have P_Ω is an $\underline{Aut}_x$-torsor. That is, if we have two isomorphisms then α, β are related by unique automorphism $\beta^{-1} \circ \alpha$! Hence here $P_\Omega \cong \underline{Aut}_x$, well done.

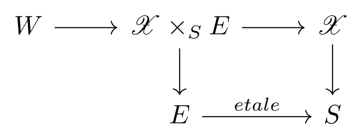
♣(Page 180, Proof of Theorem 8.3.3) In the last paragraph, we can draw a diagram to explain it:



So we get



Hence we get $W_y \neq \emptyset$ with



Hence $\coprod_y W_y \twoheadrightarrow \mathscr{X}$ being an étale covering.

References

[1] Jarod Alper. *Notes on Stacks and Moduli*. `https://sites.math.washington.edu/~jarod/moduli-7-12-22.pdf/`, 2022-7-12.

[2] The Stacks Project Authors. *The Stacks Project*. `https://stacks.math.columbia.edu/`, 2022.

[3] Alexandre Grothendieck and Michel Demazure. *Schémas en groupes (SGA 3)*. Institut des Hautes Etudes Scientifiques, 1963.

[4] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39. Springer, 2000.

[5] Martin Olsson. *Algebraic spaces and stacks*, volume 62. American Mathematical Soc., 2016.