Some Algebraic Topology

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Contents

1 The Fundamental Group and Covering Space

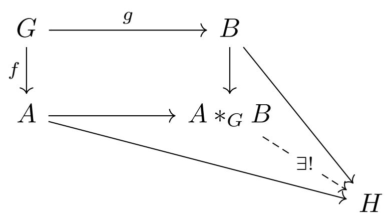
Theorem 1.1 (van Kampen)**.** *Let $X = \bigcup_\alpha A_\alpha$ where A_α are path-connected open sets with a basepoint x_0. Let all $A_\alpha \cap A_\beta$ are path-connected, then consider*

$$\begin{array}{ccc} \pi_1(A_\alpha \cap A_\beta) & \xrightarrow{i_{\alpha\beta}} & \pi_1(A_\alpha) \\ {\scriptstyle i_{\beta\alpha}}\downarrow & & \downarrow{\scriptstyle j_\alpha} \\ \pi_1(A_\beta) & \xrightarrow[j_\beta]{} & \pi_1(X) \end{array}$$

where all maps induced by inclusions. Then j_α induce $\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective. If $A_\alpha \cap A_\beta \cap A_\gamma$ are path-connected, then $\ker \Phi$ is a normal subgroup generated by all elements of form $i_{\alpha\beta}(w) i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A_\alpha \cap A_\beta)$.

Remark 1.2. *In the case of two open sets U, V with $U \cap V$ path-connected, we have the following.*

In the category of groups $\mathfrak{Grp}$, we can describe pushout of $f : G \to A$ and $g : G \to B$. We let $A *_G B$ as $A * B/(f(a)g(a)^{-1})_{a\in G}$, then we have the following universal property in $\mathfrak{Grp}$:



We call it the amalgamated product of A and B with amalgam G. So in the van Kampen theorem with U, V, we have

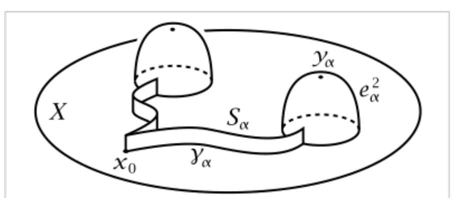
$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V).$$

Next step we need to calculate the fundamental group of a CW complex. Obviously we just need to consider X_2.

Consider 1-skeleton X, then 2-cell e^2_α attach on X via $\phi_\alpha : S^1 \to X$. Then we get Y. Note that if we fix a base point $s_0 \in S^1$, then π_α is a loop. Consider the original base point $x_0 \in X$, let γ_α is a path on X from x_0 to $\phi_\alpha(s_0)$. Then $\gamma_\alpha\phi_\alpha\overline{\gamma}_\alpha$ is a loop which is null-homotopy in Y when the 2-cell attaching. Let $N \subset \pi_1(X, x_0)$ be a normal subgroup generated by $\gamma_\alpha\phi_\alpha\overline{\gamma}_\alpha$.

Theorem 1.3. *Inclusion $X \hookrightarrow Y$ induce surjection $\pi_1(X, x_0) \twoheadrightarrow \pi_1(Y, x_0)$ with kernel N, that is, $\pi_1(Y) \cong \pi_1(X)/N$.*

Proof. Consider a larger space Z with $Z \simeq Y$:



Pick one point y_α on each 2-cells, respectively, as in the diagram. Then $A = Z - \bigcup_\alpha\{y_\alpha\}$ can be deformation retracted to X. Let $B = Z - X$ which is null-homopopic. Use van Kampen theorem to $\{A, B\}$ and we get $\pi_1(Z) \cong \pi_1(A) * \pi_1(B)/(\cdot)$ where $(\cdot)$ is the image of $\pi_1(A\cap B) \to \pi_1(A)$, which is N. □

Proposition 1.4. *Let M, N are two orientable closed surfaces. Then there exists $f : M \to N$ to be a covering space iff $g(M) = mn + 1$ and $g(N) = m + 1$ for some $m, n \geq 0$.*

Proof. Trivial. □

2 Homology

2.1 Singular Homology

Theorem 2.1 (Excision Theorem)**.** *Let $Z \subset A \subset X$ where $\mathrm{cl}(Z) \subset \mathrm{int}(A)$, then the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induce $H_n(X - Z, A - Z) \cong H_n(X, A)$.*

If now we let $B = X - Z$ we have $H_n(B, A \cap B) \cong H_n(X, A)$.

Proposition 2.2. *For good pairs (X, A), map $q : (X, A) \to (X/A, A/A)$ induce $q_* : H_n(X, A) \cong H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$.*

Proof. Let V be the open set deformation retracts into A, consider

$$\begin{array}{ccccc} H_n(X,A) & \xrightarrow{f} & H_n(X,V) & \xleftarrow{g} & H_n(X-A,V-A) \\ \downarrow{q_*} & & \downarrow{q_*} & & \downarrow{q_*} \\ H_n(X/A,A/A) & \xrightarrow{u} & H_n(X/A,V/A) & \xleftarrow{v} & H_n(X/A-A/A,V/A-A/A) \end{array}$$

f, u are isomorphisms by the long exact sequences of triples (X, V, A) and $(X/A, V/A, A/A)$. And g, v are isomorphisms directly by excision. The right hand q_* is isomorphism. So is the left. ☐

2.2 Cellular Homology

Proposition 2.3. *If $f : S^n \to S^n$ has no fixed points, then $f \simeq -\mathbf{1}$. In particular, $\deg f = (-1)^{n+1}$.*

Proof. Consider $f_t(x) = ((1-t)f(x) - tx)/|(1-t)f(x) - tx|$ and well done. ☐

Corollary 2.4. *$\mathbb{Z}/2\mathbb{Z}$ is the only non-trivial group that can act freely on S^n is n is even.*

Proof. Let G be such non-trivial group. The degree map give us homomorphism $G \to \{\pm 1\}$. Since action is free, this map sends all non-trivial elements of G to $(-1)^{n+1}$ by Proposition 2.3. When n even, kernel is trivial. Well done. ☐

Corollary 2.5. *Let $f : S^{2n} \to S^{2n}$, then there exists $x \in S^{2n}$ such that $f(x) = x$ or $f(x) = -x$. In particular, any $g : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.*

Proof. If there are no such points for f, then f and $-f$ both have no fixed points. Then by Proposition 2.3 this is impossible. ☐

Proposition 2.6. *Let $M \in O(n+1)$ which induce $f_M : S^n \to S^n$ by $x \mapsto Mx$, then $\deg f_M = \det M$.*

Proof. This follows from the fact that any orthogonal matrix can be decomposed into the reflection matrixes and rotation matrixes. ☐

Theorem 2.7 (Hairly Ball)**.** *S^n has a continuous field of nonzero tangent vectors iff n is odd.*

Proof. Consider such vector field $v(x)$ and view it as centering at origin. Let $|v(x)| = 1$ via $v(x)/|v(x)|$. Consider $f_t(x) = (\cos t)x + (\sin t)v(x)$. Then $\deg(-\mathrm{id}) = \deg(\mathrm{id}) = 1$, so $(-1)^{n+1} = 1$, so n is odd.

Conversely if $n = 2k-1$, then let $v(x_1, ..., x_{2k}) = (-x_2, -x_1, ..., -x_{2k}, -x_{2k-1})$. ☐

Proposition 2.8 (Euler Charactristic)**.** *We have:*

(a) *For finite CW complexes X, Y, we have $\chi(X \times Y) = \chi(X)\chi(Y)$.*

(b) *If a finite CW complex $X = A \cup B$ of two subcomplexes A, B, then $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.*

(c) *For an n-sheeted covering space of finite CW complexes $p : \widetilde{X} \to X$, we have $\chi(\widetilde{X}) = n\chi(X)$.*

Proof. (a)(b) are trivial. For (c), given an m-dimensional CW-complex X, one can lift the CW-structure to a CW-structure on $\widetilde{X}$ by lifting the characteristic maps $\phi_\alpha : e_\alpha^k \to X$, which can be done since $\pi_1(D^k) = 0$. There are exactly n lifts of ϕ_α to $\widetilde{X}$. So for each k-cell e^k in X, there exists n k-cells in the lifted CW-structure on $\widetilde{X}$ which are mapped homeomorphically onto e^k. Hence well done. □

Remark 2.9. *For (c) there is a generalization for Serre fiberations, see [Multiplicativity of the Euler characteristic for fibrations].*

Now we consider CW complex X with k-skeleton X_k. We have the following elementary conclusion:

Lemma 2.10. *(a) $H^k(X_n, X_{n-1})$ is zero when $k \neq n$ and free abelian with basis of n-cells of X when $k = n$;*
(b) $H_k(X^n) = 0$ for $k > n$;
(c) Inclusion $X^n \hookrightarrow X$ induces $H_k(X^n) \cong H_k(X)$ for $k < n$.

Theorem 2.11 (Cellular Boundary Formula)**.** *The map d_n in above diagram we have $d_n(e_\alpha^n) = \sum_\beta \deg(S_\alpha^{n-1} = \partial e_\alpha^n \to X^{n-1} \to S_\beta^{n-1}) e_\beta^{n-1}$ where the map is the attaching map of e_α^n with the quotient map collapsing $X^{n-1} - e_\beta^{n-1}$ to a point.*

Example 2.12. *Let M_g be the closed orientable surface of genus g, then the cellular complex is $0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \to 0$.*

Example 2.13. *Let N_g be the closed non-orientable surface of genus g, then the cellular complex is $0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z}^g \xrightarrow{0} \mathbb{Z} \to 0$ where $f : 1 \mapsto (2, ..., 2)$.*

Example 2.14. *Consider $\mathbb{R}P^n$, then the cellular complex is*

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

when n is even and

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

when n is odd.

Example 2.15 (Acyclic Space)**.** *Let X obtained from $S1 \vee S^1$ by attaching two 2-cells by words a^5b^{-3} and $b^2(ab)^{-2}$. Then the cellular complex is $0 \to \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}^2 \xrightarrow{0} \mathbb{Z} \to 0$ where $f = \begin{pmatrix} 5 & -2 \\ -3 & 1 \end{pmatrix}$.*

2.3 Mayer-Vietoris

Theorem 2.16 (Mayer-Vietoris Sequence)**.** *Let $A, B \subset X$ with $X = \text{int}(A) \cap \text{int}(B)$. Then we have*

$$0 \longrightarrow C_n(A \cap B) \xrightarrow{x \mapsto (x,-x)} C_n(A) \oplus C_n(B) \xrightarrow{(x,y) \mapsto x+y} C_n(A + B) \longrightarrow 0$$

Then induce the long exact sequence

$$\begin{array}{ccccc} \cdots \longrightarrow H_n(A \cap B) & \xrightarrow{(i_{1*}, -i_{2*})} & H_n(A) \oplus H_n(B) & \xrightarrow{g_* + j_*} & H_n(X) \\ & & & & \downarrow \partial \\ & & & \cdots \longleftarrow & H_{n-1}(A \cap B) \end{array}$$

where $i_1 : A \cap B \to A, i_2 : A \cap B \to B$ and $g : A \to X, j : B \to X$.

Theorem 2.17 (Mapping Torus and Mayer-Vietoris Sequence)**.** *Let $f, g : X \to Y$ and let $Z = X \times I/((x,0) \sim f(x), (x,1) \sim g(x))$ be the mapping torus, then we have*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(X) & \xrightarrow{f_*-g_*} & H_n(Y) & \xrightarrow{i_*} & H_n(Z) \\ & & & & & & \downarrow \\ & & & & \cdots & \longleftarrow & H_{n-1}(X) \end{array}$$

More special case, we let $f : A \cap B \to A, g : A \cap B \to B$, then we can get the traditional Mayer-Vietoris sequence.

Theorem 2.18 (Relative Mayer-Vietoris Sequence)**.** *Let $(X, Y) = (A \cup B, C \cup D)$ with $C \subset A, D \subset B$. Then we have*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(A \cap B, C \cap D) & \longrightarrow & H_n(A, C) \oplus H_n(B, D) & \longrightarrow & H_n(X, Y) \\ & & & & & & \downarrow \\ & & & & \cdots & \longleftarrow & H_{n-1}(A \cap B, C \cap D) \end{array}$$

derived by nine lemma and long exact sequence.

2.4 More Applications

2.4.1 Embedding and Homology

Theorem 2.19 (Invariance of Domain)**.** *Let M and N are both n-dimensional topological manifolds and $f : M \to N$ is one-one and continuous, then f is open.*

Proof. See [1] page 235. □

Corollary 2.20. *If $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is continuous injective map where U is open, then $m \leq n$.*

Proof. If not, we let $m > n$. Consider $g : U \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ with $x \mapsto (f(x), 0)$. By invariance of domain, the image of g, which is $f(U) \times \{0\}$, is open in $\mathbb{R}^m$ which is impossible. □

Remark 2.21. *But unfortunately, for any $m, n > 0$, there is a continuous surjective map $f : \mathbb{R}^m \to \mathbb{R}^n$. See [Existence of a continuous surjective function].*

2.4.2 Borsuk-Ulam Type Theorem

For any two-sheeted covering space $p : X' \to X$, we have exact sequence

$$0 \to C_n(X, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\tau} C_n(X', \mathbb{Z}/2\mathbb{Z}) \xrightarrow{p_\sharp} C_n(X, \mathbb{Z}/2\mathbb{Z}) \to 0$$

as $p_\sharp$ is surjective follows from homotopy lifting property and as each $\sigma : \Delta^n \to X$ has precisely two lifts σ_1', σ_2', then τ maps σ to $\sigma_1' + \sigma_2'$ holds as the coefficient is $\mathbb{Z}/2\mathbb{Z}$. Hence from this we have the long exact sequence

$$\cdots \to H_n(X, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\tau_*} H_n(X', \mathbb{Z}/2\mathbb{Z}) \xrightarrow{p_*} H_n(X, \mathbb{Z}/2\mathbb{Z}) \to \cdots.$$

This is a special case of Gysin sequence.

Theorem 2.22 (Borsuk)**.** *A map $f : S^n \to S^n$ with $f(-x) = -f(x)$ must have odd degree.*

Proof. Consider the covering space $p : S^n \to \mathbb{R}P^n$. As $f(-x) = -f(x)$, we have

$$\begin{array}{ccc} S^n & \xrightarrow{\quad f \quad} & S^n \\ \downarrow p & & \downarrow p \\ \mathbb{R}P^n & \xrightarrow{\quad \bar{f} \quad} & \mathbb{R}P^n \end{array}$$

We claim that the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_i(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\tau} & C_i(S^n, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{p_\sharp} & C_i(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow \bar{f}_\sharp & & \downarrow f_\sharp & & \downarrow \bar{f}_\sharp & & \\ 0 & \longrightarrow & C_i(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\tau} & C_i(S^n, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{p_\sharp} & C_i(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

The right square is trivial. The left square commutes since for $\sigma : \Delta^i \to \mathbb{R}P^n$ with lifts σ_1', σ_2', the two lifts of $\bar{f}\sigma$ are $f\sigma_1', f\sigma_2'$ since $f(-x) = -f(x)$.

Finally taking long exact sequence we can find that $f_* : H_n(S^n, \mathbb{Z}/2\mathbb{Z}) \to H_n(S^n, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism by induction on dimension using the trivial fact that they are isomorphisms in dimension 0. So f must have odd degree. □

Corollary 2.23 (Borsuk-Ulam)**.** *Every map $g : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $g(x) = g(-x)$.*

Proof. Let $f(x) = g(x) - g(-x)$, then f is odd. If f is nowhere vanish, we replace f by $f/|f|$ and get a morphism $f : S^n \to S^{n-1}$ which is still odd. Restrict it on the equator, which is still odd, has odd degree by the theorem of Borsuk. But this restriction is nullhomotopic as it is a restriction of $f|_{D^n}$ in the hemisphere. □

Corollary 2.24. *Whenever S^n is expressed as the union of $n+1$ closed sets $A_0, ..., A_n$, then at least one of these sets must contain a pair of antipodal points.*

Proof. We define $d_i : S^n \to \mathbb{R}, x \mapsto \inf_{y \in A_i} |x - y|$. Let $g : S^n \to \mathbb{R}^n, x \mapsto (d_1(x), ..., d_n(x))$. By Borsuk-Ulam theorem, it obtaining a pair of antipodal points $x, -x$ with $d_i(x) = d_i(-x), i = 1, ..., n$. If either of these distances is 0, then well done. If not, $x, -x \in A_0$, well done. □

Corollary 2.25 (Ham-Sandwich)**.** *Let $A_1, ..., A_n \subset \mathbb{R}^n$ are n measurable subsets, then there exists a hyperplane $H \subset \mathbb{R}^n$ such that H cut each A_i into two parts with equal volume.*

Proof. For any hyperplane H we can write it as $a_1x_1 + ... + a_nx_n + a_{n+1} = 0$ such that $a_1^2 + ... + a_{n+1}^2 = 1$. Consider the map

$$f : S^n \to \mathbb{R}^n, \quad (a_1, ..., a_{n+1}) \mapsto (m(A_i \cap H_+) - m(A_i \cap H_-))_{1 \le i \le n}$$

where $H_+ = \{\mathbf{x} \in \mathbb{R}^n : a_1x_1 + ... + a_nx_n + a_{n+1} > 0\}$ and H_- for < 0. Then by Borsuk-Ulam theorem well done. □

Proposition 2.26. *A map $f : S^n \to S^n$ with $f(-x) = f(x)$ must have even degree. Moreover if n is even , then $\deg f = 0$. If n is odd, then $\deg f$ can be any even number.*

Proof. As $f(-x) = f(x)$, then f can factors as $S^n \to \mathbb{R}P^n \to S^n$ where $S^n \to \mathbb{R}P^n$ be the double covering. Hence induce $H_n(S^n) \xrightarrow{2} H_n(\mathbb{R}P^n)$, hence even degree and when n is even we have $\deg f = 0$. When n odd, consider $f_{2k} : S^n \to \mathbb{R}P^n \to \mathbb{R}P^n/\mathbb{R}P^{n-1} \cong S^n \xrightarrow{\deg k} S^n$ and well done. □

2.4.3 The Lefschetz Fixed Point Theorem

Theorem 2.27 (Lefschetz)**.** *If X is a finite simplicial complex, or more generallya retract of a finite simplicial complex and $f : X \to X$ is a map with $\tau(f) = \sum_n (-1)^n \mathrm{tr}(f_* : H_n(X) \to H_n(X)) \neq 0$, then f has a fixed point.*

3 Cohomology

3.1 Universal Coefficient Theorem and Künneth Formula

Theorem 3.1 (Universal Coefficient Spectral Sequence)**.** *For cohomology we have*

$$E_2^{p,q} = \mathrm{Ext}_R^q(H_p(C_*), G) \Rightarrow H^{p+q}(C_*; G)$$

where R is a ring with unit, C_* is a chain complex of free modules over R, G is any (R, S)-bimodule for some ring with a unit S. The differential d^r has degree $(1 - r, r)$.

Similarly for homology

$$E^2_{p,q} = \mathrm{Tor}_q^R(H_p(C_*), G) \Rightarrow H_*(C_*; G)$$

and the differential d_r having degree $(r - 1, -r)$.

Theorem 3.2 (Universal Coefficient Theorem for Homology)**.** *Let R be a PID and let C_* a chain complex of R-modules such that C_n is free for all n and let M be an R-module. Then there is a natural short exact sequence of R-modules*

$$0 \to H_n(C_*) \otimes_R M \to H_n(C_* \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(C_*), M) \to 0$$

which is split non-naturally.

Proof. As $0 \to Z_n(C_*) \to C_n \to B_{n-1}(C_*) \to 0$ is exact with $B_{n-1}(C_*)$ free since R is PID, then this sequence split. Hence $Z_n(C_*) \otimes_R M \to C_n \otimes_R M$ is also injective. As we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} C_{n-1} \otimes_R M & \longrightarrow & Z_n(C_*) \otimes_R M & \longrightarrow & H_n(C_*) \otimes_R M & \longrightarrow & 0 \\ \downarrow = & & \downarrow & & \downarrow \alpha & & \\ C_{n-1} \otimes_R M & \longrightarrow & Z_n(C_* \otimes_R M) & \longrightarrow & H_n(C_* \otimes_R M) & \longrightarrow & 0 \end{array}$$

by some easy diagram chase we find that $\alpha : H_n(C_*) \otimes_R M \to H_n(C_* \otimes_R M)$ injective. Let's consider its cokernel.

Pick any free resolution $0 \to F_1 \to F_0 \to M \to 0$. As C_i free, we have $0 \to F_1 \otimes_R C_* \to F_0 \otimes_R C_* \to M \otimes_R C_* \to 0$ which give us the long exact sequence. Split it into short exact sequences

$$\begin{array}{c} 0 \longrightarrow \mathrm{coker}(H_n(C_* \otimes_R F_1) \to H_n(C_* \otimes_R F_0)) \\ \downarrow \\ H_n(C_* \otimes_R M) \\ \downarrow \\ \ker(H_{n-1}(C_* \otimes_R F_1) \to H_{n-1}(C_* \otimes_R F_0)) \longrightarrow 0 \end{array}$$

Actually α is trivally an isomorphisms when we consider the free module. As $\mathrm{coker}(H_n(C_*) \otimes_R F_1 \to H_n(C_*) \otimes_R F_0) \cong H_n(C_*) \otimes_R M$ and $\ker(H_{n-1}(C_*) \otimes_R F_1 \to H_{n-1}(C_*) \otimes_R F_0) \cong \mathrm{Tor}_1^R(H_{n-1}(C_*), M)$, we get the theorem. □

Theorem 3.3 (Universal Coefficient Theorem for Cohomology). *Let R be a PID and let C_* a chain complex of R-modules such that C_n is free for all n and let M be an R-module. Then there is a natural short exact sequence of R-modules*

$$0 \to \operatorname{Ext}^1_R(H_{n-1}(C_*), M) \to H^n(\operatorname{Hom}(C_*, M)) \to \operatorname{Hom}(H_n(C_*), M) \to 0$$

which is split non-naturally.

Proof. Similar as the version of homology. □

Theorem 3.4 (Algebraic Künneth Formula). *Let R be a PID and let C_*, C'_* a chain complex of R-modules such that C_n is free for all n. Then there is a natural short exact sequence of R-modules*

$$0 \to \bigoplus_{p+q=n} (H_p(C_*) \otimes_R H_q(C'_*)) \to H_n(C_* \otimes_R C'_*) \to \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(C_*) \otimes_R H_q(C'_*)) \to 0.$$

Theorem 3.5 (Topological Künneth Formula). *Let R be a PID and let X, Y are two CW complexes. Then there is a natural short exact sequence of R-modules*

$$0 \to \bigoplus_{p+q=n} (H_p(X;R) \otimes_R H_q(Y;R)) \to H_n(X \times Y; R) \to \bigoplus_{p+q=n-1} \operatorname{Tor}_1^R(H_p(X;R) \otimes_R H_q(Y;R)) \to 0.$$

Example 3.6. *Let R be a commutative ring with ideal I, J, then $\operatorname{Tor}_1^R(R/I, R/J) \cong \frac{I \cap J}{IJ}$ and $\operatorname{Ext}^1_R(R/I, M) \cong \operatorname{Hom}_R(I, M)/M_I$ where $M_I := \{g_m : i \mapsto im\} \subset \operatorname{Hom}_R(I, M)$.*

Proof. Follows from $0 \to I \to R \to R/I \to 0$. □

3.2 Cup and Cap Products

Definition 3.7 (Cross Product). *Let R be a commutative ring with unit and let X, Y be spaces. We define morphism of chain complexes*

$$C^*(X;R) \otimes_R C^*(Y;R) \to \operatorname{Hom}(C_*(X) \otimes C_*(Y), R) \to C^*(X \times Y; R)$$

where the first one is the natural map and the second is dual of the following Alexander-Whitney map:

$$C_*(X \times Y) \to C_*(X) \otimes C_*(Y), \sigma \mapsto \sum_{p+q=n} {}_p(\pi_X \circ \sigma) \otimes (\pi_Y \circ \sigma)_q$$

where ${}_p\sigma := \sigma|_{[v_0,\dots,v_p]}$ and $\sigma_q := \sigma|_{[v_{n-q},\dots,v_n]}$ when $\sigma \in C_n(-)$. This induce the map

$$\bigoplus_{p+q=n} H^p(X;R) \otimes_R H^q(Y;R) \xrightarrow{\times} H^n(X \times Y; R)$$

which is the cross product.

Definition 3.8 (Cup Product). *Let R be a commutative ring with unit and let X be a space. For $\Delta : X \to X \times X$ be the diagonal, we define*

$$\begin{array}{ccc} H^p(X;R) \otimes_R H^q(X;R) & \xrightarrow{\times} & H^{p+q}(X \times X; R) \\ & \searrow^{\cup} & \downarrow \Delta^* \\ & & H^{p+q}(X;R) \end{array}$$

to be the cup product.

Proposition 3.9. *Let R be a commutative ring and let X be a space.*

(a) Alexander-Whitney map gives an explicit product formula:

$$(\alpha \cup \beta)(\sigma) = \alpha({}_p\sigma) \cdot \beta(\sigma_q), \quad \forall \alpha \in C^p(X;R), \beta \in C^q(X;R), \sigma : \Delta^{p+q} \to X.$$

(b) $H^*(X;R)$ is a graded commutative ring with unit:

- *Let $1 \in H^0(X;R)$ be represented by the cocyle which takes every singular 0-simplex to $1 \in R$. Then $1 \cup \alpha = \alpha \cup 1 = \alpha$ for any $\alpha \in H^*(X;R)$.*
- $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$.
- *$\alpha \cup \beta = (-1)^{pq}\beta \cup \alpha$ for all $\alpha \in H^p(X;R)$ and $\beta \in H^q(X;R)$.*

(c) Let $f : X \to Y$ be a continuous map. Then

$$f^* : H^*(Y;R) \to H^*(X;R)$$

is a morphism of graded commutative rings, i.e. $f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta$.

Remark 3.10. *Actually if we define the cup product in the level of chain complex by (a), then $\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^k \alpha \cup \delta(\beta)$ for $\alpha \in C^k(X;R)$. This is coincident to the original definition since the coboundary map of complex $C^*(X;R) \otimes_R C^*(Y;R)$ has the similar formula.*

Theorem 3.11 (Künneth Formula)**.** *Assumem R is a PID, if $H^*(X;R)$ or $H^*(Y;R)$ are finitely generated free R-modules, we have an isomorphism of graded commutative rings*

$$H^*(X;R) \otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X \times Y;R)$$

where the first one we define $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$.

Definition 3.12 (Cap Product)**.** *We define*

$$C^p(X;R) \times C_{p+q}(X;R) \xrightarrow{\cap} C_q(X;R), \phi \cap \sigma = \phi({}_p\sigma)\sigma_q.$$

Then one can check that $\partial(\phi \cap \sigma) = (-1)^p(\phi \cap \partial\sigma - \delta\phi \cap \sigma)$. This induce the cap product:

$$H^p(X;R) \times H_{p+q}(X;R) \xrightarrow{\cap} H_q(X;R).$$

Proposition 3.13. *The cap product extends naturally to the relative case: for any good pair (X, A), we have*

(a) $H^p(X, A;R) \otimes_R H_{p+q}(X, A;R) \xrightarrow{\cap} H_q(X;R)$;

(b) $H^p(X;R) \otimes_R H_{p+q}(X, A;R) \xrightarrow{\cap} H_q(X, A;R)$.

More generally, we have

$$H^p(X, A;R) \otimes_R H_{p+q}(X, A \cup B;R) \xrightarrow{\cap} H_q(X, B;R).$$

Sketch. Just need to check $\cap$ induce $C^*(X, A;R) \times C_*(A + B;R) \to C_*(B)$. □

Proposition 3.14. *We have the following:*

(a) If $f : X \to Y$ continuous, then we have

$$f_*(\sigma) \cap \phi = f_*(\sigma \cap f^*\phi).$$

(b) For any $\sigma \in C_{k+l}(X;R)$, $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$, we have

$$\psi(\sigma \cap \phi) = (\phi \cup \psi)(\sigma).$$

Proof. Directly check. □

3.3 Orientations

We consider the n-manifold is T2 with locally homeomorphic to $\mathbb{R}^n$. Here we let $H_n(M|A;R) := H_n(M, M-A;R)$. We consider a sheaf both as a functor and as a topological space by the trivial choice of topological basis.

Fix any commutative ring with unit R.

Definition 3.15. *We define $\mathbb{O}_R$ be a locally constant sheaf of R-modules on M whose stalk at a point is $H_n(X|x;R)$. Of course, $\mathbb{O}_R = R \otimes_{\mathbb{Z}} \mathbb{O}_{\mathbb{Z}}$.*

Actually there is a associated (framed) bundle of $\mathbb{O}_R$ as follows: we define a principal $R^\times$-bundle $\widetilde{\mathbb{O}}_R \to M$ which send the open set $U \subset M$ to $\{$trivializations $\alpha : \underline{R}_U \cong \mathbb{O}_R|_U\}$.

Definition 3.16. *An R-orientation of M is a global section of the associated principal $R^\times$-bundle $\widetilde{\mathbb{O}}_R \to M$. In this case we have $\mathbb{O}_R \cong \underline{R}_M$ and we say M is R-orientable.*

Remark 3.17. *When $\mathbb{Z} = R$, then we will ignore R.*

Remark 3.18. *As $H_n(M|x;R) \cong H_n(M|x;\mathbb{Z})_{\mathbb{Z}}R$, we consider a subsheaf $\mathbb{O}_r \subset \mathbb{O}_R$ for each $r \in R$ consist of $\pm\mu_x \otimes r \in H_n(M|x;R)$ where μ_x is a generator of $H_n(M|x;\mathbb{Z}) \cong \mathbb{Z}$. Then as a topological space, if $r = -r$, then $\mathbb{O}_r = M$; if not, then $\mathbb{O}_r \cong \widetilde{\mathbb{O}}_{\mathbb{Z}}$.*

Hence if M is orientable, then it is R-orientable for all R. Any manifold is $\mathbb{F}_2$-orientable.

Proposition 3.19. *Consider the principal $R^\times$-bundle $\widetilde{\mathbb{O}}_R \to M$, then $\widetilde{\mathbb{O}}_R$ is always R-orientable.*

Proof. Follows from construction and $H_n(\widetilde{\mathbb{O}}_R|\mu_x;R) \cong H_n(U(\mu_B)|\mu_x;R) \cong H_n(B|x;R) \cong H_n(M|x;R)$. Well done. □

Proposition 3.20. *Let M connected. Then M is orientable if and only if $\widetilde{\mathbb{O}}_{\mathbb{Z}}$ is connected. In particular, if $\pi_1(M)$ has no subgroup of index 2, then it is orientable.*

Proof. In this case $R = \mathbb{Z}$ and $\widetilde{\mathbb{O}}_{\mathbb{Z}}$ principal $\mathbb{Z}/2\mathbb{Z}$-bundle which is indeed a two-sheeted covering space. Now the result follows directly from the following fact:

- If $p : E \to X$ is a covering space with a section $s : X \to E$, then $s(X) \subset E$ is both open and closed. Hence, it is a union of connected components of E.

Well done. □

Remark 3.21. *We can generalize this into R but I do not care about them.*

Theorem 3.22. *Let M be a manifold of dimension n and let $A \subset M$ be a compact subset. Then for any section $(x \mapsto \alpha_x) \in \Gamma(M, \mathbb{O}_R)$ there exists a unique class $\alpha_A \in H_n(M|A;R)$ whose image in $H_n(M|x;R)$ is α_x for all $x \in A$. Moreover, $H_i(M|A;R) = 0, i > n$.*

Sketch of the Proof. More details see [2].Our method is to reduce the case in to simple one.

(i) **If this hold for $A, B, A \cap B$, then this is also hold of $A \cup B$.** Use the MV-principle, we have:

$$0 = H_{n+1}(M|A \cap B) \longrightarrow H_n(M|A \cup B) \longrightarrow H_n(M|A) \oplus H_n(M|B) \longrightarrow H_n(M|A \cap B)$$

then this is easy to see;

(ii) **Reduce to the case $M = \mathbb{R}^n$.** Actually we can let $A = \bigcup_{i=1}^m A_i$ where A_i in some $\mathbb{R}^n$. Then use MV-principle and induction, well done;

(iii) **Consider the case $M = \mathbb{R}^n$ and $A = \bigcup_{i=1}^m A_i$ where A_i is convex.** Use the MV-principle as (ii) we can let A is convex. Then the result is trivial by $H_i(\mathbb{R}^n|A) \cong H_i(\mathbb{R}^n|x)$ naturally;

(iv) **Consider the case $M = \mathbb{R}^n$ and A be any compact.** Let $\alpha \in H_i(\mathbb{R}^n|A)$ represented by z and let $C \subset \mathbb{R}^n - A$ be the union of the images of the singular simplices in ∂z. Then one can cover some closed balls over A outside of C. Let K be the union of these balls and we see that the relative cycle z defines an element $\alpha_K \in H_i(\mathbb{R}^n|K)$ mapping to the given $\alpha \in H_i(\mathbb{R}^n|A)$. Use (iii) to $H_i(\mathbb{R}^n|K)$, well done. □

Theorem 3.23. *Let M be a closed connected n-manifold. Then*

(a) If M is R-orientable, then the map $H_n(M;R) \to H_n(M|x;R) \cong R$ is an isomorphism for all $x \in M$;

(b) If M is not R-orientable, then the map $H_n(M;R) \to H_n(M|x;R) \cong R$ is injective for all $x \in M$ with image $\{r \in R : 2r = 0\}$.

By the isomorphism $H_n(M;R) \to H_n(M|x;R) \cong R$, the element in $H_n(M;R)$ is called fundamental class if its image in any $H_n(M|x;R) \cong R$ is a generator.

Proof. By Theorem 3.22 for $A = M$, we have $H_n(M;R) \cong \Gamma(M, \mathbb{O}_R)$.

For (a), if M is R-orientable, then the map $H_n(M;R) \to H_n(M|x;R) \cong R$, which is just the evaluation map $e_x : \Gamma(M, \mathbb{O}_R) \to H_n(M|x;R)$, is isomorphism since $\mathbb{O}_R \cong \underline{R}_M$ canonically.

For (b), M is not R-orientable then it is not $\mathbb{Z}$-orientable. By Remark 3.18 we have $\mathbb{O}_R = \bigoplus_{r \in U} \mathbb{O}_r$ where $U = R/\{\pm 1\}$ which is well defined since $\mathbb{O}_r = \mathbb{O}_{-r}$. We know that if $r = -r$, then $\mathbb{O}_r = M$; if not, then $\mathbb{O}_r \cong \widetilde{\mathbb{O}}_{\mathbb{Z}}$. Hence as it is not $\mathbb{Z}$-orientable, there is no global section of $\widetilde{\mathbb{O}}_{\mathbb{Z}}$. As when $r = -r$ there is the trivial section of $\mathbb{O}_r = M$. Hence we get the result. □

Corollary 3.24. *Let M be a closed connected n-manifold. If M is closed and orientable, then $H_{n-1}(M - \{x\}) \cong H_{n-1}(M)$.*

Proof. Indeed, from the pair $(M, M - \{x\})$ we have

$$\cdots \to H_n(M) \to H_n(M, M-\{x\}) \to H_{n-1}(M-\{x\}) \to H_{n-1}(M) \to H_{n-1}(M, M-\{x\}) \to \cdots.$$

As M is orientable, then $H_n(M) \cong H_n(M, M - \{x\})$ by Theorem 3.23(a). Since $H_{n-1}(M, M - \{x\}) = 0$, we have $H_{n-1}(M - \{x\}) \cong H_{n-1}(M)$. □

Corollary 3.25. *Let M be a closed connected n-manifold. The torsion subgroup of $H_{n-1}(M;\mathbb{Z})$ is trivial if M is orientable and $\mathbb{Z}/2\mathbb{Z}$ if M is nonorientable.*

Proof. If M is orientable and if $H_{n-1}(M;\mathbb{Z})$ contained torsion, then for some prime p and universal coefficient, we have

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow H_n(M;\mathbb{Z}/p\mathbb{Z}) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(M), \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0$$

Then $H_n(M;\mathbb{Z}/p\mathbb{Z})$ is bigger than $\mathbb{Z}/p\mathbb{Z}$ which is impossible.

If M is nonorientable, we let $H_{n-1}(M) = F \oplus \bigoplus_j \mathbb{Z}/p_j\mathbb{Z}$, then we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & H_n(M;\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \bigoplus_j \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/p_j\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & 0 \\ & & & & & & \Vert & & \\ & & & & & & \bigoplus_j \frac{p_j\mathbb{Z} \cap 2\mathbb{Z}}{2p_j\mathbb{Z}} & & \end{array}$$

then we have $H_{n-1}(M)_{\text{tor}} = \mathbb{Z}/2\mathbb{Z}$. □

Proposition 3.26. *If M is a connected noncompact n-manifold, then $H_i(M;R) = 0$ for all $i \geq n$.*

Proof. Let z be a cycle represent an element of $H_i(M;R)$. It has a compact image and we let U be an open set cover it with compact closure. Let $V = M - \mathrm{cl}(U)$ and consider $(M, U \cup V, V)$ we have

$$\begin{array}{ccccc} 0 = H_{i+1}(M, U\cup V; R) & \longrightarrow & H_i(U\cup V, V; R) & \longrightarrow & H_i(M, V; R) = 0 \\ & & \uparrow\cong & & \uparrow \\ & & H_i(U;R) & \longrightarrow & H_i(M;R) \end{array}$$

When $i > n$ we have $H_i(U;R) = 0$ so z is a boundary in U and so in M, so $H_i(M;R) = 0$.

When $i = n$, class $[z] \in H_n(M;R)$ defines a section $x \mapsto [z]_x$ of M_R. This section determined by the value in single point since M is connected. Also consider

$$\begin{array}{ccccc} 0 = H_{n+1}(M, U\cup V; R) & \longrightarrow & H_n(U\cup V, V; R) & \longrightarrow & H_n(M, V; R) \\ & & \uparrow\cong & & \uparrow \\ & & H_n(U;R) & \longrightarrow & H_n(M;R) \end{array}$$

Then since M is noncompact and z has a compact image, there must have some point x such that $[z]_x = 0$, so $[z]_x = 0$ for all $x \in M$. Then $[z] = 0$ in $H_n(M, V; R)$, so is in $H_n(U;R)$ and then in $H_n(M;R)$. We win. □

Example 3.27. *Let M, N are both closed connected n-manifolds. Show that $M\sharp N$ is orientable if and only if both M, N are orientable. What is $H_i(M\sharp N)$?*

Analysis. If M, N are orientable, then consider pair $(M\sharp N, S^{n-1})$ with quotient $M\sharp N/S^{n-1} \cong M \vee N$. If $M\sharp N$ is not orientable, then we have injection of $\mathbb{Z}$-modules $\mathbb{Z}\oplus\mathbb{Z} \hookrightarrow \mathbb{Z}$ which is impossible.

If one of them is not orientable, we say N, then we claim that $M\sharp N$ is not orientable. Consider the pair $(M\sharp N, M - \{p\})$, we have

$$\cdots \to H_n(M - \{p\}) \to H_n(M\sharp N) \to H_n(M\sharp N, M - \{p\}) \to \cdots .$$

By Proposition 3.26 we have $H_n(M - \{p\}) = 0$. As $H_n(M\sharp N, M - \{p\}) = H_n(M\sharp N/(M - \{p\})) = H_n(N) = 0$, we find that $H_n(M\sharp N) = 0$. Hence $M\sharp N$ is not orientable.

Now we will compute $H_i(M\sharp N)$. Consider pair $(M\sharp N, S^{n-1})$ with quotient $M\sharp N/S^{n-1} \cong M \vee N$ again. We have

$$\cdots \to \widetilde{H}_i(S^{n-1}) \to \widetilde{H}_i(M\sharp N) \to \widetilde{H}_i(M \vee N) \to \widetilde{H}_{i-1}(S^{n-1}) \to \cdots .$$

Hence if $i \neq n-1, n$, then $\widetilde{H}_i(M\sharp N) \cong \widetilde{H}_i(M\vee N) \cong \widetilde{H}_i(M) \oplus \widetilde{H}_i(N)$. We consider $i = n-1, n$ and we need to consider

$$0 \to \widetilde{H}_n(M\sharp N) \to \widetilde{H}_n(M\vee N) \to \widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(M\sharp N) \to \widetilde{H}_{n-1}(M\vee N) \to 0.$$

Three cases:

If both M, N are orientable, then so is $M\sharp N$. Hence $\widetilde{H}_n(M\sharp N) \cong \mathbb{Z}$ and we have

$$0 \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to \widetilde{H}_{n-1}(M\sharp N) \to \widetilde{H}_{n-1}(M\vee N) \to 0.$$

By some analysis of topology we find that the first map is $1 \mapsto (1,1)$ and the second one is $(a,b) \mapsto a - b$. Hence $\widetilde{H}_{n-1}(M\sharp N) \cong \widetilde{H}_{n-1}(M\vee N)$.

If M is orientable but N is not, then $M\sharp N$ is not and we have $\widetilde{H}_n(M\sharp N) = 0$ and

$$0 \to 0 \to \mathbb{Z} \oplus 0 \to \mathbb{Z} \to \widetilde{H}_{n-1}(M\sharp N) \to \widetilde{H}_{n-1}(M \vee N) \to 0.$$

We know that $\mathbb{Z} \oplus 0 \to \mathbb{Z}$ induced by $(1,0) \mapsto 1$ by trivial reason. Hence $\widetilde{H}_{n-1}(M\sharp N) \cong \widetilde{H}_{n-1}(M \vee N)$.

If both M, N are not orientable, so is $M\sharp N$ and we have $\widetilde{H}_n(M\sharp N) = 0$ and

$$0 \to 0 \to 0 \to \mathbb{Z} \to \widetilde{H}_{n-1}(M\sharp N) \to \widetilde{H}_{n-1}(M \vee N) \to 0.$$

Here we need more information of these manifolds. □

Example 3.28. *Let M is a closed connected m-manifold and N is a closed connected n-manifold. Show that $M \times N$ is orientable if and only if both M, N are orientable.*

Proof. By Topological Künneth Formula, we have

$$0 \to H_m(M) \otimes H_n(N) \to H_{m+n}(M \times N) \to \mathrm{Tor}_1^R(H_m(M) \otimes H_{n-1}(N)) \oplus \mathrm{Tor}_1^R(H_{m-1}(M) \otimes H_n(N)) \to 0.$$

By Theorem 3.23 we have $H_m(M) \otimes H_n(N) \cong H_{m+n}(M \times N)$ and the result follows by Theorem 3.23 again. □

3.4 Poincaré Duality

First consider cohomology with compact supports.

Definition 3.29. *Let $C_c^i(X; G)$ be the subgroup of $C^i(X; G)$ consisting of cochains $\phi : C^i(X) \to G$ for which there exists a compact set $K = K_\phi \subset X$ such that ϕ is zero on all chains in $X - K$. Note that $\delta\phi$ is then also zero on chains in $X - K$, so $\delta\phi$ lies in $C_c^{i+1}(X; G)$ and the $C_c^i(X; G)$'s for varying i form a subcomplex of the singular cochain complex of X . The cohomology groups $H_c^i(X; G)$ of this subcomplex are the cohomology groups with compact supports.*

Another way we let compact $K \hookrightarrow L$ induce $(X, X - L) \hookrightarrow (X, X - K)$, then we have $C^i(X, X - K; G) \hookrightarrow C^i(X, X - L; G)$ and $H^i(X, X - K; G) \to H^i(X, X - L; G)$.

Proposition 3.30. *Since $K \subset X$ are compact sets form a direct system via inclusions. Then we have*

$$\varinjlim H^i(X, X - K; G) \cong H_c^i(X; G).$$

Theorem 3.31 (Poincaré Duality)**.** *Let M be a R-oriented n-manifold. First we define a map $D_M : H_c^k(M; R) \to H_{n-k}(M; R)$. Consider compact sets $K \subset L \subset M$, we have*

$$\begin{array}{ccccc}
H_n(M|L;R) & \overset{\times}{\text{———}} & H^k(M|L;R) & \xrightarrow{\frown} & H_{n-k}(M;R) \\
{\scriptstyle i_*}\downarrow & & \uparrow{\scriptstyle i^*} & \nearrow{\scriptstyle \frown} & \\
H_n(M|K;R) & \overset{\times}{\text{———}} & H^k(M|K;R) & &
\end{array}$$

By previous theorem we can find unique elements $\mu_K \in H_n(M|K; R), \mu_L \in H_n(M|L; R)$ restricting to a given orientation of M at each point of K and L, respectively.

So we have $i_*(\mu_L) = \mu_K$ and $\mu_K \frown x = i_*(\mu_L) \frown x = \mu_L \frown i^*(x)$ for all $x \in H^k(M|K; R)$. So when K vary, we also have $H^k(M|K; R) \xrightarrow{\mu_K \frown (-)} H_{n-k}(M; R)$ which induce

$$D_M : H_c^k(M; R) = \varinjlim H^i(X|K; G) \cong H_{n-k}(M; R).$$

Remark 3.32. *When M is a closed R-oriented n-manifold, if $[M]$ is the fundamental class, we have isomorphism*

$$D_M : H^k(M;R) \xrightarrow{[M]\frown(-),\cong} H_{n-k}(M;R).$$

Proposition 3.33. *A closed manifold of odd dimension has Euler characteristic zero.*

Proof. If M is orientable, then $\operatorname{rank}(H_i(M;\mathbb{Z})) = \operatorname{rank}(H^{n-i}(M;\mathbb{Z})) = \operatorname{rank}(H_{n-i}(M;\mathbb{Z}))$ by Poincaré duality and universal coefficient theorem. If n is odd, well done.

If M is not orientable, the similar argument we have $\sum_i (-1)^i \dim H_i(M;\mathbb{Z}/2\mathbb{Z}) = 0$. Now we claim that $\sum_i (-1)^i \dim H_i(M;\mathbb{Z}/2\mathbb{Z}) = \sum_i (-1)^i \operatorname{rank}(H_i(M;\mathbb{Z}))$. Each $\mathbb{Z}$ summand of $H_i(M;\mathbb{Z})$ gives $\mathbb{Z}/2\mathbb{Z}$ summand of $H_i(M;\mathbb{Z}/2\mathbb{Z})$; each $\mathbb{Z}/m\mathbb{Z}$ (where m even) of $H_i(M;\mathbb{Z})$ gives $\mathbb{Z}/2\mathbb{Z}$ summands of $H_i(M;\mathbb{Z}/2\mathbb{Z})$ and $H_{i+1}(M;\mathbb{Z}/2\mathbb{Z})$ which canceled; each $\mathbb{Z}/m\mathbb{Z}$ (where m odd) of $H_i(M;\mathbb{Z})$ contribute nothing. Well done. □

3.5 Other Duality

Example 3.34 (Euler Charactristic of Boundaries)**.** *Let W be a compact $(2m+1)$-dimensional manifold, then $\chi(\partial W) = 2\chi(W)$.*

Proof. Consider $W \times I$ as a $(2m+2)$-manifold with $\partial(W\times I) = (W\times\{0\})\cup(M\times I)\cup(W\times\{1\})$. Let $M = \partial W$. Let $U = \partial(W \times I) - (W \times \{1\})$ and $V = \partial(W \times I) - (W \times \{0\})$. Then U, V are open in $\partial(W \times I)$. Both U, V are open in $\partial(W \times I)$. Moreover $U, V \simeq W, U \cap V \simeq M$. So by MV sequence

$$\begin{array}{ccccccc} H_{i+1}(U\cup V) & \longrightarrow & H_i(U\cap V) & \longrightarrow & H_i(U)\oplus H_i(V) & \longrightarrow & H_i(U\cup V) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ H_{i+1}(\partial(W\times I)) & \longrightarrow & H_i(M) & \longrightarrow & H_i(W)\oplus H_i(W) & \longrightarrow & H_i(\partial(W\times I)) \end{array}$$

Since $\chi(\partial(W \times I)) = 0$ since $\dim \partial(W \times I)$ is odd. So

$$2\chi(W) = \chi(M) + \chi(\partial(W \times I)) = \chi(M),$$

well done. □

Corollary 3.35. *If $M = \partial W$ for some compact manifold W, then $\chi(M)$ is even.*

Example 3.36 (Boundary of Orientable Manifold is Orientable)**.** *Let M be a R-orientable n-manifold with boundary ∂M, then ∂M is R-orientable.*

Proof. See [3].Consider a coordinate $U \cong \mathbb{H}^n$ of $x \in \partial M$. Let $V = \partial U = u \cap \partial M$, and choose $y \in \operatorname{int}(U) = U - V$. We consider R-coefficient homology group, then we have

$$\begin{aligned} H_n(\operatorname{int}(M), \operatorname{int}(M) - \operatorname{int}(U)) &\xrightarrow{R-\text{orientable},\cong} H_n(\operatorname{int}(M), \operatorname{int}(M) - y) \\ &\xrightarrow{\text{Homotopy by boundary collar},\cong} H_n(M, M - y) \\ &\xrightarrow{R-\text{orientable},\cong} H_n(M, M - \operatorname{int}(U)) \\ &\xrightarrow{\partial,\cong} H_n(M - \operatorname{int}(U), M - U) \\ &\xrightarrow{\text{Homotopy by boundary collar},\cong} H_n(M - \operatorname{int}(U), M - \operatorname{int}(U) - x) \\ &\xrightarrow{\text{Excision of } \operatorname{int}(M)-\operatorname{int}(U),\cong} H_n(\partial M, \partial M - x) \\ &\xrightarrow{R-\text{orientable},\cong} H_n(\partial M, \partial M - V). \end{aligned}$$

Well done. $\square$

Remark 3.37. *In smooth case, we can calculate the transition function. See Theorem 1.3 in* `http://staff.ustc.edu.cn/~wangzuoq/Courses/21F-Manifolds/Notes/Lec24.pdf`.

Theorem 3.38 (Poincaré Duality with Boundaries)**.** *Suppose M is a compact R-orientable n-manifold whose boundary ∂M s decomposed as the union of two compact $(n-1)$ dimensional manifolds A and B with a common boundary $\partial A = \partial B = A \cap B$. Take fundamental class $[M] \in H_n(M, \partial M; R)$. Then for all k we have isomorphism $D_M : H^k(M, A; R) \xrightarrow{[M]\frown(-),\cong} H_{n-k}(M, B; R)$.*

Corollary 3.39 (Lefschetz Duality)**.** *Suppose M is a compact R-orientable n-manifold and take fundamental class $[M] \in H_n(M, \partial M; R)$. Then for all k we have isomorphism D_M : $H^k(M, \partial M; R) \xrightarrow{[M]\frown(-),\cong} H_{n-k}(M; R)$ and $D_M : H^k(M; R) \xrightarrow{[M]\frown(-),\cong} H_{n-k}(M, \partial M; R)$.*

Theorem 3.40 (Generalized Local Homology Groups)**.** *Let K be a compact, locally contractible subspace of a closed orientable n-manifold M, then*

$$H_i(M, M-K; \mathbb{Z}) \cong H^{n-i}(K; \mathbb{Z}).$$

Theorem 3.41 (Alexander Duality)**.** *If K is a compact, locally contractible subspace of S^n, then for all i and any abelian group G, we have*

$$\widetilde{H}_i(S^n - K; G) \cong \widetilde{H}^{n-i-1}(K; G).$$

Theorem 3.42 (Poincaré-Alexander-Lefschetz Duality)**.** *Let M be an n-manifold R-oriented by ϑ where R is any commutative ring with an identity element. For any R-module G, and let $L \subset K$ be compact subsets of M. Then the cap product induce the isomorphism*

$$\cap[\vartheta] : \varinjlim_{(U,V)\supset(K,L)\, open} H^p(U, V; G) \overset{\cong}{\to} H_{n-p}(M-L, M-K; G).$$

Proof. See Theorem VI.8.3 in[1]. More corollary we refer book Homology, Cohomology, and Sheaf Cohomology for Algebraic Topology, Algebraic Geometry, and Differential Geometry section 14.5. $\square$

Corollary 3.43. *Let M be an n-manifold R-oriented by ϑ where R is any commutative ring with an identity element. For any R-module G, and let $L \subset K$ be compact subsets of M. If both of them are good pair, then the cap product induce the isomorphism*

$$H^p(K, L; G) \cong \varinjlim_{(U,V)\supset(K,L)\, open} H^p(U, V; G) \overset{\cap[\vartheta]}{\to} H_{n-p}(M-L, M-K; G).$$

Proof. From Lemma 9 and Theorem 10 in chapter 6.1 in [4]. $\square$

3.6 Cohomology Rings

As before we have

$$\psi(\alpha \frown \phi) = (\phi \smile \psi)(\alpha)$$

where $\alpha \in C_{k+l}(X; R), \phi \in C^k(X; R), \psi \in C^l(X; R)$. So we have

$$\begin{array}{ccc} H^l(X;R) & \xrightarrow{h} & \mathrm{Hom}_R(H_l(X;R), R) \\ \downarrow{\scriptstyle \phi\smile} & & \downarrow{\scriptstyle (\frown\phi)^*} \\ H^{k+l}(X;R) & \xrightarrow{h} & \mathrm{Hom}_R(H_{k+l}(X;R), R) \end{array}$$

For closed R-orientable n-manifold M, consider an important pair:

$$H^k(M;R) \times H^{n-k}(M;R) \longrightarrow R$$
$$(\phi, \psi) \longmapsto (\phi \smile \psi)[M]$$

Proposition 3.44. *This pair is nonsingular for closed R-orientable manifolds when R is a field or when $R = \mathbb{Z}$ and torsion in $H^*(M;\mathbb{Z})$ is factored out.*

Proof. By the universal coefficient theorem and the Poincaré duality, we have an isomorphism

$$H^{n-k}(M;R) \xrightarrow{h} \mathrm{Hom}_R(H_{n-k}(M;R), R) \xrightarrow{D_M^*} \mathrm{Hom}_R(H^k(M;R), R).$$

Well done. □

Corollary 3.45. *If M is a connected closed orientable n-manifold, then for each element $\alpha \in H^k(M;\mathbb{Z})$ of infinite order that is not a proper multiple of another element, there exists an element $\beta \in H^{n-k}(M;\mathbb{Z})$ such that $\alpha \smile \beta$ is a generator of $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$. With coefficients in a field the same conclusion holds for any $\alpha \neq 0$.*

Proof. Follows directly from the nonsingular pair. □

Proposition 3.46. *If $R \to S$ be a ring map, then so is $H^*(X, A; R) \to H^*(X, A; S)$.*

Proof. Trivial. □

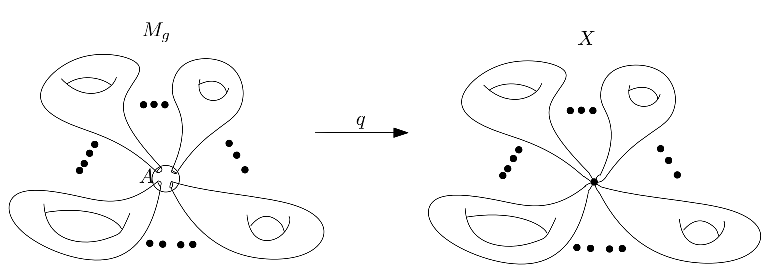
Example 3.47 (Moore Spaces)**.** *Let $M(G, n)$ is a Moore space, let G generated by g_i with relations F_j. then $H^*(M(G,n);\mathbb{Z}) \cong \mathbb{Z}[g_i]/(g_i^2, F_j)$.*

Example 3.48 (Oriented Closed Surfaces)**.** *Let $g \geq 0$ and M_g be the genus g oriented closed surface. Then*

$$H^*(M_g;\mathbb{Z}) \cong \frac{\mathbb{Z}[x_i, y_i, z]_{i=1}^g}{(x_i y_i - z, x_i x_j, y_i y_j, z^2, x_i z, y_i z, \{x_i y_k : i \neq k\})}$$

for $\deg x_i = \deg y_j = 1$ *and* $\deg z = 2$.

Proof. Let $X := \bigvee_{i=1}^g T_i$ where T_i are tori. Let $i : A \hookrightarrow M_g$ be the inclusion where $A = S^2 \setminus \coprod_{j=1}^g D^2 \simeq \bigvee_{j=1}^{g-1} S^1$ as following diagram:



where $q : M_g \to M_g/A \cong X$ be the qoutient map since $M_g \cong T_1 \sharp T_2 \sharp \cdots \sharp T_g$. Hence we get a ring map $q^* : H^*(X) \to H^*(M_g)$.

First we need to find the relation of cohomology classes via $q^* : H^*(X) \to H^*(M_g)$. The only non-trivial cases are $* = 1, 2$. For $* = 1$, we consider the exact sequence

$$H_1(A) \xrightarrow{i_*} H_1(M_g) \xrightarrow{q_*} H_1(X) \xrightarrow{\delta} H_0(A) \xrightarrow{i_*} H_0(M_g).$$

We find that $i_* : H_1(A) \to H_1(M_g)$ is zero since the loops in A will be in the commutator of $\pi_1(M_g)$ and $H_1(M_g) = \text{Abel}(\pi_1(M_g))$. Moreover, $i_* : H_0(A) \to H_0(M_g)$ is injective by the definition, so $\delta = 0$. Hence $q_* : H_1(M_g) \cong H_1(X)$ which induce $q^* : H^1(X) \cong H^1(M_g)$ by universal coefficient theorem.

For $* = 2$, we let $p_i : X \to T_i$ are projects in to the i-th torus. Then we have

$$H_2(M_g) \cong \mathbb{Z} \xrightarrow{q_*} H_2(X) \cong \mathbb{Z}^g \xrightarrow{(p_i)_*} H_2(T_i) \cong \mathbb{Z}$$

As the top cell e^2 of M_g send via q, p_i is also a top cell T_i, by cellular chain complex and its homology we know that $(p_i)_* \circ q_* : 1 \mapsto 1$. Hence $q_* : 1 \mapsto (1, ..., 1)$. So $q^* : H^2(X) \cong \mathbb{Z}^g \to H^2(M_g)$ given by $\gamma_i \mapsto 1$ for the generators of every summands.

Finally we know the work of q^*. Let $H^1(X)$ generated by $\{\alpha_i, \beta_i\}_{i=1}^g$ and $H^2(X)$ generated by $\{\gamma_i\}_{i=1}^g$. We know that $\alpha_i \cup \beta_i = \gamma_i$ and all other cup product between them will be zero by the case of tori. Hence if we let $x_i = q^*\alpha_i, y_i = q^*\beta_i$ and $z = q^*\gamma_1 = ... = q^*\gamma_g$, we have

$$H^*(M_g; \mathbb{Z}) \cong \frac{\mathbb{Z}[x_i, y_i, z]_{i=1}^g}{(x_i y_i - z, x_i x_j, y_i y_j, z^2, x_i z, y_i z, \{x_i y_k : i \neq k\})}$$

for $\deg x_i = \deg y_j = 1$ and $\deg z = 2$. Well done. □

Example 3.49 (Non-oriented Closed Surfaces)**.** *Let $g \geq 0$ and N_g be the genus g non-oriented closed surface. Then*

$$H^*(N_g; \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[x_i, y]_{i=1}^g}{(x_i^2 = y, y^2, \{x_i x_j : i \neq j\})}$$

for $\deg x_i = 1$ *and* $\deg y = 2$.

Proof. Here all the homology and cohomology groups are of $\mathbb{Z}/2\mathbb{Z}$-coefficients.

Let $X := \bigvee_{i=1}^g \mathbb{R}P^2$ and let $i : A \hookrightarrow N_g$ be the inclusion where $A = S^2 \backslash \coprod_{j=1}^g D^2 \simeq \bigvee_{j=1}^{g-1} S^1$ as the orinetable case. Hence we have $q : N_g \to M_g/A \cong X$ be the qoutient map since $N_g \cong \underbrace{\mathbb{R}P^2 \sharp \mathbb{R}P^2 \sharp \cdots \sharp \mathbb{R}P^2}_{g}$ and get the ring map $q^* : H^*(X) \to H^*(N_g)$.

First we need to find the relation of cohomology classes via $q^* : H^*(X) \to H^*(N_g)$. The only non-trivial cases are $* = 1, 2$. For $* = 1$, we consider the exact sequence

$$H_1(A) \xrightarrow{i_*} H_1(N_g) \xrightarrow{q_*} H_1(X) \xrightarrow{\delta} H_0(A) \xrightarrow{i_*} H_0(N_g).$$

We find that $i_* : H_1(A) \to H_1(N_g)$ is zero since the loops in A will be two times of the loops in $\pi_1(N_g)$ and $H_1(N_g) = \text{Abel}(\pi_1(N_g)) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. Moreover, $i_* : H_0(A) \to H_0(N_g)$ is injective by the definition, so $\delta = 0$. Hence $q_* : H_1(N_g) \cong H_1(X)$ which induce $q^* : H^1(X) \cong H^1(N_g)$ by universal coefficient theorem.

For $* = 2$, we let $p_i : X \to \mathbb{R}P^2$ are projects in to the i-th space. Then we have

$$H_2(N_g) \cong \mathbb{Z}/2\mathbb{Z} \xrightarrow{q_*} H_2(X) \cong \mathbb{Z}/2\mathbb{Z}^g \xrightarrow{(p_i)_*} H_2(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$$

As the top cell e^2 of N_g send via q, p_i is also a top cell $\mathbb{R}P^2$, by cellular chain complex and its homology we know that $(p_i)_* \circ q_* : 1 \mapsto 1$. Hence $q_* : 1 \mapsto (1, ..., 1)$. So $q^* : H^2(X) \cong \mathbb{Z}/2\mathbb{Z}^g \to H^2(N_g)$ given by $\gamma_i \mapsto 1$ for the generators of every summands.

Finally we know the work of q^*. Let $H^1(X)$ generated by $\{\alpha_i\}_{i=1}^g$ and $H^2(X)$ generated by $\{\beta_i\}_{i=1}^g$. We know that $\alpha_i^2 = \beta$ and all other cup product between them will be zero by the case of tori. Hence if we let $x_i = q^*\alpha_i$ and $y = q^*\beta_1 = ... = q^*\beta_g$, we have

$$H^*(N_g; \mathbb{Z}/2\mathbb{Z}) \cong \frac{\mathbb{Z}/2\mathbb{Z}[x_i, y]_{i=1}^g}{(x_i^2 = y, y^2, \{x_i x_j : i \neq j\})}$$

for $\deg x_i = 1$ and $\deg y = 2$. Well done. □

Proposition 3.50. *Let M_1 and M_2 be closed oriented manifolds of dimension n. If $f : M_1 \to M_2$ is a continuous map of non-zero degree, and $x \in H^k(M_2; \mathbb{Z})$ is non-torsion, then $f^*x \neq 0$.*

Proof. Let x_i denote the oriented generator of $H^n(M_i; \mathbb{Z})$, then $f^*x_2 = \deg(f)x_1$. As $x \in H^k(M_2; \mathbb{Z})$ is non-torsion, by Poincaré duality there is $y \in H^{n-k}(M_2; \mathbb{Z})$ with $x \cup y \neq 0$, so $x \cup y = rx_2$ for $r \neq 0$. Now $f^*x \cup f^*y = f^*(x \cup y) = r \deg(f)x_1 \neq 0$. Hence $f^*x \neq 0$. □

Corollary 3.51. *If there exists a map $M_1 \to M_2$ of non-zero degree, then for all k we have* $\operatorname{rank} H^k(M_1) \geq \operatorname{rank} H^k(M_2)$. *In particular, if $M_g \to M_h$ of non-zero degree, then $g \geq h$.*

Proof. Directly from the Proposition. □

Example 3.52 (Complex Projective Spaces)**.** *We have the ring isomorphisms*

$$H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1}), \quad H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha], \quad |\alpha| = 2.$$

Proof. Inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ induce the same cohomology group of degree less than $2n-2$, so by induction on n we have $H^{2i}(\mathbb{C}P^n; \mathbb{Z})$ is generated by α^i for $i < n$. By the Corollary 3.45 we can find $m\alpha^{i-1}$ such that $\alpha \smile m\alpha^{n-1} = m\alpha^n$ generates $H^{2n}(\mathbb{C}P^n; \mathbb{Z})$, so $m = \pm 1$, well done. For the case of $\mathbb{C}P^\infty$, this follows from the cellular cohomology. □

Remark 3.53. *Similarly, we have $H^*(\mathbb{H}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$ and $H^*(\mathbb{H}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha]$ for $|\alpha| = 4$.*

Example 3.54 (Real Projective Spaces)**.** *We have the ring isomorphisms*

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1}), \quad H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha], \quad |\alpha| = 1.$$

Furthermore, we have $H^*(\mathbb{R}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha)$ for $|\alpha| = 2$ and

$$H^*(\mathbb{R}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1}), |\alpha| = 2, & n = 2k; \\ \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta), |\alpha| = 2, |\beta| = n, & n = 2k+1. \end{cases}$$

Here β is a generator of $H^{2k+1}(\mathbb{R}P^{2k+1}; \mathbb{Z}) \cong \mathbb{Z}$.

Proof. For the coefficient of $\mathbb{Z}/2\mathbb{Z}$ this is similar as complex projective space.

By Proposition 3.46 we have a ring map $H^*(\mathbb{R}P^\infty; \mathbb{Z}) \to H^*(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$. Consider the cellular cochain as follows:

$$\begin{array}{ccccccccc} \cdots \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \xleftarrow{2} & \mathbb{Z} & \xleftarrow{0} & \mathbb{Z} & \longleftarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \xleftarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xleftarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xleftarrow{0} & \mathbb{Z}/2\mathbb{Z} & \xleftarrow{0} & \mathbb{Z}/2\mathbb{Z} & \longleftarrow 0 \end{array}$$

Hence the ring map $H^*(\mathbb{R}P^\infty;\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}[\alpha]$ is injective in the positive dimension with the image in the even part of $\mathbb{Z}/2\mathbb{Z}[\alpha]$. Hence $H^*(\mathbb{R}P^\infty;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha] = \mathbb{Z}[\alpha]/(2\alpha)$ for $|\alpha| = 2$.

For $n = 2k$ this is the same, hence $H^*(\mathbb{R}P^{2k};\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1})$ for $|\alpha| = 2$. For $n = 2k+1$, note that the top cohomology is $\mathbb{Z}$. We let a generator of it is β, hence α, β generated by α, β. But in this case $\beta^2 = 0$ and $\alpha\beta = 0$ by dimension reason. Hence $H^*(\mathbb{R}P^{2k+1};\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta)$ for $|\alpha| = 2, |\beta| = 2k+1$. □

Example 3.55 (Torus)**.** *We have*

$$H^*(T^n;R) \cong \bigwedge_R[\alpha_1, ..., \alpha_n]$$

for generators $\alpha_i \in H^1(S^1;R)$.

More generally, we have

$$H^*(S^{k_1} \times \cdots \times S^{k_n};R) \cong \bigwedge_R[\alpha_1, ..., \alpha_n]$$

when all k_i odd for generators $\alpha_i \in H^1(S^{k_i};R)$.

Proof. Follows directly from Künneth formula. □

Example 3.56 (Same Cohomology Group with Different Ring)**.** *The spaces $\mathbb{C}P^2$ and $S^2 \vee S^4$ has the same cohomology groups but with the different ring structure.*

Proof. We have $\widetilde{H}^*(S^2 \vee S^4;\mathbb{Z}) \cong \widetilde{H}^*(S^2;\mathbb{Z}) \oplus \widetilde{H}^*(S^4;\mathbb{Z})$. Now as $\alpha^2 = 0$ for $\alpha \in H^2(S^2;\mathbb{Z})$ which is impossible for $\mathbb{C}P^2$, then well done. □

Example 3.57 (Same Cohomology with Additive Structure but Different Cup Product)**.** *The spaces $\mathbb{C}P^3$ and $S^2 \times S^4$ has the same cohomology groups but with the different ring structure.*

Proof. By Künneth formula we have $H^*(S^2 \times S^4;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta]/(\beta^4)$ for $|\alpha| = 2$ and $|\beta| = 4$ with signature-different product. Now as $\alpha^2 = 0$ for $|\alpha| = 2$ which is impossible for $\mathbb{C}P^3$, then well done. □

Remark 3.58. *Another example is $\mathbb{C}P^2 \sharp \mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$. To compute $H^*(\mathbb{C}P^2 \sharp \mathbb{C}P^2;\mathbb{Z})$, you may consider the map $\mathbb{C}P^2 \sharp \mathbb{C}P^2 \to \mathbb{C}P^2 \vee \mathbb{C}P^2$.*

Example 3.59 (Same $\mathbb{Z}$-Cohomology Ring with Different $\mathbb{Z}/2\mathbb{Z}$-Ones)**.** *We have ring isomorphism $H^*(\mathbb{R}P^{2k+1};\mathbb{Z}) \cong H^*(\mathbb{R}P^{2k} \vee S^{2k+1};\mathbb{Z})$ but this is not true for $\mathbb{Z}/2\mathbb{Z}$-coefficient rings.*

Proof. The isomorphism $H^*(\mathbb{R}P^{2k+1};\mathbb{Z}) \cong H^*(\mathbb{R}P^{2k} \vee S^{2k+1};\mathbb{Z})$ is trivial. But for the generator $\alpha \in H^1(\mathbb{R}P^{2k+1};\mathbb{Z}/2\mathbb{Z})$ we have $\alpha^{2k+1} \neq 0$. This is impossible for $\mathbb{R}P^{2k} \vee S^{2k+1}$. □

4 Applications in the Classical Results

Example 4.1 (Jordan Curve)**.** *Actually we view $S^1 \subset \mathbb{R}^2$ as one-point compactification $S^1 \subset S^2$, then we use Alexander duality as*

$$\widetilde{H}_0(S^2 - S^1;\mathbb{Z}) \cong \widetilde{H}^1(S^1;\mathbb{Z}) \cong \mathbb{Z},$$

so $H_0(S^2 - S^1;\mathbb{Z}) \cong \mathbb{Z}^2$, well done.

Example 4.2 (Jordan-Brouwer Separation Theorem)**.** *If $S \subset \mathbb{R}^n$ be a connected compact hypersurface, then $\mathbb{R}^n - S$ has two components.*

Proof. Also we let it as in one-point compactification $S \subset S^n$. Now we didn't know whether S is orientable or not, we consider $\mathbb{Z}/2\mathbb{Z}$ as coefficient, then we use Alexander duality and Poincaré duality

$$\widetilde{H}_0(S^n - S; \mathbb{Z}/2\mathbb{Z}) \cong \widetilde{H}^{n-1}(S; \mathbb{Z}/2\mathbb{Z}) \cong H_0(S; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$

well done. □

Example 4.3 (Compact Hypersurface as Boundary)**.** *If $S \subset \mathbb{R}^n$ be a connected compact hypersurface, then S be the boundary of some domain in $\mathbb{R}^n$.*

Proof. Trivial by Jordan-Brouwer Separation Theorem. □

Proposition 4.4. *Let $X \subset \mathbb{R}^n$ be a compact and locally contractible, then $H_i(X; \mathbb{Z}) = 0$ for $i \geq n$ and torsion-free for $i = n-1, n-2$.*

Proof. View $X \subset S^n$ by one-point compactification. Hence by Alexander duality we have $\widetilde{H}_{n-i-1}(S^n - X; \mathbb{Z}) \cong \widetilde{H}^i(X; \mathbb{Z})$. Then using universal coefficient theorem as in the following exmaples we can get the result. □

Example 4.5 (Compact Hypersurface in $\mathbb{R}^n$ is Orientable)**.** *If S be a connected compact hypersurface S in $\mathbb{R}^n$ is orientable.*

Proof 1. Follows from Proposition 4.4 directly. But here we will go through the proof of that proposition.

Since $\dim S = n-1$, we have to calculate $H_{n-2}(S; \mathbb{Z})$. Also we let it as in one-point compactification $S \subset S^n$. WLOG we let $n > 1$. If S is not orientable, we have $H_{n-1}(S; \mathbb{Z}) = 0$ and $H_{n-2}(S; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, then we have

$$\begin{aligned}\mathbb{Z} &\cong \widetilde{H}_0(S^n - S; \mathbb{Z}) \cong H^{n-1}(S) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(H_{n-1}(S; \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-2}(S; \mathbb{Z}), \mathbb{Z}) \\ &\cong \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-2}(S; \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}\end{aligned}$$

which is impossible. Well done. □

Proof 2. Here we give another method. Take $x \in S$ and $u \in N_x(\mathbb{R}^n/S)$ with $\|u\| = 1$. By Jordan-Brouwer separation theorem we may let u always in the same component when x is varying on S. Consider a non-trivial vector field $X(x) = u(x)$. Now $i_X(\mathrm{vol})$ restricted to S is a volume form on S where vol is the canonical volume form on $\mathbb{R}^n$. □

Proof 3. Moreover we could prove that the normal bundle of S is trivial. See `https://math.stackexchange.com/questions/863960/orientation-of-hypersurface`. □

Example 4.6. *We show that $\mathbb{R}P^n$ can be embedded in $\mathbb{R}^{n+1}$ if and only if $n = 1$ and $\mathbb{C}P^n$ can be embedded in $\mathbb{R}^{2n+1}$ if and only if $n = 1$.*

Proof. The same proof holds for $\mathbb{C}P^n$ and we only consider $\mathbb{R}P^n$. Here we only consider the $\mathbb{Z}/2\mathbb{Z}$-coefficient groups.

If $n = 1$, then $\mathbb{R}P^1 \cong S^1$. Hence it can be embedded in $\mathbb{R}^2$.

Conversely we let $n \geq 2$. If $\mathbb{R}P^n$ can be embedded in $\mathbb{R}^{n+1}$, then we have embedding $\mathbb{R}P^n \hookrightarrow S^{n+1}$. Let $H^*(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}[x]/(x^{n+1})$. Consider the good pair $(S^{n+1}, \mathbb{R}P^n)$ we have $H^{n+1}(S^{n+1}, \mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}^2$. By Corollary 3.43 we have $H^{n+1}(S^{n+1}, \mathbb{R}P^n) \cong H_0(S^{n+1} - \mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}^2$. Hence $S^{n+1} - \mathbb{R}P^n$ has two connected components. Let A, B are closure of these components in S^{n+1}, hence $A \cup B = S^{n+1}$ and $A \cap B = \mathbb{R}P^n$. Hence we get $H^{n+1}(A) \oplus$

$H^{n+1}(B) \cong H^{n+1}(S^{n+1} - \mathbb{R}P^n)$. As by Corollary 3.43 again we have $H^{n+1}(S^{n+1} - \mathbb{R}P^n) \cong H_0(S^{n+1}, \mathbb{R}P^n) = 0$, hence $H^{n+1}(A) = H^{n+1}(B) = 0$.

Now since $n \geq 2$, we have $H^1(S^{n+1}) = H^2(S^{n+1}) = 0$. By MV-sequence for (A, B) we have $H^1(A) \oplus H^1(B) \cong H^1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$ via canonical pullback. WLOG we let $H^1(B) = 0$. Let $i : \mathbb{R}P^n \subset A$ and $i^*\alpha = x$, then $i^*\alpha^n = x^n$ generates $H^n(\mathbb{R}P^n)$. MV-sequence again:

$$0 \to H^n(A) \oplus H^n(B) \to H^n(\mathbb{R}P^n) \to H^{n+1}(S^{n+1}) \to H^{n+1}(A) = H^{n+1}(B) = 0.$$

This is impossible since $H^n(A) \to H^n(\mathbb{R}P^n)$ surjective but $H^{n+1}(S^{n+1}) \cong \mathbb{Z}/2\mathbb{Z}$. □

Remark 4.7. *Actually you can show this using Stiefel-Whitney class for $\mathbb{R}^n$ and total Pontryagin class for $\mathbb{C}P^n$, see EmbeddedProj.*

5 Some Other Things

Proposition 5.1. *The infinity sphere S^∞ is contractible.*

Proof. **Step 1.** Consider $f_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ as $(x_1, x_2, ...) \mapsto (1-t)(x_1, x_2, ...) + t(0, x_1, x_2, ...)$.
Step 2. Consider $g_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ as $(x_1, x_2, ...) \mapsto (1-t)(0, x_1, ...) + t(1, 0, ...)$.
Step 3. Consider $\frac{g_t}{|g_t|} \circ \frac{f_t}{|f_t|}$ which give us $\mathbf{1}_{S^\infty} \simeq (x \mapsto (1, 0, ...))$. □

Proposition 5.2. *Show $S^1 \cong \mathrm{SO}(2)$ and $\mathbb{R}P^3 \cong \mathrm{SO}(3)$.*

Proof. We have $f : S^1 \to \mathrm{SO}(2)$ by $e^{i\theta} \mapsto \theta$. Well done.

Consider $f : D^3 \to \mathrm{SO}(3)$ send $\boldsymbol{x}$ into the rotation through angle $|\boldsymbol{x}|\pi$ about the axis formed by the line through the origin in the direction of $\boldsymbol{x}$. This is surjective. The only non injective points are the antipodal points over $\partial D^3 = S^2$. This give us the homeomorphism $\bar{f} : \mathbb{R}P^3 \cong D^3/S^2 \cong \mathrm{SO}(3)$. □

Remark 5.3. *We also have $\mathrm{SO}(4) \cong S^3 \times \mathrm{SO}(3)$ and more information we refer Section 3.D in [2].*

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