Notes on Spectral Sequences

Xiaolong Liu

Shandong University, Taishan Collage

Abstract

This is a note about the basic spectral sequences , including spectral sequences of exact couples, filered complexes and double complexes. Moreover, we will make some examples to show how them work. Furthermore, we also introduce Cartan-Eilenberg Resolutions and lts most important application, Grothendieck spectral sequences and its applications such as Leray spectral sequences.

Contents

1 Preliminaries

Difinition 1.1. *Let $\mathcal{A}$ be an additive category, a double complex in it is a system $\{A^{p,q}, d_1^{p,q}, d_2^{p,q}\}_{p,q\in\mathbb{Z}}$ where $d_1^{p,q} : A^{p,q} \to A^{p+1,q}, d_2^{p,q} : A^{p,q} \to A^{p,q+1}$ satisfies*

(1) $d_1^{p+1,q} \circ d_1^{p,q} = 0$;

(2) $d_2^{p,q+1} \circ d_2^{p,q} = 0$;

(3) $d_1^{p,q+1} \circ d_2^{p,q} = d_2^{p+1,q} \circ d_1^{p,q}$.

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & A^{p,q+1} & \xrightarrow{d_1^{p,q+1}} & A^{p+1,q+1} & \longrightarrow & \cdots \\
 & & {\scriptstyle d_2^{p,q}}\uparrow & & \uparrow{\scriptstyle d_2^{p+1,q}} & & \\
\cdots & \longrightarrow & A^{p,q} & \xrightarrow[d_1^{p,q}]{} & A^{p+1,q} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & &
\end{array}$$

The associated total complex as

$$sA^n = \mathrm{Tot}^n(A^{*,*}) = \bigoplus_{p+q=n} A^{p,q}$$

with $d^n_{\mathrm{Tot}} = \sum_{p+q=n}(d_1^{p,q} + (-1)^p d_2^{p,q})$.

Difinition 1.2. *Let $\mathcal{A}$ be an abelian category.*

(1) A filtered object of $\mathcal{A}$ is a pair (A, F) where $A \in Obj(\mathcal{A})$ and $F = (F^nA)$ where

$$A \supset \cdots \supset F^nA \supset F^{n+1}A \supset \cdots \supset 0;$$

(2) A morphism $f : (A, F) \to (B, F)$ as $f(F^iA) \subset F^iB$;

(3) Let $X \subset A$, then the induced filtration as $F^nX = X \cap F^nA$;

(4) It is called finite if there exists m, n such that $F^nA = A, F^mA = 0$;

(5) It is called separated if $\bigcap F^iA = 0$, called exhaustive if $\bigcup F^iA = A$.

Difinition 1.3. *Let $\mathcal{A}$ be an abelian category.*

(1) A spectral sequence is a system $(E_r, d_r)_{r\geq s}$ such that $d_r^2 = 0$ with $E_{r+1} = \ker(d_r)/\mathrm{Im}(d_r)$;

(2) A morphism $f : (E_r, d_r)_{r\geq s} \to (E'_r, d'_r)_{r\geq s}$ as $f_r \circ d_r = d'_r \circ f_r$ and such that f_{r+1} induced by f_r via $E_{r+1} = \ker(d_r)/\mathrm{Im}(d_r)$ and $E'_{r+1} = \ker(d'_r)/\mathrm{Im}(d'_r)$.

Remark 1.4. *Given a spectral sequence $(E_r, d_r)_{r\geq s}$ we will define*

$$0 = B_s \subset \cdots \subset B_r \subset \cdots \subset Z_r \subset \cdots \subset Z_s = E_s$$

by the following simple procedure. Set $B_{s+1} = \mathrm{Im}(d_s)$ and $Z_{s+1} = \ker(d_s)$. Then it is clear that $d_{s+1} : Z_{s+1}/B_{s+1} \to Z_{s+1}/B_{s+1}$. Hence we can define B_{s+2} as the unique subobject of E_s containing B_{s+1} such that B_{s+2}/B_{s+1} is the image of d_{s+1}. Similarly we can define Z_{r+2} as the unique subobject of E_s containing B_{s+1} such that Z_{s+2}/B_{s+1} is the kernel of d_{s+1}. And so on and so forth. In particular we have $E_r = Z_r/B_r$.

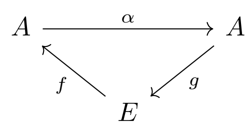
Difinition 1.5. *Let $\mathcal{A}$ be an abelian category and let a spectral sequence $(E_r, d_r)_{r\geq s}$.*

(1) If the subobjects $Z_\infty = \bigcap Z_r$ and $B_\infty = \bigcup B_r$ exists, then we define the limit of the spectral sequence is $E_\infty = Z_\infty/B_\infty$;

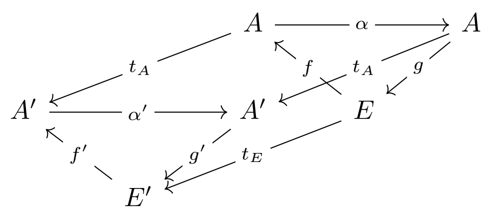
(2) We say that the spectral sequence $(E_r, d_r)_{r\geq s}$ degenerates at E_r if $d_r = \cdots = 0$.

2 Spectral Sequences of exact comples

Difinition 2.1. *Let $\mathcal{A}$ be an abelian category.*
(1) An exact couple is a datum (A, E, α, f, g) with

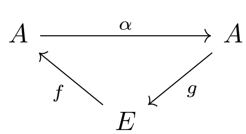


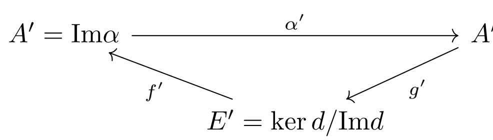
such that is a exact sequence;
(2) A morphism $t : (A, E, \alpha, f, g) \to (A', E', \alpha', f', g')$ as



where $\alpha' \circ t_A = t_A \circ \alpha$, $f' \circ t_E = t_A \circ f$ and $g' \circ t_A = t_E \circ g$.

Theorem 2.2. *Let (A, E, α, f, g) be an exact couple, let*
(1) $d := g \circ f : E \to E$, so that $d^2 = 0$;
(2) $E' = \ker d/\mathrm{Im} d$, $A' = \mathrm{Im}\alpha$;
(3) $\alpha' : A' \to A'$ induced by α;
(4) $f' : E' \to A'$ induced by f;
(5) $g' : A' \to E'$ induced by $g \circ \alpha^{-1}$.





Then we have
(a) $\ker d = f^{-1}(\ker g) = f^{-1}(\mathrm{Im}\alpha)$;
(b) $\mathrm{Im} d = g(\mathrm{Im} f) = g(\ker \alpha)$;
(c) $(A', E', \alpha', f', g')$ be an exact couple.

Proof. We will show how α', f', g' work. Actually α' works by trivial reason. Next, $\ker d = \ker g \circ f = f^{-1}(\ker g) = f^{-1}(\mathrm{Im}\alpha)$, then f maps $\ker d$ into $\mathrm{Im}\alpha$ and since $f \circ g = 0$, we can induce to $f' : E' \to A'$. Finally, since $\mathrm{Im} d = \mathrm{Im} g \circ f = g(\mathrm{Im} f) = g(\ker \alpha)$, we choose $a, b \in \alpha^{-1}(s)$, then $g(a) - g(b) \in \mathrm{Im} d$. So we can induce to $g' : A' \to E'$. It's easy to see that $(A', E', \alpha', f', g')$ be an exact couple. □

So if we let $E_1 = E, d_1 = d, E_2 = E', d_2 = g' \circ f'$ and so on, we get (E_r, d_r).

Difinition 2.3. *Let (A, E, α, f, g) be an exact couple. We say the spectral sequence associated to exact couple is defined (E_r, d_r) as above.*

Remark 2.4. *So in this case we can let $B_{r+1} = g(\ker \alpha^r), Z_{r+1} = f^{-1}(\mathrm{Im}\alpha^r)$. Let*

$$B_\infty = g\left(\bigcup_r \ker \alpha^r\right) \subset Z_\infty = f^{-1}\left(\bigcap_r \mathrm{Im}\alpha^r\right)$$

and $E_\infty = Z_\infty / B_\infty$.

3 Spectral Sequences of filtered complexes

3.1 Main Results

Difinition 3.1. *Let $\mathcal{A}$ be an abelian category. A filtered complex K^* of $\mathcal{A}$ is a complex of filtered objects.*

So it seems as follows.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longleftarrow & F^{p-1}K^{n+1} & \longleftarrow & F^{p}K^{n+1} & \longleftarrow & F^{p+1}K^{n+1} & \longleftarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longleftarrow & F^{p-1}K^{n} & \longleftarrow & F^{p}K^{n} & \longleftarrow & F^{p+1}K^{n} & \longleftarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longleftarrow & F^{p-1}K^{n-1} & \longleftarrow & F^{p}K^{n-1} & \longleftarrow & F^{p+1}K^{n-1} & \longleftarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

We now assume the category $\mathcal{A}$ has countable direct sums and countable direct sums are exact. Next we will construct the spectral sequence associated to it.

Let K^* be the filtered complex and let $E_0 = \bigoplus_{p,q} E_0^{p,q}, E_0^{p,q} = \mathrm{gr}^p K^{p+q}$ where for a filtered object A, we denotes $\mathrm{gr}^p(A) = F^pA/F^{p+1}A, \mathrm{gr}(A) = \bigoplus_p \mathrm{gr}^p(A)$. We call p the filtration degree, and q is called the complementary degree.

Let $d_0 = \bigoplus d_0^{p,q}, d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$. Now we define

$$Z_r^{p,q} = \frac{F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

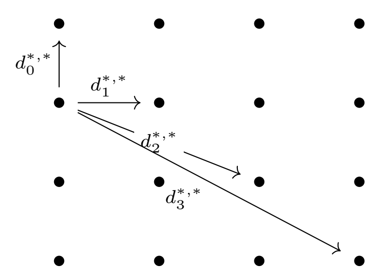
and

$$B_r^{p,q} = \frac{F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}{F^{p+1}K^{p+q}}$$

where d be the differential of K^*. So

$$E_r^{p,q} = Z_r^{p,q}/B_r^{p,q} = \frac{F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q}}{F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}}.$$

Also, we let $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ as $z + F^{p+1}K^{p+q} \mapsto dz + F^{p+r+1}K^{p+q+1}$.



Theorem 3.2. *Let $\mathcal{A}$ be an abelian category with exact countable direct sums. Let K^* be a filtered complex of $\mathcal{A}$. There is a spectral sequence defined as above. Further more, we have $E_1^{p,q} = H^{p+q}(\mathrm{gr}^p(K^*))$.*

Proof. Trivial by the discussion above. In this case $E_{r+1}^{p,q} \cong \frac{\ker d_r^{p,q}}{\mathrm{Im} d_r^{p-r,q+r-1}}$. □

Proposition 3.3. *Let $\mathcal{A}$ be an abelian category with countable direct sums. Let K^* be a filtered complex of $\mathcal{A}$. Let the spectral sequence associated to K^* is (E_r, d_r). Then the map $d_1^{p,q} : E_1^{p,q} = H^{p+q}(\mathrm{gr}^p(K^*)) \to E_1^{p+1,q} = H^{p+q+1}(\mathrm{gr}^{p+1}(K^*))$ is equal to the boundary map of the following short exact sequence*

$$0 \to \mathrm{gr}^{p+1}(K^*) \to F^pK^*/F^{p+2}K^* \to \mathrm{gr}^p(K^*) \to 0.$$

Proof. This is just a diagram chase. □

If we let K^* be a filtered complex, then the induced filtration on $H^n(K^*)$ defined by $F^pH^n(K^*) = \mathrm{Im}(H^n(F^pK^*) \to H^n(K^*))$. Then

$$F^pH^n(K^*, d) = \frac{\ker d \cap F^pK^n + \mathrm{Im} d \cap K^n}{\mathrm{Im} d \cap K^n} \text{ and } \mathrm{gr}^pH^n(K^*) = \frac{\ker d \cap F^pK^n}{\ker d \cap F^{p+1}K^n + \mathrm{Im} d \cap F^pK^n}.$$

Proposition 3.4. *Let $\mathcal{A}$ be an abelian category and let K^* be a filtered complex of $\mathcal{A}$. If $Z_\infty^{p,q}, B_\infty^{p,q}$ exist, then*

(1) The limit E_∞ exists and with bigraded object with $E_\infty^{p,q} = Z_\infty^{p,q}/B_\infty^{p,q}$;

(2) $\mathrm{gr}^pH^n(K^*)$ is a subquotient of $E_\infty^{p,n-p}$.

Proof. (1) is trivial and now we have

$$E_\infty^{p,q} = \frac{\bigcap_r (F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q})}{\bigcup_r (F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q})}.$$

For (2) we let $q = n - p$, then we have

$$\ker d \cap F^pK^n + F^{p+1}K^n \subset \bigcap_r (F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q})$$

and

$$\bigcup_r (F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}) \subset \mathrm{Im} d \cap F^pK^n + F^{p+1}K^n,$$

then a subquotient of $E_\infty^{p,n-p}$ is

$$\frac{\ker d \cap F^pK^n + F^{p+1}K^n}{\mathrm{Im} d \cap F^pK^n + F^{p+1}K^n} = \frac{\ker d \cap F^pK^n}{\ker d \cap F^{p+1}K^n + \mathrm{Im} d \cap F^pK^n},$$

we win. □

Difinition 3.5. *Let $\mathcal{A}$ be an abelian category and let K^* be a filtered complex of $\mathcal{A}$. Let the spectral sequence associated to K^* is $(E_r, d_r)_{r \geq r_0}$. We say it is*

(1) regular if for all p, q there exists $b = b(p, q)$ such that $d_r^{p,q} = 0$ for all $r \geq b$;

(2) coregular if for all p, q there exists $b = b(p, q)$ such that $d_r^{p-r,q+r-1} = 0$ for all $r \geq b$;

(3) bounded if for all n there are only a finite number of nonzero $E_{r_0}^{p,n-p}$; bounded below (resp. above) if there exists $b = b(n)$ such that $E_{r_0}^{p,n-p} = 0$ for $p \geq b$(resp. $p \leq B$);

(4) weakly converges to $H^*(K^*)$ if $\mathrm{gr}^pH^n(K^*) = E_\infty^{p,n-p}$;

(5) abuts to $H^*(K^*)$ if it weakly converges to $H^*(K^*)$ and $\bigcap_p F^pH^n(K^*) = 0$ and $\bigcup_p F^pH^n(K^*) = H^n(K^*)$ for all n;

(6) converges to $H^*(K^*)$ if it is regular, abuts to $H^*(K^*)$ and $H^n(K^*) = \lim_p H^n(K^*)/F^pH^n(K^*)$.

Theorem 3.6. *Let $\mathcal{A}$ be an abelian category and let K^* be a filtered complex of $\mathcal{A}$. Let the spectral sequence associated to K^* is (E_r, d_r). If for all n each filtration on K^n is finite, then the spectral sequence (E_r, d_r) is bounded, the filtration on each $H^n(K^*)$ is finite and (E_r, d_r) converges to $H^*(K^*)$.*

Proof. The first two statements are trivial. Finally since for $r \gg 0$ we have $F^{p+r}K^n = 0$ and $F^{p-r+1}K^{n-1} = K^{n-1}$, then we have the equality

$$\ker d \cap F^pK^n + F^{p+1}K^n = \bigcap_r (F^pK^{p+q} \cap d^{-1}(F^{p+r}K^{p+q+1}) + F^{p+1}K^{p+q})$$

and

$$\bigcup_r (F^pK^{p+q} \cap d(F^{p-r+1}K^{p+q-1}) + F^{p+1}K^{p+q}) = \mathrm{Im} d \cap F^pK^n + F^{p+1}K^n,$$

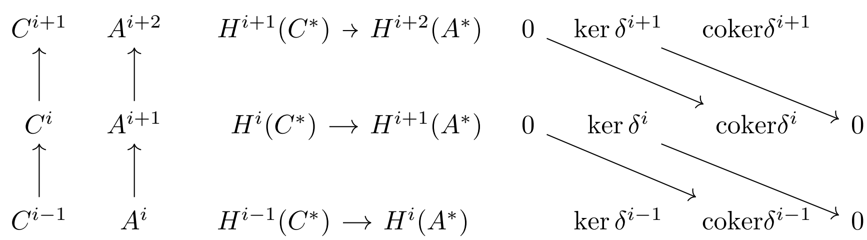
so (E_r, d_r) weakly converges to $H^*(K^*)$. Since the filtration on each $H^n(K^*)$ is finite, so it is abuts and converges to $H^*(K^*)$. □

3.2 An Application

Example 1 (Exact sequence from short one to long one)**.** *Consider a short exact sequence of complex $0 \to A^* \to B^* \to C^* \to 0$, show that there exists a exact sequence*

$$\cdots \to H^i(A^*) \to H^i(B^*) \to H^i(C^*) \to H^{i+1}(A^*) \to \cdots.$$

Proof. Consider the filtration $B^* \supset A^* \supset 0$, then we have $F^0B^i = B^i, F^1B^i = A^i, F^2B^i = 0$. So we have $E_0^{0,i} = C^i, E_0^{1,i} = A^{i+1}$. For $r = 1$ we have $E_1^{0,i} = H^i(\mathrm{gr}^0(B^*)) = H^i(C^*)$ and $E_1^{1,i} = H^{i+1}(\mathrm{gr}^1(B^*)) = H^{i+1}(A^*)$. As we said, $d_1^{p,q}$ as the boundary map, then we have $E_2^{0,i} = \ker \delta^i, E_2^{1,i} = \mathrm{coker}\delta^i$. So the spectral sequence as the following diagram:



In this case $E_2 = E_\infty$. By the diagram above, we have

$$0 \to \ker \delta^i \to H^i(C^*) \to H^{i+1}(A^*) \to \mathrm{coker}\delta^i \to 0.$$

By the theorem we know that the spectral sequence converge to $H^*(C^*)$, thus

$$H^i(B^*)/F^1H^i(B^*) = \mathrm{gr}^0H^i(B^*) = E_\infty^{0,i} = E_2^{0,i} = \ker \delta^i$$

and $F^1H^i(B^*) = \mathrm{gr}^1H^i(B^*) = E_\infty^{1,i-1} = E_2^{1,i-1} = \mathrm{coker}\delta^i$, so we have

$$0 \to \mathrm{coker}\delta^i \to H^i(B^*) \to \ker \delta^{i-1} \to 0$$

is exact. So we combine these exact sequences and then we have the results. □

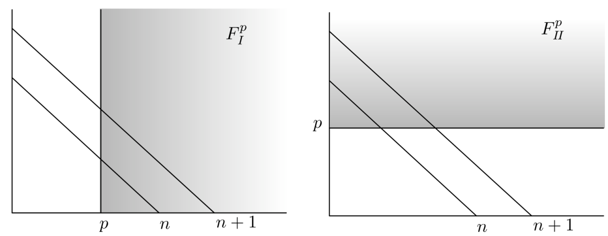
4 Spectral Sequences of double complexes

4.1 Main Results

Consider a double complex $K^{*,*}$, then there are two natrul filterations on $\mathrm{Tot}(K^{*,*})$ as

$$F_I^p(\mathrm{Tot}^n(K^{*,*})) = \bigoplus_{i+j=n, i\geq p} K^{i,j}, F_{II}^p(\mathrm{Tot}^n(K^{*,*})) = \bigoplus_{i+j=n, j\geq p} K^{i,j}.$$

See the following diagram.



Then we get two filtered complexes

$$\begin{array}{ccccccc}
\vdots & & & & & & \\
\uparrow & & & & & & \\
\mathrm{Tot}^{n+1}(K^{*,*}) & \xrightarrow{\supset} & F_{I/II}^1(\mathrm{Tot}^{n+1}(K^{*,*})) & \xrightarrow{\supset} & F_{I/II}^2(\mathrm{Tot}^{n+1}(K^{*,*})) & \xrightarrow{\supset} & \cdots \\
\uparrow & & & & & & \\
\mathrm{Tot}^{n}(K^{*,*}) & \xrightarrow{\supset} & F_{I/II}^1(\mathrm{Tot}^{n}(K^{*,*})) & \xrightarrow{\supset} & F_{I/II}^2(\mathrm{Tot}^{n}(K^{*,*})) & \xrightarrow{\supset} & \cdots \\
\uparrow & & & & & & \\
\mathrm{Tot}^{n-1}(K^{*,*}) & \xrightarrow{\supset} & F_{I/II}^1(\mathrm{Tot}^{n-1}(K^{*,*})) & \xrightarrow{\supset} & F_{I/II}^2(\mathrm{Tot}^{n-1}(K^{*,*})) & \xrightarrow{\supset} & \cdots \\
\uparrow & & & & & & \\
\vdots & & & & & &
\end{array}$$

and can associated them two spectral sequences $('E_r, 'd_r)$ and $(''E_r, ''d_r)$.

We now denote $H_I^p(K^{*,*})$ as the collections of $\ker d_1^{p,q}/\mathrm{Im} d_1^{p-1,q}$ and $H_{II}^q(K^{*,*})$ as the collections of $\ker d_2^{p,q}/\mathrm{Im} d_2^{p,q-1}$. So we have $H_{II}^q(H_I^p(K^{*,*}))$ and $H_I^p(H_{II}^q(K^{*,*}))$.

Theorem 4.1. *Let $\mathcal{A}$ be an abelian category and let $K^{*,*}$ be a double complex, then*

(1) $'E_0^{p,q} = K^{p,q}$ with $'d_0^{p,q} = (-1)^p d_2^{p,q} : K^{p,q} \to K^{p,q+1}$;
(2) $''E_0^{p,q} = K^{q,p}$ with $''d_0^{p,q} = d_1^{q,p} : K^{q,p} \to K^{q+1,p}$;
(3) $'E_1^{p,q} = H^q(K^{p,*})$ with $'d_1^{p,q} = H^p(d_1^{p,*})$;
(4) $''E_1^{p,q} = H^q(K^{*,p})$ with $''d_1^{p,q} = (-1)^q H^q(d_2^{*,p})$;
(5) $'E_2^{p,q} = H_I^p(H_{II}^q(K^{*,*}))$ and $''E_2^{p,q} = H_{II}^p(H_I^q(K^{*,*}))$.

Proof. By the previous section this is easy to see. □

Now we say $('E_r, 'd_r)$ and $(''E_r, ''d_r)$ weakly converge to, abuts to and converge to is Similar as the previous section. So we see that $('E_r, 'd_r)$ and $(''E_r, ''d_r)$ weakly converge if and only if for all n we have

$$\mathrm{gr}_{F_I}(H^n(\mathrm{Tot}(K^{*,*}))) = \bigoplus_{p+q=n} {'E_\infty^{p,q}}, \mathrm{gr}_{F_II}(H^n(\mathrm{Tot}(K^{*,*}))) = \bigoplus_{p+q=n} {''E_\infty^{p,q}}.$$

Theorem 4.2. *Let $\mathcal{A}$ be an abelian category and let $K^{*,*}$ be a double complex. Assmue that for all n there are only finitely many nonzero $K^{p,q}$ with $p+q=n$. Then*

(1) two spectral sequences $('E_r,' d_r)$ and $(''E_r,'' d_r)$ are all bounded;

(2) two filtrations F_I, F_{II} on $H^n(\mathrm{Tot}(K^{*,*}))$ are finite;

(3) two spectral sequences $('E_r,' d_r)$ and $(''E_r,'' d_r)$ all converge to $H^*(\mathrm{Tot}(K^{*,*}))$.

Proof. This is just the restatement of the theorem in the previous section. □

4.2 Some Applications

Example 2 (Snake Lemma)**.** *Consider the following commutative diagram with exact rows.*

$$\begin{array}{ccccccccc}
 & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle r} & & \\
0 & \longrightarrow & X & \xrightarrow{k} & Y & \xrightarrow{h} & Z & &
\end{array}$$

Then we have the following exact sequence

$$\ker s \longrightarrow \ker t \longrightarrow \ker r \longrightarrow \mathrm{coker} s \longrightarrow \mathrm{coker} t \longrightarrow \mathrm{coker} r$$

Proof. We may think it as a double complex as the following

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{k} & Y & \xrightarrow{h} & Z & \dashrightarrow & 0 \\
\uparrow & & {\scriptstyle s}\uparrow & & {\scriptstyle t}\uparrow & & {\scriptstyle r}\uparrow & & \uparrow \\
0 & \dashrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0
\end{array}$$

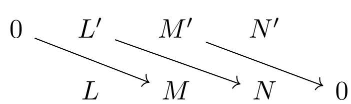
where the dotted arrow means they are not be the part of exactness. Then the second spectral sequence as the following

$$\begin{array}{ccccccc}
C & & Z & \qquad & 0 & \longrightarrow & \mathrm{coker} h \\
{\scriptstyle g}\uparrow & & {\scriptstyle h}\uparrow & & & & \\
B & & Y & & 0 & \longrightarrow & 0 \\
{\scriptstyle f}\uparrow & & {\scriptstyle k}\uparrow & & & & \\
A & & X & & \ker f & \longrightarrow & 0
\end{array}$$

Then $''E_\infty = ''E_1$. So $\ker f = ''E_\infty^{0,0} = \mathrm{gr}^0(H^0(\mathrm{Tot}))$, so $H^0(\mathrm{Tot}) = \ker f$ since $F_{II}^1 H^0(\mathrm{Tot}) = 0$. Similarly, we have that the cohomological group of Tot are $\ker f, 0, 0, \mathrm{coker} h$.

The first spectral sequence as the following

$$\begin{array}{ccccccccc}
X & Y & Z & \qquad & \mathrm{coker} s & \longrightarrow & \mathrm{coker} t & \longrightarrow & \mathrm{coker} r \\
{\scriptstyle s}\uparrow & {\scriptstyle -t}\uparrow & {\scriptstyle r}\uparrow & & & & & & \\
A & B & C & & \ker s & \longrightarrow & \ker t & \longrightarrow & \ker r
\end{array}$$



$$\begin{array}{ccc}
E & M' & N' \\
L & M & F
\end{array}$$

So since the result of the second spectral sequence, we have $L = \ker f, M = M' = 0, N' = \mathrm{coker} h$ and

$L' \to N$ is an isomorphism since $E = F = 0$. So finally the first spectral sequence as

$$\begin{array}{ccc} X & Y & Z \\ {\scriptstyle s}\uparrow & {\scriptstyle -t}\uparrow & {\scriptstyle r}\uparrow \\ A & B & C \end{array} \qquad \begin{array}{ccccc} \operatorname{coker} s & \longrightarrow & \operatorname{coker} t & \longrightarrow & \operatorname{coker} r \\ \\ \ker s & \longrightarrow & \ker t & \longrightarrow & \ker r \end{array}$$

$$\begin{array}{ccc} L' & 0 & \operatorname{coker} h \\ & \searrow^{\cong} & \\ \ker f & 0 & N \end{array} \qquad \begin{array}{ccc} 0 & 0 & \operatorname{coker} h \\ \\ \ker f & 0 & 0 \end{array}$$

Since $L' \cong N$, we find that $\ker(\operatorname{coker} s \to \operatorname{coker} t) = \operatorname{coker}(\ker t \to \ker r)$ in the second diagram of the first spectral sequence. Similarly, we have $\ker(\ker s \to \ker t) = \ker f$ and $\operatorname{coker}(\operatorname{coker} s \to \operatorname{coker} t) = \operatorname{coker} h$ and $\ker s \to \ker t \to \ker r$ and $\operatorname{coker} s \to \operatorname{coker} t \to \operatorname{coker} r$ are exact. So we can combine them into a long exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker f & \longrightarrow & \ker s & \longrightarrow & \ker t & \longrightarrow & \ker r \\ & & & & & & & & \downarrow \\ 0 & \longleftarrow & \operatorname{coker} h & \longleftarrow & \operatorname{coker} r & \longleftarrow & \operatorname{coker} t & \longleftarrow & \operatorname{coker} s \end{array}$$

And we win. □

Example 3 (Balanced Tor and Ext)**.** *We works in modules categroy for some ring R (we can replace it to be some enough projective or injective abelian categroies). Let A, B be one of it and pick projective resolutions $P^* \to A \to 0$ and $Q^* \to B \to 0$. So we have a double complex $P^* \otimes Q^*$. Then we claim that $H^n(\operatorname{Tot}(P^* \otimes Q^*)) = \operatorname{Tor}_n^R(A, B)$.*

Actually we have the spectral sequence as following

$$\begin{array}{ccc} P_{i-1} \otimes Q_{i+1} & P_1 \otimes Q_{i+1} & P_{i+1} \otimes Q_{i+1} \\ \downarrow & \downarrow & \downarrow \\ P_{i-1} \otimes Q_i & P_i \otimes Q_i & P_{i+1} \otimes Q_i \\ \downarrow & \downarrow & \downarrow \\ P_{i-1} \otimes Q_{i-1} & P_i \otimes Q_{i-1} & P_{i+1} \otimes Q_{i-1} \end{array} \qquad \begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \\ P_{i-1} \otimes B & \to & P_i \otimes B & \to & P_{i+1} \otimes B \end{array}$$

$$\begin{array}{ccc} 0 & 0 & 0 \\ \\ 0 & 0 & 0 \\ & \nearrow & \\ \operatorname{Tor}_0^R(A, B) & \operatorname{Tor}_1^R(A, B) & \operatorname{Tor}_2^R(A, B) \end{array}$$

Then the claim is right. Ext is similar.

Example 4 (The Frölicher Spectral Sequence)**.** *Consider X be a compact complex manifold and let $(\mathscr{A}^{*,*}(X), \partial, \overline{\partial})$ be a double complex where $\mathscr{A}^{p,q}(X)$ be the space of p, q-forms of X. So the spectral sequence associated to $(\mathscr{A}^{*,*}(X), \partial, \overline{\partial})$ is called the Frölicher spectral sequence.*

This spectral sequence converges since it is bounded. So we can see that ${}'E_1^{p,q} = H^{p,q}(X)$ and it is converge to $H^*(\operatorname{Tot}(\mathscr{A}^{p,q}(X))) = H^*(\Omega^*(X)) = H^*(X, \mathbb{C})$. So the limit term ${}'E_\infty^{p,q} = \operatorname{gr}_{F_I}^p H^{p+q}(X, \mathbb{C})$.

In the case of compact Kähler manifold, the Frölicher spectral sequence degenerates at ${}'E_1$. This because we now have r such that ${}'E_\infty^{p,q} =' E_r^{p,q} = \operatorname{gr}_{F_I}^p H^{p+q}(X, \mathbb{C})$. Since $\dim {}'E_i^{p,q} \leq \dim {}'E_{i-1}^{p,q}$ and the equality holds if and only if $d_i = 0$ and since by the basic Hodge theory we know that $\dim {}'E_\infty^{p,q} = \dim {}'E_1^{p,q}$, so it is degenerates at ${}'E_1$.

Example 5 (Čech-de Rham Spectral Sequence)**.** *Let M be a manifold and $\mathfrak{U} = \{U_i\}$ be a good cover over M. Consider the double complex*

$$K^{p,q} = C^p(\mathfrak{U}, \Omega^q) = \prod_{i_0<\cdots<i_p} \Gamma(U_{i_0\cdots i_p}, \Omega^q)$$

as

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & \\
0 & \to & \Omega^1(M) & \overset{r}{\to} & \prod_{i_0} \Gamma(U_{i_0}, \Omega^1) & \overset{\delta}{\to} & \prod_{i_0<i_1} \Gamma(U_{i_0 i_1}, \Omega^1) & \overset{\delta}{\to} \cdots \\
 & & \uparrow d & & \uparrow d & & \uparrow d & \\
0 & \to & \Omega^0(M) & \overset{r}{\to} & \prod_{i_0} \Gamma(U_{i_0}, \Omega^0) & \overset{\delta}{\to} & \prod_{i_0<i_1} \Gamma(U_{i_0 i_1}, \Omega^0) & \overset{\delta}{\to} \cdots \\
 & & & & \uparrow i & & \uparrow i & \\
 & & & & C^0(\mathfrak{U}, \mathbb{R}) & \xrightarrow{\delta} & C^1(\mathfrak{U}, \mathbb{R}) & \xrightarrow{\delta} \cdots \\
 & & & & \uparrow & & \uparrow & \\
 & & & & 0 & & 0 &
\end{array}$$

where $U_{i_0\cdots i_p} = \bigcap_{j=i_0<\cdots<i_p} U_j$. In the diagram $r : \Omega^*(M) \to K^{*,0}$ are the restriction and $C^*(\mathfrak{U}, \mathbb{R})$ be the kernel of bottom d and i be the inclution. Moreover, d is the usual differential between differential forms and δ defined as

$$(\delta\omega)_{i_0\cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \omega_{i_0\cdots i_{j-1} j_{j+1}\cdots i_{p+1}}, \omega \in \prod_{i_0<\cdots<i_p} \Gamma(U_{i_0\cdots i_p}, \Omega^q).$$

We first get the second spectral sequence as

$$\begin{array}{ccc}
\vdots & \vdots & \\
\uparrow & \uparrow & \\
\prod_{i_0<i_1} \Gamma(U_{i_0 i_1}, \Omega^0) & \prod_{i_0<i_1} \Gamma(U_{i_0 i_1}, \Omega^1) & \cdots \\
\uparrow & \uparrow & \\
\prod_{i_0} \Gamma(U_{i_0}, \Omega^0) & \prod_{i_0} \Gamma(U_{i_0}, \Omega^1) & \cdots
\end{array}
\qquad
\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\Omega^0(M) & \to & \Omega^1(M) & \to & \cdots
\end{array}$$

$$\begin{array}{ccc}
0 & 0 & \cdots \\
 & \searrow & \\
H^0_{DR}(M) & H^1_{DR}(M) & \cdots
\end{array}$$

So $''E_\infty = {}''E_2$ for now. So we get the classical result that $H^n_{DR}(M) = \bigoplus_{p+q=n} H^{p,q}_D(C^*(\mathfrak{U}, \Omega^*))$ where $H^{p,q}_D(C^*(\mathfrak{U}, \Omega^*))$ as the cohomological group worked with $D = d + (-1)^p\delta$ and on $C^p(\mathfrak{U}, \Omega^q)$.

Now we consider the first spectral sequence as

$$\begin{array}{ccc}
\vdots & \vdots & \\
\uparrow & \uparrow & \\
\prod_{i_0} \Gamma(U_{i_0}, \Omega^1) & \prod_{i_0<i_1} \Gamma(U_{i_0 i_1}, \Omega^1) & \cdots \\
\uparrow & \uparrow & \\
\prod_{i_0} \Gamma(U_{i_0}, \Omega^0) & \prod_{i_0<i_1} \Gamma(U_{i_0 i_1}, \Omega^0) & \cdots
\end{array}
\qquad
\begin{array}{ccccc}
0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
C^0(\mathfrak{U}, \mathbb{R}) & \to & C^1(\mathfrak{U}, \mathbb{R}) & \to & \cdots
\end{array}$$

$$\begin{array}{ccccc}
0 & & 0 & & \cdots \\
 & \searrow & & \searrow & \\
H^0(\mathfrak{U}, \mathbb{R}) & & H^1(\mathfrak{U}, \mathbb{R}) & & \cdots
\end{array}$$

So $'E_\infty = {'E_2}$ *and* $H^k(\mathfrak{U}, \mathbb{R}) = H^k_D(C^*(\mathfrak{U}, \Omega^*))$. *In summary, we now have* $H^k(\mathfrak{U}, \mathbb{R}) = H^n_{DR}(M)$.

5 Grothendieck Spectral Sequence and It's Applications

5.1 Cartan-Eilenberg Resolutions

Consider a complex C^* and an injective resolution as follows

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & I^{n-1,1} & \xrightarrow{d_1^{n-1,1}} & I^{n,1} & \xrightarrow{d_1^{n,1}} & I^{n+1,1} & \longrightarrow & \cdots \\
 & & {\scriptstyle d_2^{n-1,0}}\uparrow & & {\scriptstyle d_1^{n,0}}\uparrow & & {\scriptstyle d_2^{n+1,0}}\uparrow & & \\
\cdots & \longrightarrow & I^{n-1,0} & \xrightarrow{d_1^{n-1,0}} & I^{n,0} & \xrightarrow{d_1^{n,0}} & I^{n+1,0} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & C^{n-1} & \xrightarrow{d^{n-1}} & C^{n} & \xrightarrow{d^{n}} & C^{n+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

where each $C^n \to I^{n,*}$ be an injective resolution and $I^{*,m}$ be complexes.

Now let $Z^p(C^*) = \ker d^p, B^p(C^*) = \mathrm{Im} d^{p+1}$ and $Z^p(I^{*,q}) = \ker d_1^{p,q}, B^p(I^{*,q}) = \mathrm{Im} d_1^{p+1,q}$. Then we have three complexes as following

$$0 \longrightarrow Z^p(C^*) \longrightarrow Z^p(I^{*,0}) \longrightarrow Z^p(I^{*,1}) \longrightarrow \cdots$$

$$0 \longrightarrow B^p(C^*) \longrightarrow B^p(I^{*,0}) \longrightarrow B^p(I^{*,1}) \longrightarrow \cdots$$

$$0 \longrightarrow H^p(C^*) \longrightarrow H^p(I^{*,0}) \longrightarrow H^p(I^{*,1}) \longrightarrow \cdots$$

Difinition 5.1. *We say the injective resolution* $C^* \to I^{*,*}$ *is a Cartan-Eilenberg Resolutions (also called fully injective resolution) if the previous three sequences are injective resolutions.*

Theorem 5.2. *Let* $\mathcal{A}$ *be an abelian category with enough injectives. Then every complex* C^* *admits a Cartain-Eilenberg resolution.*

Proof. Actually we can use the horseshoe lemma (See any books about homological algebra like [Rot]) twice at the following two exact sequences, respectively.

$$0 \longrightarrow B^n(C^*) \longrightarrow Z^n(C^*) \longrightarrow H^n(C^*) \longrightarrow 0$$

$$0 \longrightarrow Z^n(C^*) \longrightarrow C^n \longrightarrow B^{n+1}(C^*) \longrightarrow 0$$

Then we can combine it into a resolution of C^* by the universal property of injective objects. □

Remark 5.3. *Cartan-Eilenberg resolutions are in some sense the most correct type of injective resolutions and they are play an important role in the construction of the Grothendieck spectral sequence.*

5.2 Grothendieck Spectral Sequence

Theorem 5.4 (Grothendieck Spectral Sequence)**.** *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be additive functors between abelian categories where $\mathcal{A}, \mathcal{B}$ have enough injectives and $\mathcal{C}$ is cocomplete (that is, colimits always exist), and suppose that F sends injectives to G-acyclics. Then for any object $A \in \mathcal{A}$ there is a first quadrant spectral sequence E starting on page zero, such that*

$$E_2^{p,q} = R^pG(R^qF(A)) \Rightarrow R^{p+q}(G \circ F)(A).$$

Proof. Let the complex C^* be an injective resolution of A, and let the bicomplex $I^{*,*}$ be a Cartan-Eilenberg resolution of the complex FC^* with $I^{p,q} = 0$ unless $p, q \geq 0$.

Consider the double complex $GI^{*,*}$ and we have two first quadrant spectral sequences $'E_r,'' E_r$ associated to it both converging to the cohomology of $\mathrm{Tot}(GI^{*,*})$. Moreover, we have

$$'E_2^{p,q} = H_I^p(H_{II}^q(GI^{*,*})) = H^p(R^qG(FC^*)).$$

But C^* is a complex of injectives and F sends injectives to G-acyclics, so for $q > 0$ the complex $R^qG(FC^*) = 0$, and for $q = 0$ it is canonically isomorphic to GFC^*. So we have $'E_2$ page as

$$\begin{array}{ccccccc} \cdots & & 0 & & 0 & & 0 & & \cdots \\ & \searrow & & \searrow & & \searrow & & \searrow & \\ \cdots & & R^{p-1}(GF)(A) & & R^p(GF)(A) & & R^{p+1}(GF)(A) & & \cdots \end{array}$$

So $'E_2 = {'E_\infty}$. So we have

$$H^n(\mathrm{Tot}(GI^{*,*})) = {'E_\infty^{n,0}} \cong R^n(GF)(A).$$

Now we consider the second spectral sequence $''E_r$. We have the exact sequence

$$0 \longrightarrow Z^p(I^{*,q}) \longrightarrow I^{p,q} \longrightarrow B^{p+1}(I^{*,q}) \longrightarrow 0$$

Since $I^{*,*}$ be a Cartan-Eilenberg resolution of the complex FC^*, so $Z^p(I^{*,q})$ is an injective object. So this exact sequence is split. So

$$0 \longrightarrow GZ^p(I^{*,q}) \longrightarrow GI^{p,q} \longrightarrow GB^{p+1}(I^{*,q}) \longrightarrow 0$$

is split too since G is additive. So

$$GZ^p(I^{*,q}) \cong \ker(GI^{p,q} \to GB^{p+1}(I^{*,q})) = Z^p(GI^{*,q}), GB^{p+1}(I^{*,q}) = B^{p+1}(GI^{*,q}).$$

Consider another split exact sequence

$$0 \longrightarrow B^p(I^{*,q}) \longrightarrow Z^p(I^{*,q}) \longrightarrow H^p(I^{*,q}) \longrightarrow 0$$

So as the same reason, we have the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & GB^p(I^{*,q}) & \longrightarrow & GZ^p(I^{*,q}) & \longrightarrow & GH^p(I^{*,q}) & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \vdots & & \\ 0 & \longrightarrow & B^p(GI^{*,q}) & \longrightarrow & Z^p(GI^{*,q}) & \longrightarrow & H^p(GI^{*,q}) & \longrightarrow & 0 \end{array}$$

Use five-lemma, we have $GH^p(I^{*,q}) \cong H^p(GI^{*,q})$. For now we have

$$''E_2^{p,q} = H_{II}^p(H_I^q(GI^{*,*})) = H_{II}^p(GH_I^q(I^{*,*})).$$

We also have injective resolution $R^qF(A) = H^q(FC^*) \to H_I^q(I^{*,*})$, so $''E_2^{p,q} = R^pG(R^qF(A))$. So $''E_r$ is what we want. ◻

5.3 Some Applications

5.3.1 The Leray Spectral Sequence

Since we know that the category $\mathfrak{AbSh}_X$ of abelian sheaves on X has enough injectives, we can define the higher direct image. Let $f : X \to Y$ is continuous and $\mathscr{F} \in \mathfrak{AbSh}_X$, we can choose an injective resolution $\mathscr{F}[0] \to \mathscr{I}^*$. Then the higher direct image $R^i f_* \mathscr{F} = H^i(f_* \mathscr{I}^*)$.

Lemma 5.5. *The functor $f_* : \mathfrak{AbSh}_X \to \mathfrak{AbSh}_Y$ sends injective sheaves to flabby sheaves.*

Proof. Actually we can prove that $f_* : \mathfrak{AbSh}_X \to \mathfrak{AbSh}_Y$ sends injective sheaves to injective sheaves. Consider the exact sequence $0 \to \mathscr{A} \to \mathscr{B}$ on Y, then we have exact $0 \to f^{-1}\mathscr{A} \to f^{-1}\mathscr{B}$ on X. See the following diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathfrak{AbSh}_X}(f^{-1}\mathscr{B}, \mathscr{I}) & \longrightarrow & \mathrm{Hom}_{\mathfrak{AbSh}_X}(f^{-1}\mathscr{A}, \mathscr{I}) & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \\ \mathrm{Hom}_{\mathfrak{AbSh}_Y}(\mathscr{B}, f_*\mathscr{I}) & \longrightarrow & \mathrm{Hom}_{\mathfrak{AbSh}_Y}(\mathscr{A}, f_*\mathscr{I}) & \longrightarrow & 0 \end{array}$$

by the adjointness of (f^{-1}, f_*). Well done. □

Example 6 (The Leray Spectral Sequence)**.** *In the case of Grothendieck spectral sequence, we let $\mathcal{A} = \mathfrak{AbSh}_X, \mathcal{B} = \mathfrak{AbSh}_Y, \mathcal{C} = \mathfrak{Ab}$. Let $F = f_*, G = \Gamma(X, -)$. Then there exists a spectral sequence such that $E_2^{p,q} = H^p(Y, R^q f_* \mathscr{F}) \Rightarrow H^{p+q}(X, \mathscr{F})$. This spectral sequence is called Leray spectral sequence.*

Remark 5.6. *It shows how we can approximate the sheaf cohomology on X by looking at the sheaf cohomology on Y with respect to the higher direct images.*

5.3.2 The Čech-to-derived Functor Spectral Sequence

Let X be a topological space and let $\mathscr{F}$ be a sheaf on X. Let $\mathfrak{U} = \{U_i\}$ be an open cover of X. Let $\mathscr{H}^q(X, \mathscr{F})$ be the presheaf with $U \mapsto H^q(U, \mathscr{F})$. For any presheaf $\mathscr{P}$ we define the Čech cohomology $\check{H}^p(\mathfrak{U}, \mathscr{P})$ is cohomology repect to the following complex

$$\textstyle\prod_{i_0} \mathscr{P}(U_{i_0}) \longrightarrow \prod_{i_0<i_1} \mathscr{P}(U_{i_0 i_1}) \longrightarrow \cdots \longrightarrow \prod_{i_0<\cdots<i_k} \mathscr{P}(U_{i_0\cdots i_k}) \longrightarrow \cdots$$

where the map as in the case of Čech-de Rham spectral sequence.

Example 7 (The Čech-to-derived functor spectral sequence)**.** *In the case of Grothendieck spectral sequence, we let $\mathcal{A} = \mathfrak{AbSh}_X, \mathcal{B} = \mathfrak{AbPSh}_X, \mathcal{C} = \mathfrak{Ab}$. Let $F = \iota, G = \check{H}^0(\mathfrak{U}, -)$. Then there exists a spectral sequence such that $E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathscr{H}^q(X, \mathscr{F})) \Rightarrow H^{p+q}(X, \mathscr{F})$. This spectral sequence is called the Čech-to-derived functor spectral sequence.*

Example 8 (Application: Affine open cover of a scheme)**.** *Let X is a quasi-compact and separated scheme. So we can take a finite affine open cover $\mathfrak{U} = \{U_i\}$. Take $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Since X is separated, $U_{i_0\cdots i_k}$ are all affine open. By the Serre theorem, we have $H^i(U_{i_0\cdots i_k}, \mathscr{F}) = 0, i > 0$. So $E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathscr{H}^q(X, \mathscr{F})) = 0, q > 0$. So the E_2 page of it is*

$$\begin{array}{ccccc} \cdots & 0 & 0 & 0 & \cdots \\ \cdots & \check{H}^{n-1}(\mathfrak{U}, \mathscr{F}) & \check{H}^{n}(\mathfrak{U}, \mathscr{F}) & \check{H}^{n+1}(\mathfrak{U}, \mathscr{F}) & \cdots \end{array}$$

So we have $\check{H}^n(\mathfrak{U}, \mathscr{F}) \cong H^n(X, \mathscr{F})$.

5.3.3 The Local-to-Global Ext Spectral Sequence

Now we consider a ringed space $(X, \mathscr{O}_X)$ and for open $U \subset X$ we define a sheaf

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})(U) = \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U).$$

Similarly, since $\mathfrak{Mod}_X$ has enough injective objects, we can define $\mathscr{E}xt^p_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ as the sheafification of the derived functors of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$.

Lemma 5.7. *Let $(X, \mathscr{O}_X)$ is a ringed space.*
(1) If $\mathscr{F} \in \mathfrak{Mod}_X$ is flat and $\mathscr{I} \in \mathfrak{Mod}_X$ is injective, then $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I})$ is injective.
(2) If $\mathscr{F}, \mathscr{I} \in \mathfrak{Mod}_X$ with $\mathscr{I}$ is injective, then $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I})$ is $\Gamma(X, -)$-acyclic.

Proof. (1) Use the fact that $\mathrm{Hom}_{\mathscr{O}_X}(-, \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I})) = \mathrm{Hom}_{\mathscr{O}_X}(- \otimes_{\mathscr{O}_X} \mathscr{F}, \mathscr{I})$ and both two sides are exact. Well done.

(2) We can find an exact sequence $0 \to \mathscr{X} \to \mathscr{Y} \to \mathscr{F} \to 0$ where $\mathscr{Y}$ is flat (See [DM2], Lemma 47). Then we get the following exact sequence

$$0 \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{X}, \mathscr{I}) \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{Y}, \mathscr{I}) \longrightarrow \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{I}) \longrightarrow 0$$

By (1) the middle term is injective, then the use the long exact sequence we could have the conclusion. □

Example 9 (The Local-to-Global Ext Spectral Sequence)**.** *Let $(X, \mathscr{O}_X)$ is a ringed space and let $\mathscr{F}, \mathscr{G}$ be the sheaf of $\mathscr{O}_X$-modules. In the case of Grothendieck spectral sequence, we let $\mathcal{A} = \mathcal{B} = \mathfrak{Mod}_X$ and $\mathcal{C} = \mathfrak{Ab}$. Let $F = \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, -)$ and $G = \Gamma(X, -)$. Then we have a spectral sequence such that*

$$E_2^{p,q} = H^p(X, \mathscr{E}xt^q_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})) \Rightarrow \mathrm{Ext}^{p+q}(\mathscr{F}, \mathscr{G}).$$

References

[BT] Raoul Bott, Loring W. Tu, *Differential Forms in Algebraic Topology*, Springer, 1982.

[CV] Claire Voisin, *Hodge Theory and Complex Algebraic Geometry, I*, Cambridge, 2002.

[DM1] Daniel Murfet, *Spectral Sequences*, `http://therisingsea.org/notes/SpectralSequences.pdf`, 2006.

[DM2] Daniel Murfet, *Derived Categories of Sheaves*, `http://therisingsea.org/notes/DerivedCategoriesOfSheaves.pdf`, 2006.

[JPD] Jean-Pierre Demailly, *Complex Analytic and Differential Geometry*, `https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`, 2012.

[Rot] Joseph J. Rotman, *An Introduction to Homological Algebra, Second Edition*, Springer, 2009.

[St] Stacks project collaborators, *Stacks project*, `https://stacks.math.columbia.edu/`.

[Wb] Charles A. Weibel, *An Introduction to Homological Algebra*, Cambridge, 1994.

[XR] Xiong Rui, *Spectral Sequence, My Homological Saw*, `https://www.cnblogs.com/XiongRuiMath/p/14992978.html`, 2021.