Notes on Birational Geometry

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Contents

1 Introduction

First you need to read [9]. For basic things we refer [10] and [3].

2 Preliminaries

2.1 Basic Notations and Definitions

Definition 2.1. *A scheme will assume to a scheme of finite type over a field k where k will be a algebraically closed field of characteristic 0. A veriety will assume to a integral scheme.*

Definition 2.2 (Basic notations and definitions)**.** *For convenience, we summary some definitions and standard notations here. Let X be a normal variety.*

- *A prime divisor P on X is a codimension 1 irreducible and reduced subvariety of X. The group of Weil divisors on X, $\mathrm{WDiv}(X)$ is the torsion free $\mathbb{Z}$-module given by the set of all formal linear combinations $D = \sum d_i P_i$ of prime divisors on X with integral coefficients. We let $\mathrm{WDiv}_{\mathbb{Q}}(X) := \mathrm{WDiv}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{WDiv}_{\mathbb{R}}(X) := \mathrm{WDiv}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.*

- *A divisor $D \in \mathrm{WDiv}_{\mathbb{R}}(X)$ is effective, i.e., $D \geq 0$ if $D = \sum d_i D_i$ and $d_i \geq 0$ for all i. If $D = \sum d_i D_i \in \mathrm{WDiv}_{\mathbb{R}}(X)$, then we define $\|D\| = \sup\{|d_i|\}$.*

- *The support of a divisor $D = \sum d_i D_i \in \mathrm{WDiv}_{\mathbb{R}}(X)$ is $\mathrm{supp}(D) := \bigcup_{d_i \neq 0} D_i \subset X$.*

- *A divisor $D = \sum a_i D_i \in \mathrm{WDiv}_{\mathbb{R}}(X)$ is called reduced if $\mathrm{coeff}_D D_i \in \{0, 1\}$ for every D_i.*

- *A principal divisor is a divisor of the form $D = (f)$ where $f \in K(X)$ is a rational function on X and (f) is the divisor given by the difference between the zeroes and poles of f.*

- *For any $D \in \mathrm{WDiv}(X)$ the associated Weil divisorial sheaf is the $\mathscr{O}_X$-module $\mathscr{O}_X(D)$ defined by*
$$\Gamma(U, \mathscr{O}_X(D)) := \{D + (f) \geq 0, f \in K(X)\}.$$

- *A divisor $D \in \mathrm{WDiv}(X)$ is Cartier if it is locally principal and let $\mathrm{Div}(X) \subset \mathrm{WDiv}(X)$ be their group.*

- *We let $\mathrm{Div}_{\mathbb{Q}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{Div}_{\mathbb{R}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.*

- *If $\mathrm{Div}(X) = \mathrm{WDiv}(X)$, then we say that X is factorial. If $\mathrm{Div}_{\mathbb{Q}}(X) = \mathrm{WDiv}_{\mathbb{Q}}(X)$, then we say that X is $\mathbb{Q}$-factorial. If $D \in \mathrm{Div}(X)$, then $\mathscr{O}_X(D)$ is invertible.*

- *Two divisors $D_1, D_2 \in \mathrm{WDiv}(X)$ are* linearly equivalent *$D_1 \sim D_2$ if $D_1 - D_2 = (f)$ for some $f \in K(X)$. Note that $D_1 \sim D_2$ if and only if $\mathscr{O}_X(D_1) \cong \mathscr{O}_X(D_2)$.*

- *Two divisors $D_1, D_2 \in \mathrm{WDiv}_{\mathbb{Q}}(X)$ are* $\mathbb{Q}$-linearly equivalent *$D_1 \sim_{\mathbb{Q}} D_2$ if $D_1 - D_2 = \sum a_i(f_i)$ for some $f_i \in K(X)$ and $a_i \in \mathbb{Q}$. Two divisors $D_1, D_2 \in \mathrm{WDiv}_{\mathbb{R}}(X)$ are* $\mathbb{R}$-linearly equivalent *$D_1 \sim_{\mathbb{R}} D_2$ if $D_1 - D_2 = \sum a_i(f_i)$ for some $f_i \in K(X)$ and $a_i \in \mathbb{R}$.*

- *Let* complete linear series *of D is $|D| = \{D' \geq 0 : D' \sim D\} \cong \mathbb{P}H^0(X, \mathscr{O}_X(D))$. The* linear series *is a subspace of $|D|$.*

- *If $D \in \mathrm{WDiv}(X)$ and $V \subset |D|$ is a linear series, then the* base locus *of V is*

$$\mathrm{Bs}(V) := \bigcap_{C \in V} \mathrm{supp}(C).$$

  If $\mathrm{Bs}(V) = \emptyset$, we say that V is basepoint-free.

- *Two divisor $D_1, D_2 \in \mathrm{Div}_{\mathbb{R}}(X)$ are* numerically equivalent *$D_1 \equiv D_2$ if for any curve C we have $(D_1 - D_2) \cdot C = 0$.*

- *The* Néron-Severi group *of X is $N^1(X) = \mathrm{Div}(X)/\equiv$ is a free abelian group of finite rank. The rank of this group is $\varrho(X)$, the* Picard number *of X.*

- *Let $D = \sum d_i D_i \in \mathrm{WDiv}_{\mathbb{R}}(X)$, then we define $\lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$ and $\lceil D \rceil = \sum \lceil d_i \rceil D_i$. Let $D = \sum d_i D_i, D' = \sum d_i' D_i \in \mathrm{WDiv}_{\mathbb{R}}(X)$, then $D \wedge D' = \sum \min\{d_i, d_i'\} D_i$.*

- *Let $f : Y \to X$ be a birational morphism, then $\mathrm{Exc}(f)$ be the exceptional locus. Let $D \subset X$, then we let $f_*^{-1}D$ be the strict transformation of D.*

Definition 2.3 (Some relative version)**.** *Let $f : X \to Z$ be a proper morphism of normal varieties.*

- *Let $D_i \in \mathrm{WDiv}_{\mathbb{R}}(X)$, then they are* $\mathbb{R}$-linear equivalent over *Z, i.e. $D_1 \sim_{\mathbb{R},Z} D_2$ if $D_1 - D_2 = \sum r_i(f_i) + f^*C$ where $r_i \in \mathbb{R}$, $f_i \in K(X)$ and C be a $\mathbb{R}$-Cartier divisor on Z.*

- *Let $D_i \in \mathrm{Div}_{\mathbb{R}}(X)$, then they are* numerically equivalent over *Z, i.e. $D_1 \equiv_Z D_2$ if $(D_1 - D_2) \cdot C = 0$ where $r_i \in \mathbb{R}$, $f_i \in K(X)$ and C be a $\mathbb{R}$-Cartier divisor on Z.*

- Real linear series *over Z associated to a $\mathbb{R}$-divisor D on X is*

$$|D/Z|_{\mathbb{R}} := \{D' \geq 0 : D' \sim_{\mathbb{R},Z} D\}.$$

  The stable base locus *of D over Z is*

$$\mathbf{B}(D/Z) := \bigcap_{C \in |D/Z|_{\mathbb{R}}} \mathrm{supp}(C).$$

2.2 Reflexive sheaves

Here we select some foundamental results of reflexive sheaves (see Tag 0AVT and Tag 0EBK).

Definition 2.4. *Here we work over the general schemes.*

(i) Let X be an integral locally Noetherian scheme. Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. The reflexive hull *of $\mathscr{F}$ is the $\mathscr{O}_X$-module*

$$\mathscr{F}^{**} := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X), \mathscr{O}_X).$$

And $\mathscr{F}$ is called reflexive *if the canonical map $j : \mathscr{F} \to \mathscr{F}^{**}$ is an isomorphism.*

(ii) Define the m-th reflexive power *is $\mathscr{F}^{[m]} := (\mathscr{F}^{\otimes m})^{**}$ and* reflexive tensor product *is $\mathscr{F} \otimes_{\mathrm{Rf},\mathscr{O}_X} \mathscr{G} := (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})^{**}$.*

(iii) Let $\mathscr{L}$ be a rank 1 reflexive sheaf over X, then the index *of it defined as the smallest $m > 0$ such that $\mathscr{L}^{[m]}$ is invertible.*

Proposition 2.5. *Fix X be an integral locally Noetherian scheme.*

(i) Reflexiveness of coherent sheaves can checking on the stalks;

(ii) reflexiveness of coherent sheaves stable under pullback along a flat morphism;

(iii) reflexive coherent sheaves are torsion free;

(iv) if X is normal, then $\mathscr{F}$ is reflexive if and only if $\mathscr{F}$ is torsion free and S_2, if and only if there exists an open subscheme $j : U \to X$ such that $\operatorname{codim}_X(X \backslash U) \geq 2$ and $j^*\mathscr{F}$ is finite locally free and $\mathscr{F} = j_* j^* \mathscr{F}$.

Proof. See Tag 0AY2, Tag 0AY3, Tag 0EBF and Tag 0AY6. □

Lemma 2.6. *Let X be an integral locally Noetherian normal scheme. For $\mathscr{F}$ and $\mathscr{G}$ coherent reflexive $\mathscr{O}_X$-modules the map*

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{O}_X) \otimes_{\mathrm{Rf},\mathscr{O}_X} \mathscr{G} \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$$

is an isomorphism. The group law $(\mathscr{F}, \mathscr{G}) \mapsto \mathscr{F} \otimes_{\mathrm{Rf},\mathscr{O}_X} \mathscr{G}$ defines an abelian group law on the set of isomorphism classes of rank 1 coherent reflexive $\mathscr{O}_X$-modules.

Proof. See Tag 0EBL. □

Theorem 2.7. *Let X be an integral locally Noetherian normal scheme. The Weil divisor class group $\mathrm{Cl}(X)$ is isomorphic to the group of rank 1 coherent reflexive $\mathscr{O}_X$-modules by $D \mapsto \mathscr{O}_X(D)$.*

Proof. See Tag 0EBM. □

Remark 2.8. *Hence $D_1 + D_2$ correspond to $\mathscr{O}_X(D_1 + D_2) = \mathscr{O}_X(D_1) \otimes_{\mathrm{Rf},\mathscr{O}_X} \mathscr{O}_X(D_2)$ and $\mathscr{O}_X(mD) = \mathscr{O}_X(D)^{[m]}$.*

Some times we may using this lemma:

Lemma 2.9. *Let R be a Noetherian domain. Let $f : M \to N$ be a map of R-modules. Assume M is finite, N is torsion free, and that for every prime $\mathfrak{p}$ of R one of the following happens*

(a) $M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ s an isomorphism, or

(b) $\operatorname{depth}(M_{\mathfrak{p}}) \geq 2$,

then f is an isomorphism.

Proof. See Tag 0AV9. □

2.3 Cone of curves and contractions

Definition 2.10. *Let $k = \mathbb{Q}$ or $\mathbb{R}$ and V is a K-vector space. A cone $N \subset V$ is a subset with $0 \in N$ and closed under closed under multiplication by positive scalars.*

A subcone $M \subset N$ is called extremal if $u, v \in N, u + v \in M$ imply that $u, v \in M$. M is also called an extremal face of N. A 1-dimensional extremal subcone is called an extremal ray.

Definition 2.11. *Let X be a variety with proper $f : X \to Z$.*

- *Let $N_1(X/Z)$ be a group of 1-cycle contracted by f is a formal linear combination of integral proper curves contracted by f, up to the numerically equivalent, as a $\mathbb{R}$-vector space.*
- *Let $\operatorname{NE}(X/Z) := \{\sum a_i[C_i] : d_i \in \mathbb{R}_{\geq 0}\} \subset N_1(X/Z)$ and let $\overline{\operatorname{NE}}(X/Z)$ be its closure.*
- *For a divisor $D \in \operatorname{Div}_{\mathbb{R}}$, set $D_{\geq 0} := \{x \in N_1(X) : x \cdot D \geq 0\}$ (and for $>, <, \leq$) and $D^{\perp} = \{x : x \cdot D = 0\}$. Let $\overline{\operatorname{NE}}(X/Z)_{D\geq 0} := \overline{\operatorname{NE}}(X/Z) \cap D_{\geq 0}$ (and for $>, <, \leq$).*
- *The relative Néron-Severi group $N^1(X/Z)_{\mathbb{R}} := \operatorname{Div}_{\mathbb{R}}(X)/\equiv_Z$.*

2.4 Kodaria Dimension

Definition 2.12. *Let $D \in \operatorname{WDiv}_{\mathbb{R}}(X)$ on a normal projective variety X, define the Kodaira dimension of D as the largest integral $\kappa(D)$ such that*

$$\limsup_{m\to\infty} \frac{h^0(X, \lfloor mD \rfloor)}{m^{\kappa(D)}} > 0$$

if $h^0(X, \lfloor mD \rfloor) \neq 0$ for $m > 0$, otherwise let $\kappa(D) = \infty$. We define the Kodaira dimension of X is $\kappa(X) := \kappa(K_X)$.

Proposition 2.13. *Let $D \in \operatorname{WDiv}_{\mathbb{Q}}(X)$ on a normal projective variety X.*

(1) $\kappa(D) = \kappa(aD)$ for any $a \in \mathbb{Q}_{>0}$, and

(2) $\kappa(D) = \kappa(D')$ if $D \sim_{\mathbb{Q}} D'$.

Proof. (1) Just need to show the case that aD is integral where $a \in \mathbb{Z}_{>0}$. First we know that $\kappa(aD) \leq \kappa(D)$. Conversely, if $\kappa(D) = \infty$, then so is aD. So let $\kappa(D) \geq 0$. If $h^0(X, \lfloor mD \rfloor) = 0$ for some $m > 0$, then $h^0(X, \lfloor maD \rfloor) \geq h^0(X, \lfloor mD \rfloor)$. If $h^0(X, \lfloor mD \rfloor) \neq 0$ for some $m > 0$, then $\lfloor mD \rfloor \sim G$ for some integral $G \geq 0$. Hence $mD \sim G + \{mD\}$ and

$$\lfloor maD \rfloor = maD \sim aG + a\{mD\}$$

and then $a\{mD\}$ is effective and integral. Hence

$$h^0(X, \lfloor maD \rfloor) \geq h^0(X, aG) \geq h^0(X, G) = h^0(X, \lfloor mD \rfloor).$$

Hence $\kappa(aD) \geq \kappa(D)$. Well done.

(2) Trivial by (1). □

Proposition 2.14. *Let $f : Y \to X$ be a contraction of normal projective varieties and D be a $\mathbb{Q}$-Cartier divisor on X, then $\kappa(D) = \kappa(f^*D)$.*

Proof. By Proposition 2.13 (1) we may let D is Cartier. Hence as f is a contraction we get $f_*\mathscr{O}_Y = \mathscr{O}_X$. Hence $f_*f^*\mathscr{O}_X(D) = f_*\mathscr{O}_Y \otimes \mathscr{O}_X(D) = \mathscr{O}_X(D)$. Hence

$$H^0(Y, f^*\mathscr{O}_X(D)) = H^0(X, f_*f^*\mathscr{O}_X(D)) = H^0(X, \mathscr{O}_X(D))$$

and then $\kappa(D) = \kappa(f^*D)$. □

Corollary 2.15. *If H is ample, then $\kappa(H) = \dim X$. Moreover, for any $D \in \mathrm{WDiv}_{\mathbb{Q}}(X)$ we have $\kappa(D) \leq \dim X$.*

Proof. By Proposition 2.13 (1) we may let H is very ample. By

$$0 \to \mathscr{O}_X((m-1)H) \to \mathscr{O}_X(mH) \to \mathscr{O}_H(mH) \to 0$$

and Serre's vanishing theorem and induction, well done. □

Theorem 2.16 (Iitaka fibration)**.** *Let D be a $\mathbb{Q}$-Cartier divisor on a normal projective variety X with $\kappa(D) \geq 0$. Then there are projective morphisms $f : W \to X$ and $g : W \to Z$ from a smooth W such that*

- *f is birational and g is a contraction;*
- *$\kappa(D) = \dim Z$;*
- *if V is the generic fiber of g, then $\kappa(f^*D|_V) = 0$.*

Proof. Omitted, see Theorem 2.1.33 in [12]. □

Remark 2.17. *We can define the Kodaira dimension of non-normal varieties using normalization (see section 2.1 in [12]), but we does not use it any more.*

2.5 Some Fundamental Results of Positivity

Definition 2.18 (Absolute version)**.** *Let X be a normal variety with $D \in \mathrm{WDiv}_{\mathbb{R}}(X)$.*

- *$D \in \mathrm{WDiv}(X)$ is called very ample if $|D|$ induce a embedding;*
- *$D \in \mathrm{Div}_{\mathbb{R}}(X)$ is called ample if $D = \sum a_i D_i$ where $a_i > 0$ and D_i ample;*
- *$D \in \mathrm{Div}_{\mathbb{R}}(X)$ is called nef if $D \cdot C \geq 0$ for any curve C;*
- *D is called big if $D = \sum a_i D_i$ where $a_i > 0$ and D_i are integral big divisors. An integral divisor D called big is $\kappa(D) = \dim X$;*
- *$D \in \mathrm{Div}_{\mathbb{R}}(X)$ is called pseudo-effective if it is in the closure of the big-locus in $N^1(X)$;*
- *$D \in \mathrm{WDiv}(X)$ is called free if $\mathrm{Bs}(D) \neq \emptyset$;*
- *$D \in \mathrm{Div}_{\mathbb{R}}(X)$ is called semiample if $D \sim_{\mathbb{R}} g^*H$ for some projective morphism $g : X \to Y$ and H is ample on Y.*

Definition 2.19 (Relative version)**.** *Let $f : X \to Z$ be a projective morphism from a normal variety with $D \in \mathrm{WDiv}_{\mathbb{R}}(X)$.*

- *$D \in \mathrm{Div}(X)$ is called f-very ample if there is an embedding $i : X \to \mathbb{P}_Z(\mathscr{F})$ for some coherent sheaf $\mathscr{F}$ with $\mathscr{O}_X(D) \cong i^*\mathscr{O}_{\mathbb{P}_Z(\mathscr{F})}(1)$;*
- *$D \in \mathrm{Div}_{\mathbb{R}}(X)$ is called f-ample if for any $p \in Z$ there is an affine neighborhood U such that D is ample on $f^{-1}(U)$;*
- *$D \in \mathrm{Div}_{\mathbb{R}}(X)$ is called f-nef if $D \cdot C \geq 0$ for any curve C contracted by f;*
- *$D \in \mathrm{Div}_{\mathbb{R}}(X)$ is called f-big if*

  $$\limsup_{m \to \infty} \frac{\mathrm{rank} f_*\mathscr{O}_X(\lfloor mD \rfloor)}{m^{\dim f}} > 0$$

  where $\dim f$ is the dimension of the generic fiber;
- *$D \in \mathrm{Div}_{\mathbb{R}}(X)$ is called f-pseudo-effective if it is in the closure of the big-locus in $N^1(X/Z)$;*
- *$D \in \mathrm{WDiv}(X)$ is called f-free if $f^*f_*\mathscr{O}_X(D) \to \mathscr{O}_X(D)$ is surjective;*
- *$D \in \mathrm{Div}_{\mathbb{R}}(X)$ is called f-semiample if $D \sim_{\mathbb{R},Z} g^*H$ for some projective morphism $g : X \to Y$ over Z and H is ample over Z in Y.*

Here we will give several criterions of several positivity properties which is very important in birational geometry and higher dimensional algebraic geometry.

Lemma 2.20 (Asymptotic Riemann-Roch Theorem)**.** *Let X be a proper scheme of dimension n and $D_1, ..., D_r \in \mathrm{Div}(X)$ with coherent sheaf $\mathscr{F}$, then*

$$\chi(X, \mathscr{F}(k_1D_1 + ... + k_rD_r)) = \mathrm{rank}(\mathscr{F})\frac{(k_1D_1 + ... + k_rD_r)^n}{n!} + \textit{lower degree terms.}$$

Proof. We just prove the case of proper smooth case. Using the Hirzebruch-Riemann-Roch Theorem, we get

$$\chi(X, \mathscr{F}(k_1D_1 + ... + k_rD_r)) = \int_X \mathrm{ch}(\mathscr{F}(k_1D_1 + ... + k_rD_r)) \cdot \mathrm{td}(T_X) \cap [X]$$
$$= \int_X \mathrm{ch}(\mathscr{F})\mathrm{ch}(k_1D_1 + ... + k_rD_r) \cdot \mathrm{td}(T_X) \cap [X]$$
$$= \mathrm{rank}(\mathscr{F})\frac{c_1(k_1D_1 + ... + k_rD_r)^n}{n!} + \text{lower degree terms.}$$

For the general case, we refer [8] Corollary 18.3.11. □

Theorem 2.21 (Triditional cohomology criterions of amplitude)**.** *Let $\mathscr{L}$ be a line bundle over a proper scheme X, then the following are equivalent:*

(i) $\mathscr{L}$ is ample;

(ii) for any coherent sheaf $\mathscr{F}$ on X, there exists a positive number $M = M(\mathscr{F})$ such that $H^i(X, \mathscr{F} \otimes \mathscr{L}^{\otimes m}) = 0$ for all $i > 0$ and $m > M$;

(iii) for any coherent sheaf $\mathscr{F}$ on X, there exists a positive number $M' = M'(\mathscr{F})$ such that $\mathscr{F} \otimes \mathscr{L}^{\otimes m}$ generated by global sections for any $m > M'$;

(iv) there exists $M'' > 0$ such that $\mathscr{L}^{\otimes m}$ is very ample for every $m > M''$.

Proof. This is classical, see [9] Proposition III.5.3. □

Theorem 2.22 (Absolute case for amplitude/nefness)**.** *Let X be a proper scheme and $D \in \mathrm{Div}_{\mathbb{R}}(X)$, then*

(i)[Nakai-Moishezon-Fujino-Miyamoto] if D correspond to some $\mathbb{R}$-line bundle $\mathscr{L}$ (e.g. X is a normal variety), then D is ample if and only if $\mathscr{L}^{\dim V} \cdot V = D^{\dim V} \cdot V > 0$ for every positive-dimensional closed integral subscheme $V \subset X$;

(ii)[Nakai-Moishezon] if X is projective or $D \in \mathrm{Div}_{\mathbb{Q}}(X)$, then D is ample if and only if $D^{\dim V} \cdot V > 0$ for every positive-dimensional closed integral subscheme $V \subset X$;

(iii)[Kleiman] if X is projective, then D is ample if and only if $\overline{\mathrm{NE}}(X)\backslash\{0\} \subset D_{>0}$;

(iv)[Kleiman] if D is nef, then $D^{\dim V} \cdot V \geq 0$ for every positive-dimensional closed integral subscheme $V \subset X$.

Proof. (i) See Theorem 1.3/1.4 in [7] for details, I will omit it; (ii) the case $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ is trivial by the integral case; the case X is projective see [2] Theorem 2.3.18. The integral case we refer [10] Theorem 1.37; (iii) We refer [12] Theorem 1.4.29 for details.

Note that (iii) is not right if X is proper but not projective and we have a counter-example;

(iv) For the integral case we just give a sketch (for details see [4] Theorem 1.26). Using Chow's lemma and induction, we just need to prove $D^n \geq 0$ for projective X. Let H be an ample divisor on X and set $D_t = D + tH$. Consider the polynomial $P(t) = (D_t)^n$ and we need to show $P(0) \geq 0$. Assume the contrary and since the leading coefficient of P is positive, it has a largest positive real root t_0 and $P(t) > 0$ for $t > t_0$. For every subvariety $Y \subset X$ of positive dimension $r < n$, $D|_Y$ is nef. Easy to show that $D^s \cdot H^{r-s} \cdot Y \geq 0$. Hence $D_t^r \cdot Y > 0$ for $t > 0$. Then use this we can show that $D^r \cdot H^{n-r} \geq 0$.

Let $Q(t) = D_t^{n-1} \cdot D$ and $R(t) = tD_t^{n-1} \cdot H$. By Nakai-Moishezon (ii) we get D_t is ample for t rational and $t > t_0$, we can show that $Q(t) > 0$ for all $t > t_0$. By $D^r \cdot H^{n-r} \geq 0$ and this we get $R(t_0) \geq H^n t_0^n > 0$. Hence $0 = P(t_0) \geq R(t_0) > 0$ which is impossible. □

Theorem 2.23 (Triditional cohomology criterions of relative amplitude)**.** *Let $\mathscr{L}$ be a line bundle over scheme X and $f : X \to Y$ is proper, then the following are equivalent:*

(i) $\mathscr{L}$ is f-ample;

(ii) for any coherent sheaf $\mathscr{F}$ on X, there exists a positive number $M = M(\mathscr{F})$ such that $R^i f_*(X, \mathscr{F} \otimes \mathscr{L}^{\otimes m}) = 0$ for all $i > 0$ and $m > M$;

(iii) for any coherent sheaf $\mathscr{F}$ on X, there exists a positive number $M' = M'(\mathscr{F})$ such that $\mathscr{F} \otimes \mathscr{L}^{\otimes m}$ generated by global sections for any $m > M'$;

(iv) there exists $M'' > 0$ such that $\mathscr{L}^{\otimes m}$ is f-very ample for every $m > M''$.

Proof. See [12] Theorem 1.7.6. □

Theorem 2.24 (Relative case for amplitude/nefness)**.** *Let $f : X \to Y$ be a proper morphism and $D \in \mathrm{Div}_{\mathbb{R}}(X)$, then*

(i) if D if a $\mathbb{R}$-line bundle, then D is f-ample if and only if $D^{\dim V} \cdot V > 0$ for every positive-dimensional closed integral subscheme $V \subset X$ such that $f(V)$ is a point;

(ii) if f is projective, then $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is f-ample if and only if $\overline{\mathrm{NE}}(X/Y) \backslash \{0\} \subset D_{>0}$;

(iii) if $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is f-nef, then $D^{\dim V} \cdot V \geq 0$ for every positive-dimensional closed integral subscheme $V \subset X$ such that $f(V)$ is a point.

Proof. See [1] for the proofs. □

Proposition 2.25 (Cone-relations)**.** *Let $f : X \to Y$ be a projective morphism, then*

(i) $\mathrm{Nef}(X/Y) = \overline{\mathrm{Amp}(X/Y)}$ *and* $\mathrm{Amp}(X/Y) = \mathrm{int}(\mathrm{Nef}(X/Y))$*;*

(ii) $\overline{\mathrm{NE}}(X/Y)$ *and* $N_1(X/Y)$ *are dual, that is,*

$$\overline{\mathrm{NE}}(X/Y) = \{\gamma \in N_1(X)_{\mathbb{R}} : \delta \cdot \gamma \geq 0 \textit{ for all } \delta \in \mathrm{Nef}(X/Y)\}.$$

Proof. For the absolute case, we refer [12] Theorem 1.4.23 and Proposition 1.4.28. For the relative case, one can see [1] Theorem V.23. □

Theorem 2.26 (Semi-amplitude)**.** *Let $f : X \to Y$ be a proper morphism of schemes. Let $D \in \mathrm{Div}(X)$ then D is f-semiample if and only if there exists $m > 0$ such that $f^*f_*\mathscr{O}_X(D) \to \mathscr{O}_X(D)$ is surjective.*

Proof. **Need to check and need to add.** □

Lemma 2.27 (Kodaira Lemma I)**.** *Let X be a proper normal variety and let $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is big. Let $M \in \mathrm{Div}_{\mathbb{Q}}(X)$, then there exist a positive integer ℓ and $0 \leq E \in \mathrm{Div}_{\mathbb{R}}(X)$ such that*

$$\ell D \sim M + E.$$

Proof. We may let $D \in \mathrm{Div}(X)$ and X projective. Pick a sufficiently ample H, then we get

$$0 \to \mathscr{O}_X(\ell D - H) \to \mathscr{O}_X(\ell D) \to \mathscr{O}_H(\ell D) \to 0.$$

Now $h^0(X, \mathscr{O}_X(\ell D - H)) > 0$ by bigness of D and dimension of H. Hence so is $h^0(X, \mathscr{O}_X(\ell D - L)) > 0$! □

Theorem 2.28 (Absolute bigness)**.** *Let X be a normal projective variety and $D \in \mathrm{Div}_{\mathbb{R}}(X)$. Then the following are equivalent:*

(i) D is big;

(ii) $D \sim_{\mathbb{R}} A + E$ where A is ample and $E \geq 0$;

(iii) $D \equiv A + E$ where A is ample and $E \geq 0$.

Proof. See [12] Proposition 2.2.22 and [5] Lemma 7.10. □

Lemma 2.29 (Kodaira Lemma I)**.** *Let X be a projective normal variety and let $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ is big. Let $M \in \mathrm{Div}_{\mathbb{Q}}(X)$, then there exist a positive integer ε such that $D - \varepsilon L$ is big.*

Proof. We may let $D \in \mathrm{Div}(X)$. Pick a sufficiently ample H, then we get

$$0 \to \mathscr{O}_X(\ell D - H) \to \mathscr{O}_X(\ell D) \to \mathscr{O}_H(\ell D) \to 0.$$

Now $h^0(X, \mathscr{O}_X(\ell D - H)) > 0$ by bigness of D and dimension of H. Hence $\ell D \sim H + E$ for some $E \geq 0$. Hence let $\varepsilon > 0$ so small such that $H' := H - \ell\varepsilon L$ ample, then $D - \varepsilon L \sim_{\mathbb{Q}} \frac{1}{\ell}(H' + E)$ is big by Theorem 2.28. □

Proposition 2.30. *Let $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ is nef on a projective normal variety X. Then the following are equivalent:*

(i) D is big;

(ii) $D^n > 0$;

(iii) $D \sim_{\mathbb{Q}} H_m + \frac{1}{m}E$ *for all* $m \gg 0$ *where* $H_m \in \mathrm{Div}_{\mathbb{Q}}(X)$ *is ample and* $E \in \mathrm{Div}_{\mathbb{Q}}(X)$ *is effective;*

(iv) $D \equiv H_m + \frac{1}{m}E$ *for all* $m \gg 0$ *where* $H_m \in \mathrm{Div}_{\mathbb{Q}}(X)$ *is ample and* $E \in \mathrm{Div}_{\mathbb{Q}}(X)$ *is effective.*

Proof. See [10] Proposition 2.61 and Corollary 5.14 in C. Birkar's note. □

Theorem 2.31 (Relative bigness)**.** *Let* $f : X \to Z$ *be a projective morphisms of normal varieties and* $D \in \mathrm{Div}_{\mathbb{R}}(X)$*. Then* D *is* f*-big if and only if* $D \sim_{\mathbb{R},Z} A + E$ *where* A *is* f*-ample and* $E \geq 0$*.*

Lemma 2.32 (Compare to the classical)**.** *Let* X *be a projective normal variety, then effective cone* $\overline{\mathrm{Eff}}(X)$ *is the closure of the cone* $\mathrm{Eff}(X)$ *of all effective divisors.*

Proof. Pick $\eta \in \overline{\mathrm{Eff}(X)}$, then we have $\eta_k \in \mathrm{Eff}(X)$ such that $\eta = \lim_k \eta_k$. Fix an ample element α, then $\eta = \lim_k(\eta_k + \frac{1}{k}\alpha)$ which is the limit of big elements. Hence $\overline{\mathrm{Eff}(X)} \subset \overline{\mathrm{Eff}}(X)$. Conversely, by Theorem 2.28 we get $\mathrm{Big}(X) \subset \overline{\mathrm{Eff}(X)}$, well done. □

Proposition 2.33 (Cone-relations)**.** *Let* $f : X \to Y$ *be a projective morphism of normal varieties, then* $\overline{\mathrm{Eff}}(X/Y) = \overline{\mathrm{Big}(X/Y)}$ *and* $\mathrm{Big}(X/Y) = \mathrm{int}(\overline{\mathrm{Eff}}(X/Y))$*.*

Proof. The first one is the definition and the second one follows from the openness of bigness, using Theorem 2.31 and the openness of amplitude. □

2.6 Basic Duality Theory

2.7 Cyclic Covers

Here we will disciss some general fact of cyclic covers ($\boldsymbol{\mu}_m$-covers). And for general ramified covers we refer section 2.3 in [11].

Definition 2.34 (Unramified cyclic cover)**.** *Let* X *be a normal variety and* $\mathscr{L}$ *be a line bundle. Let* $\mathscr{L}^{\otimes m} \cong \mathscr{O}_X$*, then consider the* $\mathscr{O}_X$*-algebra* $\mathscr{A} = \bigoplus_{j=0}^{m-1} \mathscr{L}^{\otimes(-j)}$ *with* $i + j \geq m$ *multiplication defined by*

$$\mathscr{L}^{\otimes(-i)} \otimes \mathscr{L}^{\otimes(-j)} \cong \mathscr{L}^{\otimes(-i-j)} \cong \mathscr{L}^{\otimes(m-i-j)}$$

and consider $\sigma : X_{m,\mathscr{L}} = \underline{\mathrm{Spec}}_X \mathscr{A} \to X$ *is the corresponding unramified cyclic cover.*

Proposition 2.35. *Let* X *be a normal variety and* $\mathscr{L}$ *be a line bundle with* $\mathscr{L}^{\otimes m} \cong \mathscr{O}_X$*, then* $\sigma : X_{m,\mathscr{L}} \to X$ *is étale and* $\sigma^*\mathscr{L} \cong \mathscr{O}_X$*.*

Proof. Easy to see that σ is étale, we need to show that $\sigma^*\mathscr{L} \cong \mathscr{O}_X$. Easy to see that $\sigma_*\mathscr{O}_{X_{m,\mathscr{L}}} = \bigoplus_{i=0}^{m-1} \mathscr{L}^{-i}$, let $\mathscr{M} \in \ker \sigma^*$ and we get

$$\sigma_*\mathscr{O}_{X_{m,\mathscr{L}}} = \sigma_*\sigma^*\mathscr{M} = \mathscr{M} \otimes \sigma_*\mathscr{O}_{X_{m,\mathscr{L}}} = \bigoplus_{i=0}^{m-1} \mathscr{M} \otimes \mathscr{L}^{-i}.$$

By the Krull-Schmidt theorem, the decomposition of a vector bundle in a direct sum of indecomposable ones is unique up to permutation of the summands, so we get $\mathscr{M} \cong \mathscr{L}^i$ for all i. Hence $\sigma^*\mathscr{L} \cong \mathscr{O}_X$. □

Definition 2.36 ((Ramified) cyclic covers)**.** *Let X be a normal variety and $\mathscr{L}$ be a line bundle. Fix $m > 0$ and $s \in H^0(X, \mathscr{L}^{\otimes m})$ be a non-trivial section with zero divisor $D = (s)_0$. Then $(\mathscr{L}|_{X\backslash D})^{\otimes m} \cong \mathscr{O}_X$ and we can get $\sigma' : Z' \to X\backslash D$. Actually we can extend it to $\sigma : Z \to X$ by let $\mathscr{A} = \bigoplus_{j=0}^{m-1} \mathscr{L}^{\otimes(-j)}$ with $i + j \geq m$ multiplication defined by*

$$\mathscr{L}^{\otimes(-i)} \otimes \mathscr{L}^{\otimes(-j)} \cong \mathscr{L}^{\otimes(-i-j)} \xrightarrow{1\otimes s} \mathscr{L}^{\otimes(-i-j)} \otimes \mathscr{L}^{\otimes m} \cong \mathscr{L}^{\otimes(m-i-j)}$$

and consider $\sigma : X_{m,\mathscr{L}} = \underline{\mathrm{Spec}}_X \mathscr{A} \to X$ is the corresponding cyclic cover.

Consider the general case that $\mathscr{L}$ be a rank 1 reflexive sheaf and fix $m > 0$ be an integer. Then pick a section $s \in H^0(X, \mathscr{L}^{[m]})$, we again define $\mathscr{A} = \bigoplus_{j=0}^{m-1} \mathscr{L}^{[-j]}$ with $i + j \geq m$ multiplication defined by

$$\mathscr{L}^{[-i]} \otimes \mathscr{L}^{[-j]} \cong \mathscr{L}^{[-i-j]} \xrightarrow{1\otimes s} \mathscr{L}^{[-i-j]} \otimes \mathscr{L}^{[m]} \cong \mathscr{L}^{[m-i-j]}$$

and consider $\sigma : X_{m,\mathscr{L}} = \underline{\mathrm{Spec}}_X \mathscr{A} \to X$ is the corresponding cyclic cover.

Remark 2.37. *Note that when $\mathscr{L}$ is a $\mathbb{Q}$-line bundle of index $\mathrm{index}(\mathscr{L})|m$, then we can take $s' \in \Gamma(X_{m,\mathscr{L}}, \sigma^*\mathscr{L})$ such that $(s')^m = \sigma^* s$ as the construction give us $\bigoplus_{j=0}^{m-1} \mathscr{L}^{[-j]} \cong \frac{\bigoplus_{j=0}^{\infty} \mathscr{L}^{[-j]} t^j}{t^m - s}$. For more general case (even not over a normal variety, it is over a demi-normal scheme), we refer [11] 2.44.*

Proposition 2.38. *The general $\sigma : X_{m,\mathscr{L}} \to X$ where $\mathscr{L}$ is of rank 1 reflexive sheaf such that $\mathscr{L}^{[m]} \cong \mathscr{O}_X$, then $(\sigma^*\mathscr{L})^{[1]} \cong \mathscr{O}_X$.*

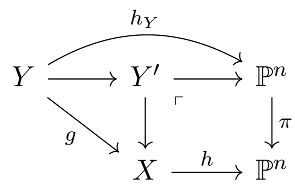
Proof. Pick a locus $X_0 \subset X$ of codimension larger than 2 such that $\mathscr{L}|_{X_0}$ is finite locally free. Then by previous proposition we get $\sigma^*(\mathscr{L}|_{X_0}) \cong \mathscr{O}_{\sigma^{-1}(X_0)}$. This can automatically extend to whole $X_{m,\mathscr{L}}$. □

Remark 2.39. *For a divisor $D \in \mathrm{Div}_{\mathbb{Q}}(X)$, if $\mathscr{O}_X(rD) \cong \mathscr{O}_X$ (at least always holds locally), then $\sigma : Z \to X$ is étale outside of singular locus. Hence $\sigma^* K_X = K_Z$. Note that in this case σ is independent of s as s/s' is nowhere zero function which take r-th square.*

When $\mathscr{L} = \mathscr{O}_X(K_X)$, the cover $\sigma : X_{m,K_X} \to X$ is called (local) index 1 cover of X.

Theorem 2.40 (Bloch-Gieseker covers)**.** *Let X be a quasi-projective variety and $D \in \mathrm{Div}(X)$ and $m \in \mathbb{N}$. Then there exists a finite cover $g : Y \to X$ and $D' \in \mathrm{Div}(Y)$ such that $g^D \sim mD'$. Moreover, if X is smooth and $\sum F_j$ is a snc divisor on X, then we can choose Y and g^*F_j are smooth and $\sum g^*F_j$ is a snc divisor.*

Proof. Consider $\pi : \mathbb{P}^n \to \mathbb{P}^n$ given by $(x_0 : ... : x_n) \mapsto (x_0^m : ... : x_n^m)$, then $\pi^*\mathscr{O}(1) \cong \mathscr{O}(m)$. Pick a very ample divisor H on X which induce $h : X \to \mathbb{P}^n$ such that $h^*\mathscr{O}(1) \cong \mathscr{O}_X(H)$ and we get the following



where $Y \to Y'$ be the normalization. Hence easy to see that $g^*\mathscr{O}_X(H) \cong h_Y^*\mathscr{O}(m)$. So if $D = H$, then well done. If not, we can let $D = H - H'$ and do the same thing. For the more things given by Kleiman's Bertini-type theorem. □

2.8 Log Resolution of Singularities

Theorem 2.41 (Hironaka)**.** *Let X be a variety over $\mathrm{char}(k) = 0$ with a divisor D on it. Then a log resolution of (X, D) exists, that is, there exists a projective birational morphism $f : Y \to X$ such that Y is smooth and $\mathrm{Exc}(f) \cup f^{-1}(\mathrm{supp}(D))$ is a strict normal crossing divisor.*

2.9 Some Fundamental Vanishing Theorems, a Sketch

Theorem 2.42 (Kodaira Vanishing Theorem)**.** *Let X be a smooth projective variety and let H be an ample Cartier divisor on X. Then*

$$H^i(X, \mathscr{O}_X(K_X + H)) = 0$$

for all $i > 0$.

Proof. We can using GAGA and cyclic cover (see [10] Theorem 2.47). □

Theorem 2.43 (Kawamata-Viehweg Vanishing Theorem)**.** *Let X be a smooth variety with a proper surjective morphism $f : X \to S$ where S be a variety. Let $D \in \mathrm{WDiv}_{\mathbb{R}}(X)$ with*

(i) D is f-nef and f-big;

(ii) $\{D\}$ has support with snc divisor.

Then $R^i f_*(K_X + \lceil D \rceil) = 0$ for any $i > 0$.

Proof. We refer [6] Theorem 3.2.1 for general case and [10] for absolute case. □

Remark 2.44. *For more general case, we can even let $\{D\}$ has support with normal crossing divisor. See [6] Theorem 3.2.1.*

Corollary 2.45 (Grauert-Riemenschneider Vanishing Theorem)**.** *Let $f : X \to Y$ be a generically finite morphism from a smooth variety X, then $R^i f_* \mathscr{O}_X(K_X) = 0$ for any $i > 0$.*

Proof. As $K_X - K_X$ is f-nef and f-big since f is generically finite, then by Theorem 2.43 and well done. □

Appendix A. Bend and Break

3 For Curves and Surfaces

3.1 Curves

Note that as the normalization of projective curves if the unique smooth model, the minimal model and resolution of singularities is trivial for dimension 1.

Let X be a smooth projective curve. Let $\kappa(X)$ be the Kodaira dimension of X. From the Riemann-Roch we have the following types of smooth projective curves:

- $g(X) = 0 \iff \deg K_X < 0 \iff X \cong \mathbb{P}^1 \iff \kappa(X) = -\infty$;
- $g(X) = 1 \iff \deg K_X = 0 \iff X$ is elliptic $\iff \kappa(X) = 0$;
- $g(X) \geq 2 \iff \deg K_X > 0 \iff X$ is of general type $\iff \kappa(X) = 1$.

For the positivity of divisors over smooth curves, we have the following well-known results:

Proposition 3.1 (See [9] Corollary IV.3.2 and Corollary IV.3.3)**.** *Let D be a divisor over a smooth projective curve X of genus g, we have*

(a) D is ample if and only if $\deg D > 0$*;*

(b) D is base-point free if $\deg D \geq 2g$*;*

(c) D is very ample if $\deg D \geq 2g + 1$*.*

Now we can related to the moduli theory of curves.

Corollary 3.2 (Results related to moduli)**.** *For a smooth projective curve X of genus $g \geq 2$, we have* *(a) K_X is base point free;*

(b) $3K_X$ is very ample, so defines an embedding $X \hookrightarrow \mathbb{P}^{5g-6}$.

Proof. (a) If $x \in X$, then by Riemann-Roch we get $h^0(X, K_X - x) - h^0(X, x) = g - 2$. First we claim that $h^0(X, x) = 1$, otherwise there exists a non-constant $f \in K(X)$ such that $(f) + x \geq 0$. Then $(f) + x$ consists of just one point and hence give a linear equivalence of two distinct points. Hence X is rational which is impossible! $h^0(X, K_X - x) = g - 1 = h^0(X, K_X) - 1$. By [9] Proposition IV.3.1.(a), well done.

(b) By Riemann-Roch this is trivial. □

Now we going to the moduli theory! Roughly speaking, we need to find a variety such that any point on it correspond to a smooth curves of given genus $g \geq 2$. But this is impossible as they have non-trivial automorphisms (actually it is so called a smooth Deligne-Mumford stack $\mathscr{M}_g$). But if we just consider the coarse moduli space (that is, consider closed points as smooth curves up to isomorphisms), we can get a variety M_g (but not smooth) and $\dim M_g = 3g - 3$.

How to construct the structure of $\mathscr{M}_g$? Actually using Corollary 3.2, we can consider the subschemes in $\mathbb{P}^{5g-6}$ of the same Hilbert polynomial, which forms a Hilbert scheme H'. We find that it is a locally closed subscheme $H \subset H'$ corresponding the smooth curves. After quotients the automorphisms of $\mathbb{P}^{5g-6}$, that is, $\mathrm{PGL}(5g-5)$, we can get $\mathscr{M}_g \cong [H/\mathrm{PGL}(5g-5)]$, the stack quotient.

But these spaces is not proper, we need some compactification (so called Deligne-Mumford compactification) $\overline{\mathscr{M}}_g$ and $\overline{M}_g$. As here the boundary $\overline{\mathscr{M}}_g \backslash \mathscr{M}_g$ is not smooth, our aim is to find some kind of singularities and stability of curves to get $\overline{\mathscr{M}}_g$. Deligne and Mumford find that we need to use the nodal singularities with stable condition: ω ample (i.e. finite automorphisms). They showed that $\overline{\mathscr{M}}_g$ is a proper smooth Deligne-Mumford stack of dimension $3g - 3$. By the theory of Keel-Mori, we get the coarse moduli space $\overline{M}_g$. Then Janos Kollár shows that $\overline{M}_g$ is projective!

Then many people study the geometric properties of M_g and $\overline{M}_g$, such as line bundles and divisors on them, the Kodaira dimension of them and the canonical and minimal models of them.

For the higher dimension, many birational geometrier want to generalized this into higher dimension, that is, the foundation of the moduli theory of varieties of general type. This theory related to the minimal model program of log general type and the finiteness of automorphisms of varieties of general type (Hacon-McKernan-Xu). As the case of curves, we need some singularities to get the compactification! Several mathematicians develop this theory, called the moduli theory of KSBA-stable varieties of general type, using the singularities called semi-log-canonical (slc) singularities which we will dicuss later.

3.2 General Surfaces

A famous theorem of blowing down (-1)-curves of Castelnuovo is needed here:

Theorem 3.3. *Let X be a smooth projective surface, $E \subset X$ a curve. Then E is a (-1)-curve if and only if E is the exceptional curve of a blowing up.*

Proof. See Theorem 3.30 in [2] for any characteristic. □

Theorem 3.4 (Classical MMP for surfaces)**.** *Let X be a smooth projective surface and $R \subset \overline{\mathrm{NE}}(X)$ an extremal ray such that $R \cdot K_X < 0$, then the contraction $\mathrm{cont}_R : X \to Z$ exists and is one of following types:*

(i) Z is smooth surface and X is obtained from Z by blowing up a closed point with $\varrho(X/Z) = 1$;

(ii) Z is a smooth curve and X is a minimal ruled surface over Z with $\varrho(X) = 2$;

(iii) Z is a point and $\varrho(X) = 1$ with $-K_X$ is ample (in fact $X \cong \mathbb{P}^2$).

Hence there is sequence of contractions $X \to \cdots \to X'$ such that X' is one of the following types:

(a) $K_{X'}$ is nef;

(b) X' is a minimal ruled surface over a smooth curve C;

(c) $X' \cong \mathbb{P}^2$.

Proof. Pick an irreducible curve C in R and consider C^2.

If $C^2 > 0$, by Lemma 3.5 we get $[C] \in \overline{\mathrm{NE}}(X)$ is an interior point. As it generate a extremal ray, then $N_1(X) \cong \mathbb{R}$. As $C \cdot K_X < 0$ we get $-K_X$ ample by Kleiman's criterion.

If $C^2 < 0$, then by adjunction formula we get C is a (-1)-curve. Hence by Theorem 3.3 we get the results.

If $C^2 = 0$, by Lemma V.1.7 in [9] we get $H^2(X, \mathscr{O}_X(mC)) = 0$ for $m \gg 1$. By Riemann-Roch theorem we get

$$h^0(X, \mathscr{O}(mC)) \geq \chi(X, \mathscr{O}(mC)) = \frac{-mC \cdot K_X}{2} + \chi(\mathscr{O}_X) \geq 2$$

for $m \gg 1$. Hence taking $mC \in H^0(X, \mathscr{O}(mC))$ and pick another independent section $s \in H^0(X, \mathscr{O}(mC))$. Then the base locus of $\{s, mC\}$ is some multiple of C since $C^2 = 0$. Hence we can find some $m' > 0$ such that we can take $s' \in H^0(X, \mathscr{O}_X(m'C))$ such that $\{s', m'C\}$ have empty base locus. Hence we get $f : X \to \mathbb{P}^1$. Taking the Stein factorization we get $\mathrm{cont}_R : X \to R$. Let $\sum a_i C_i$ be a fiber of cont_R, then $[C] = \sum a_i [C_i]$ as cont_R is flat and the general fiber have this property. Hence $[C_i] \in R$ and we get $C_i^2 = 0$ and $C_i \cdot K_X < 0$. By adjunction formula we get $C_i \cong \mathbb{P}^1$ and $C_i \cdot K_X = -2$. Hence

$$-2 = C \cdot K_X = K_X \cdot \sum a_i C_i = -2 \sum a_i,$$

hence $\sum a_i C_i$ is $\mathbb{P}^1$. □

Lemma 3.5. *Let X be an irreducible and projective surface with ample divisor H. Then the set $Q := \{z \in N_1(X) : z^2 > 0\}$ has two connected components $Q^+ = \{z \in Q : z \cdot H > 0\}$ and $Q^- = \{z \in Q : z \cdot H < 0\}$ with $Q^+ \in \overline{\mathrm{NE}}(X)$.*

Proof. Taken from Corollary 1.21 in [10]. By Hodge index theorem, we can take suitable basis such that the intersection form on $N_1(X)$ is $x_1^2 - \sum_{i\geq 2} x_i^2$ and $[H] = (\sqrt{H \cdot H}, 0, ..., 0)$. Hence $Q^+ = \{x_1 > (\sum_{i\geq 2} x_i^2)^{1/2}\}$ and $Q^- = \{x_1 < (\sum_{i\geq 2} x_i^2)^{1/2}\}$. By Corollary V.1.8 in [9], we get D or $-D$ effective where $[D] \in Q$. As effective curve has positive intersection with H, we get $Q^+ \in \overline{\mathrm{NE}}(X)$. □

4 Singularities of Pairs

5 The Basic Main Results for klt Pairs

6 Existence of Minimal models and Mori Fiber Space

7 Several Kind of Varieties

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References

[1] Oscar A. Felgueiras. The ample cone of a morphism. *Doctor thesis*, 2008.

[2] Lucian Bădescu. *Algebraic Surfaces*. Springer, 2001.

[3] Christopher D. Hacon and Sándor Kovács. *Classification of Higher Dimensional Algebraic Varieties*. Birkhäuser Basel, 2010.

[4] Olivier Debarre. *Higher-Dimensional Algebraic Geometry*. Springer, 2001.

[5] Osamu Fujino. Fundamental theorems for semi log canonical pairs. *Algebraic Geometry 1*, 2:194–228, 2014.

[6] Osamu Fujino. *Foundations of the Minimal Model Program*. World Scientific Book, 2017.

[7] Osamu Fujino and Keisuke Miyamoto. Nakai-moishezon ampleness criterion for real line bundles. *Math. Ann*, 385:459–470, 2023.

[8] William Fulton. *Intersection Theory, 2nd*. Springer, 1998.

[9] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer, 1977.

[10] Janos Kollár and Shigefumi Mori. *Birational Geometry of Algebraic Varieties*. Cambridge University Press, 1998.

[11] János Kollár. *Singularities of the Minimal Model Program*. Cambridge University Press, 2013.

[12] Robert Lazarsfeld. *Positivity in Algebraic Geometry I*. Springer, 2004.