

Reading Notes: Moduli Spaces of Curves

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Part I

The basic facts of curves

Chapter 1

Basic facts of general curves

1.1 Standard results

Definition 1.1.1. A curve over k is a pure one-dimensional scheme C of finite type over k . If C is proper, we define the arithmetic genus (simply the genus) of C as $g(C) := g_a(C) = 1 - \chi(C, \mathcal{O}_C)$. By Review A.3.1, if C is geometrically connected and geometrically reduced, this is equal to $h^1(C, \mathcal{O}_C)$.

Theorem 1.1.2 (St 0B5Y). Let k be a field. Let C be a proper scheme of dimension ≤ 1 over k . Let L be an invertible \mathcal{O}_X -module. Let C_i be the irreducible components of dimension 1. Then L is ample if and only if $\deg(L|_{C_i}) > 0$ for all i .

Theorem 1.1.3 (Serre duality of smooth curves). Let C be a smooth projective curve over k with canonical bundle $\omega_C = \Omega_C$, then for any vector bundle F we get

$$H^0(C, F^\vee \otimes \omega_C) \cong H^1(C, F)^\vee.$$

If we define the geometrical genus $g_e(C) = h^0(C, \omega_C)$ and if C is smooth projective curve which is geometrically connected and geometrically reduced, then $h^0(C, \mathcal{O}_C) = 1$. Hence by serre-duality we get $g_e(C) = g_a(C)$.

Theorem 1.1.4 (Riemann-Roch for smooth curves). Let C be a smooth projective curve over k with a line bundle L , then

$$\chi(C, L) = h^0(C, L) - h^0(C, \omega_C \otimes L^\vee) = \deg L + 1 - g.$$

Theorem 1.1.5 (Positivity of divisors on smooth curves). Let C be a smooth projective curve over k of genus g with a line bundle L , then

- (a) if $\deg L \geq 2g$, then L is base-point-free;
- (b) if $\deg L \geq 2g + 1$, then L is very ample;
- (c) if $\deg L > 0$, then L is ample.
- (d) if $\deg L < 0$, then $h^0(C, L) = 0$.

Proof. See the section IV.3 of [58] for the proof when k is algebraic closed. This is also right when k is not algebraic closed, see section 20.2 in [80].

Here we use another method to show (d) as a special case of [58] Ex.III.7.1. We just consider the case C is integral. If $\deg L < 0$, then L^{-1} is ample. Let $h^0(C, L) > 0$ and take a nonzero $s \in H^0(C, L)$. As $H^0(C, L) = \text{Hom}(\mathcal{O}_C, L)$, we can get $- \times s : \mathcal{O}_C \rightarrow L$. As C integral, s must nonzero at the generic point, hence $- \times s : \mathcal{O}_C \rightarrow L$ is injective. Hence we get $L^{-1} \subset \mathcal{O}_C$. Let n such that L^{-n} generated by global sections, we get $L^{-n} \subset \mathcal{O}_C$. Hence $H^0(C, L^{-n}) \subset H^0(C, \mathcal{O}_C)$. Consider hilbert polynomial $\chi(L^{-n}) = \alpha n + \beta$ as $\deg \chi(L^{-n}) = \dim \text{supp}(L^{-1}) = \dim C = 1$. By Serre's vanishing theorem, we get for $n \rightarrow \infty$, we have $\chi(L^{-n}) = h^0(C, L^{-n}) \rightarrow \infty$. This is impossible since $h^0(C, \mathcal{O}_C) < \infty$. \square

Theorem 1.1.6 (Riemann-Hurwitz Theorem, St 0C1B). Let $f : X \rightarrow Y$ be a separable morphism of smooth proper curves over a field k and if $k = H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$ and X and Y have genus g_X and g_Y , then

$$2g_X - 2 = (2g_Y - 2) \deg(f) + \deg R$$

where R be the ramified divisor. Moreover, $\deg R = \sum_x d_x [\kappa(x) : k]$ where $d_x = \text{length}_{\mathcal{O}_{X,x}} \Omega_{X/Y,x}$. Of course if $\mathcal{O}_{X,x}$ is tamely ramified over $\mathcal{O}_{Y,f(x)}$ then $d_x = e_x - 1$. If not, we only have $d_x > e_x - 1$ where e_x is the ramification index.

1.2 Automorphisms of curves

Here we only consider smooth connected projective curves of genus g over an algebraically closed field k .

Proposition 1.2.1. *For $g = 0$, we get $\text{Aut}(\mathbb{P}_k^1) \cong \text{PGL}_2$. Moreover, if we consider all automorphisms fixed n points, then this group is finite if and only if $n \geq 3$.*

Proof. See [58] Example II.7.1.1, we get $\text{Aut}(\mathbb{P}_k^1) \cong \text{PGL}_2$. Moreover, all automorphisms fixed n points is finite if and only if $n \geq 3$ by easy linear algebra. \square

Proposition 1.2.2. *For curve C with $g = 1$, we get $\text{Aut}(C)$ is infinite group. Moreover, if we consider all automorphisms fixed n points, then this group is finite if and only if $n \geq 1$.*

Proof. In this case C is actually a group scheme of dimension 1 (by Picard varieties) and C can then act on C . Hence $C \subset \text{Aut}(C)$, hence infinite. Moreover, by [58] Corollary IV.4.7 (for $\text{char}(k) \neq 2$), if we fixed one point P_0 , then $\text{Aut}(C; P_0)$ is finite. Indeed, let $f : X \rightarrow \mathbb{P}^1$ such that $f(P_0) = \infty$, branched over $0, 1, \lambda, \infty$. Let $\sigma \in \text{Aut}(C; P_0)$, then there exists an automorphism $\tau \in \text{Aut}(\mathbb{P}^1; \infty)$ such that $f \circ \sigma = \tau \circ f$. Hence τ sends $\{0, 1, \lambda\}$ to $\{0, 1, \lambda\}$ in some order.

(i) If $\tau = \text{id}$, then $\sigma = \text{id}$ or interchanging two sheets of f ;

(ii) If $\tau \neq \text{id}$, then τ permutes $\{0, 1, \lambda\}$ and the orbit of λ is of that six forms.

In both cases is finite, well done. \square

Proposition 1.2.3 (Hurwitz). *For curve C with $g \geq 2$, the group $\text{Aut}(C)$ is finite. Moreover, if k has characteristic 0, we have $\#(\text{Aut}(C)) \leq 84g - 84$.*

Proof. See [58] Ex.V.1.11 and Hurwitz's Automorphism Theorem. \square

Lemma 1.2.4 (St 0E67). *Let X be a smooth, proper, connected curve over k of genus g .*

(a) *If $g \geq 2$, then $\text{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$;*

(b) *If $g = 1$ and $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ is nonzero, then D does not fix any closed point of X ;*

(c) *If $g = 0$ and $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ is nonzero, then D can fix at most 2 closed points of X .*

Remark 1.2.5. *We will say an element $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ fixes x if $D(\mathcal{I}) \subset \mathcal{I}$ where \mathcal{I} is the ideal sheaf of x .*

Sketch. As we have the canonical derivation $d : \mathcal{O}_X \rightarrow \Omega_{X/k}$, taking any $D \in \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ we get $D = f \circ d$ where $f \in \text{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$ and $\deg(\Omega_{X/k}) = 2g - 2$.

(a) If $g \geq 2$, then $\deg(\Omega_{X/k}) > 0$. Hence

$$\text{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X) = \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, T_{X/k}) = \Gamma(X, T_{X/k}) = 0,$$

hence $f = 0$;

(b)(c) We claim that the vanishing of f at $x \in X$ is equivalent to the statement that D fixes x . Indeed, by St 0C1E we get for the uniformizer $z \in \mathcal{O}_{X,x}$, dz is a basis of $\Omega_{X,x}$. Since $D(z) = f(dz)$, we conclude the claim.

If $g = 1$, then a nonzero f does not vanish anywhere. Hence by the claim, D does not fix any closed point of X . If $g = 0$, then a nonzero f vanishes in a divisor of degree 2. Hence by the claim, D can fix at most 2 closed points of X . \square

Lemma 1.2.6. *Let X be a proper scheme over a field k of dimension ≤ 1 , then the following are equivalent*

(i) $\underline{\text{Aut}}(X)$ is geometrically reduced over k and has dimension 0;

(ii) $\underline{\text{Aut}}(X) \rightarrow \text{Spec}(k)$ is unramified;

(iii) $\text{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = 0$.

Proof. See St 0DSW and St 0E6G. Note that these two lemmas can also give the results about automorphism groups of smooth connected curves. \square

Proposition 1.2.7. *Let C be a curve of genus g over a field k of characteristic 0, then for any non-trivial automorphism of C fixed at most $2g + 2$ points.*

Proof. See I.F-4 in [9] for now. To add. \square

Chapter 2

Families of curves

2.1 Families of general curves

Lemma 2.1.1 (Dualizing sheaves of the families of curves). *Let $(S, f : C \rightarrow S)$ in \mathcal{M}_g (or more general) for $g \geq 2$.*

- (i) $f_*\mathcal{O}_C = \mathcal{O}_S$;
- (ii) For $k > 1$ the sheaf $f_*(\Omega_{C/S}^1)^{\otimes k}$ is locally free of rank $(2k-1)(g-1)$ on S , and for any $g : S' \rightarrow S$, we get an isomorphism $g^*f_*(\Omega_{C/S}^1)^{\otimes k} \cong f'_*(\Omega_{C'/S'}^1)^{\otimes k}$. Moreover, $R^i f_*(\Omega_{C/S}^1)^{\otimes k} = 0, i > 0$;
- (iii) The sheaf $f_*\Omega_{C/S}^1$ is locally free of rank g on S , and for any $g : S' \rightarrow S$, we get an isomorphism $g^*f_*\Omega_{C/S}^1 \cong f'_*\Omega_{C'/S'}^1$. Moreover, $R^1 f_*(\Omega_{C/S}^1) = \mathcal{O}_S$ and $R^i f_*(\Omega_{C/S}^1) = 0, i > 1$;
- (iv) For $k \geq 3$, $(\Omega_{C/S}^1)^{\otimes k}$ is relative very ample.

Proof. (i) By definition, for all $s \in S$ the C_s is proper geometrically connected and geometrically reduced, then by Review A.3.1 we get $H^0(C_s, \mathcal{O}_s) = \kappa(s)$, hence $\phi_s^0 : f_*\mathcal{O}_C \otimes \kappa(s) \rightarrow H^0(C_s, \mathcal{O}_s)$ is surjective. By Review A.1.1 with $i = 0$, we get ϕ_s^0 is an isomorphism and $f_*\mathcal{O}_C$ is a line bundle. Now consider the natural map $\mathcal{O}_S \rightarrow f_*\mathcal{O}_C$ induce a surjective fiber map $\kappa(s) \rightarrow f_*\mathcal{O}_C \otimes \kappa(s)$ by seen

$$\kappa(s) \rightarrow f_*\mathcal{O}_C \otimes \kappa(s) \rightarrow H^0(C_s, \mathcal{O}_s) = \kappa(s).$$

Thus $\mathcal{O}_S \rightarrow f_*\mathcal{O}_C$ is surjective, hence an isomorphism.

- (ii) For all $s \in S$ and $k > 1$ we get $H^1(C_s, (\Omega_{C_s/\kappa(s)}^1)^{\otimes k}) = H^0(C_s, (\Omega_{C_s/\kappa(s)}^1)^{\otimes(1-k)})^\vee = 0$ as $(\Omega_{C_s/\kappa(s)}^1)^{\otimes(1-k)}$ is anti-ample. Hence $H^i(C_s, (\Omega_{C_s/\kappa(s)}^1)^{\otimes k}) = 0$ for $i > 0$. Now use Review A.1.1 we get $R^i f_*(\Omega_{C/S}^1)^{\otimes k} = 0, i > 0$.

On the other hand, by Riemann-Roch theorem, we get

$$h^0(C_s, (\Omega_{C_s/\kappa(s)}^1)^{\otimes k}) = \deg((\Omega_{C_s/\kappa(s)}^1)^{\otimes k}) + 1 - g = (2k-1)(g-1).$$

Use Review A.1.1 again, we get $f_*(\Omega_{C/S}^1)^{\otimes k}$ is locally free of rank $(2k-1)(g-1)$ on S .

- (iii) By Review A.1.1 and the fact $H^i(C_s, \Omega_{C/S}^1 \otimes \kappa(s)) = 0, i > 1$ implies $R^i f_*\Omega_{C/S}^1 = 0, i > 1$. Now we use the duality $f_*\mathcal{H}om(F, \Omega_{C/S}^1) \cong \mathcal{H}om(R^1 f_*F, \mathcal{O}_S)$, then let $F = \Omega_{C/S}^1$. We get $f_*\mathcal{O}_C \cong (R^1 f_*\Omega_{C/S}^1)^*$. Hence $R^1 f_*\Omega_{C/S}^1 \cong (f_*\mathcal{O}_C)^* = \mathcal{O}_S^* \cong \mathcal{O}_S$.

By Review A.1.1(ii) with $i = 1$, we get $\phi_s^0 : f_*\Omega_{C/S}^1 \otimes \kappa(s) \rightarrow H^0(C_s, \Omega_{C_s/\kappa(s)}^1)$ is surjective, hence an isomorphism. Then apply Review A.1.1(i)-(ii) with $i = 0$ to imply $f_*\Omega_{C/S}^1$ is locally free of rank $h^0(C_s, \Omega_{C_s/\kappa(s)}^1) = g$.

- (iv) Easy to see for any $s \in S$ the fiber $(\Omega_{C_s/\kappa(s)}^1)^{\otimes k}$ is very ample as $\deg((\Omega_{C_s/\kappa(s)}^1)^{\otimes k}) = k(2g-2) \geq 2g+1$. Using noetherian approximation, we may let S is noetherian. Then use Review A.3.2 and well done. \square

Remark 2.1.2. *Note that we can be generalized these statements into more general families of curves, such as nodal curves and so on, without any modification.*

Proposition 2.1.3 (Flatness Criterion over Smooth Curves). *Let C be an integral and regular scheme of dimension 1 (e.g. the spectrum of a DVR or a smooth connected curve over a field) and $X \rightarrow C$ a qcqs morphism of schemes. A quasi-coherent \mathcal{O}_X -module F is flat over C if and only if every associated point of F maps to the generic point of C .*

2.2 Families of elliptic curves

This section are some preliminaries of the coarse moduli space of $\mathcal{M}_{1,1}$. Here we follows [75] 13.1 and for the basic theory of single elliptic curves, we refer [58] IV.4.

Chapter 3

Singularities of curves

3.1 δ -invariant

The the more details, see St 0C3Q and St 0C3Z.

Lemma 3.1.1 (St 0C3S). *Let (A, \mathfrak{m}) be a reduced 1-dimensional local ring of finite type over a field k . Let A' be the integral closure of A in the total ring of fractions of A . Then A' is a normal with $A \rightarrow A'$ is finite, and A'/A has finite length as an A -module.*

Definition 3.1.2. *Let A be a reduced 1-dimensional local ring of finite type over a field k . The δ -invariant of A is $\text{length}_A(A'/A)$ where A' is as in Lemma.*

Let X be a scheme locally of finite type over k . Let $x \in X$ such that $\mathcal{O}_{X,x}$ is reduced with dimension 1. The δ -invariant of X at x is the δ -invariant of $\mathcal{O}_{X,x}$.

Proposition 3.1.3 (St 0C3V). *Let A be a reduced 1-dimensional local ring of finite type over a field k . Then \hat{A} has the same δ -invariant as A and $A' \otimes_A \hat{A}$ is the integral closure of \hat{A} in its total ring of fractions.*

Proposition 3.1.4 (St 0C1R). *Let X be a reduced scheme locally finite type over a field of dimension 1 with normalization $f : \tilde{X} \rightarrow X$. Then $\mathcal{O}_X \subset f_*\mathcal{O}_{\tilde{X}}$ and $f_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X$ is a direct sum of skyscraper sheaves \mathcal{Q}_x in the singular points x and $\mathcal{Q}_x = (f_*\mathcal{O}_{\tilde{X}})_x/\mathcal{O}_{X,x}$ has finite length equal to the δ -invariant of X at x .*

3.2 Some singularities of curves

Definition 3.2.1. *Let C be a curve over k . Here we let k algebraically closed and if not, we consider the base-change.*

- (a) *We say that $p \in C(k)$ is a node is we have $\hat{\mathcal{O}}_{C,p} \cong k[[x, y]]/(xy)$;*
- (b) *We say that $p \in C(k)$ is a cusp is we have $\hat{\mathcal{O}}_{C,p} \cong k[[x, y]]/(y^2 - x^3)$;*
- (c) *We say that $p \in C(k)$ is a tacnode is we have $\hat{\mathcal{O}}_{C,p} \cong k[[x, y]]/(y^2 - x^4)$.*

Definition 3.2.2. *A curve C has locally planar singularities at p if $\hat{\mathcal{O}}_{C,p} \cong k[[x, y]]/(f)$ for a reduced series $f \in k[[x, y]]$.*

Proposition 3.2.3. *If a curve have worst locally planar singularities, then it is Gorenstein. Hence nodal, cuspidal and tacnodal curves are all Gorenstein.*

Chapter 4

The varieties associated to curves

In most of cases of this chapter we focus on the proper reduced curves over an algebraically closed field k (may not irreducible and smooth). But when we let C smooth, we will automatically let C irreducible. When we consider \mathbb{C} we can use the language of Riemann surfaces via Serre's GAGA-principle.

4.1 Jacobian variety of curves

4.1.1 Analytic approach

We let C a smooth projective curve of genus g over \mathbb{C} . We follows [9].

► **Approach 1.** If ω, ω' are holomorphic forms on C , then checking locally we get $\omega \wedge \omega' = 0$ and $\int_C \sqrt{-1} \omega \wedge \bar{\omega} > 0$ ($\omega \neq 0$). Moreover we have $d\omega = 0$, hence $[\omega] \in H_{DR}^1(C) \cong H^1(C, \mathbb{C})$.

Choose a basis $\omega_1, \dots, \omega_g \in H^0(C, K)$ and $\gamma_1, \dots, \gamma_{2g} \in H_1(C, \mathbb{Z})$. We can define the period matrix

$$\Omega = (\Omega_1, \dots, \Omega_{2g})_{g \times 2g}, \Omega_i = \begin{pmatrix} \int_{\gamma_i} \omega_1 \\ \vdots \\ \int_{\gamma_i} \omega_g \end{pmatrix}.$$

Hence by construction we can see that $\Omega_1, \dots, \Omega_{2g}$ generates a lattice Λ in \mathbb{C}^g . Hence we define $J(C) := \mathbb{C}^g / \Lambda$ as the Jacobian variety of C .

► **Approach 2.** Or equivalently, consider $H_1(C, \mathbb{Z}) \hookrightarrow H^0(C, K)^\vee$ by $\gamma \mapsto \int_\gamma$, then $J(C) = H^0(C, K)^\vee / H_1(C, \mathbb{Z})$. Sometimes we call this Albanese torus when X be a compact Kähler manifold of higher dimension (See [62], for example).

► **Approach 3.** Let $\text{Pic}^0(C)$ be the degree 0 line bundles (or the kernel of the first Chern class in higher dimension). We can consider the exponential sequence and since C is compact, then $H^1(C, \mathbb{Z}) \rightarrow H^1(C, \mathcal{O}_C)$ is injective, hence $\text{Pic}^0(C) \cong H^1(C, \mathcal{O}_C) / H^1(C, \mathbb{Z})$. By some easy argument of Hodge theory (as [62] Corollary 3.3.6), it is a complex torus.

Proposition 4.1.1. *These three approaches defined the same variety associated to C .*

Proof. The first two approaches are the same trivially.

Now we consider the second and the third approaches. Actually this is by Serre duality $H^0(C, K)^\vee \cong H^1(C, \mathcal{O}_C)$ and Poincaré duality $H^1(C, \mathbb{Z}) \cong H_1(C, \mathbb{Z})$ which is compatible by trivial reasons. \square

4.1.2 Algebraic approach

Here we follows [58] and let C a smooth projective curve over an algebraically closed field k . Actually this is just the special case of the Picard variety.

Let T be a scheme over k and we let $\text{Pic}^0(C \times_k T)$ be the subgroup of $\text{Pic}(C \times_k T)$ consisting of invertible sheaves whose restrict to any fibers C_t for $t \in T$ has degree 0. Hence we can define $\text{Pic}^0(C/T) := \text{Pic}^0(C \times_k T) / p^* \text{Pic} T$ where $p : C \times_k T \rightarrow T$.

We can show that (omitted here) the functor

$$J(C) : (\text{Sch}/\text{Speck})^{\text{opp}} \rightarrow (\text{Sets}), T \mapsto \text{Pic}^0(C/T)$$

is represented by a k -scheme, we also denoted $J(C)$. Here we find that for any points $x \in C(k)$, we find that $x : \text{Speck} \rightarrow J(C)$ correspond to an element of $\text{Pic}^0(C)$. Hence this notation make sense.

Proposition 4.1.2. *The Jacobian variety $J(C)$ is a group variety over k .*

Proof. We define $e : \text{Speck} \rightarrow J(C)$ correspond to $0 \in \text{Pic}^0(C/k)$ be the identity. Let $i : J(C) \rightarrow J(C)$ correspond to $\mathcal{L}_{\text{univ}}^{-1} \in \text{Pic}^0(C/J(C))$ be the inverse. Let $\mu : J(C) \times_k J(C) \rightarrow J(C)$ correspond to $p_1^* \mathcal{L}_{\text{univ}} \otimes p_2^* \mathcal{L}_{\text{univ}} \in \text{Pic}^0(C/J(C) \times_k J(C))$ be the multiple. Then the axiom of the group varieties is easy to check. \square

Proposition 4.1.3. *The Zariski tangent space $T_{J(C),0} \cong H^1(C, \mathcal{O}_C)$.*

Proof. Consider the dual number $T = k[\varepsilon]/(\varepsilon^2) \rightarrow J(C)$ by sending Speck to 0. By [58] Ex.III.4.6, we get

$$0 \rightarrow H^1(C, \mathcal{O}_C) \rightarrow \text{Pic}C[\varepsilon] \rightarrow \text{Pic}C \rightarrow 0.$$

Hence we win. \square

Proposition 4.1.4. *The Jacobian variety $J(C)$ is proper and nonsingular over k .*

Proof. Using valuation criterion, we need to extend the line bundle at a codimension 2 point of $C \times \text{Spec}R$. This is trivial. It is nonsingular by [58] Remark IV.4.10.9. \square

Proposition 4.1.5. *When C is a elliptic curve, then $J(C) \cong C$. In particular, C has a group structure.*

Proof. Omitted, see [58] Theorem IV.4.11. \square

4.2 Picard varieties of curves

For more detail about the general Picard scheme, we refer [68]. Here we focus on the theory on curves over an algebraically closed field k . We follows [11] in St 0B92.

Definition 4.2.1 (Picard functor). *Let $f : X \rightarrow S$ be a morphism in the big fppf-site $(\text{Sch})_{\text{fppf}}$. Consider the functor*

$$\text{Pic}_{X/S}^{\text{Psh}} : (\text{Sch}/S)_{\text{fppf}} \rightarrow (\text{Sets}), T \mapsto \text{Pic}(X_T).$$

Let $\text{Pic}_{X/S} := (\text{Pic}_{X/S}^{\text{Psh}})^{\text{fppf}}$ the sheafification in fppf-topology.

Proposition 4.2.2. (St 0B9N) *Let $f : X \rightarrow S$ as in the definition which admits a section σ . Assume that $\mathcal{O}_T \cong f_{T,*} \mathcal{O}_{X_T}$ for all $T \in \text{Ob}((\text{Sch}/S)_{\text{fppf}})$, then we have*

$$0 \longrightarrow \text{Pic}T \longrightarrow \text{Pic}X_T \xrightarrow{\sigma_T^*} \text{Pic}_{X/S}(T) \longrightarrow 0$$

is exact and split by σ_T^* .

sketch. • The left-exactness don't need the σ : WLOG we let $S = T$. If $f^*N \cong \mathcal{O}_X$, then $f_*f^*N \cong f_*\mathcal{O}_X \cong \mathcal{O}_S$ by assumption. Since N is locally trivial, we see that the canonical map $N \rightarrow f_*f^*N$ is locally an isomorphism (because $\mathcal{O}_S \rightarrow f_*f^*\mathcal{O}_S$ is an isomorphism by assumption). Hence we conclude that $N \rightarrow f_*f^*N \rightarrow \mathcal{O}_S$ is an isomorphism and we see that N is trivial. This proves the first arrow is injective.

The exactness in the middle is easy by fppf-descent of quasi-coherent sheaves.

• The right-exactness need the σ : Let $K(T) := \ker(\sigma_T^*)$, hence $\text{Pic}(X_T) \cong \text{Pic}T \oplus K(T)$ and $K(T) \subset \text{Pic}_{X/S}(T)$. As $\text{Pic}_{X/S}$ is the sheafification of K , we just need to show that K is a fppf-sheaf. I omitted here. \square

Lemma 4.2.3. *If C be a smooth projective curve over an algebraically closed field k , then the hypotheses of the previous Proposition are satisfied.*

Proof. We of course have a k -rational point (hence a section). Moreover, as $H^0(C, \mathcal{O}_C) = k$, by cohomology and base change we get $\mathcal{O}_T \rightarrow f_{T,*}\mathcal{O}_{C_T}$ is an isomorphism. \square

If C be a smooth projective curve over an algebraically closed field k with a closed point σ . Consider the functor

$$\mathrm{Pic}_{C/k,\sigma} : (\mathrm{Sch}/k)^{\mathrm{opp}} \rightarrow (\mathrm{Sets}), T \mapsto \ker(\sigma_T^* : \mathrm{Pic}(C_T) \rightarrow \mathrm{Pic}(T)),$$

which is isomorphic to $\mathrm{Pic}_{C/k}$ before by the previous propositions. Hence we denote it by $\mathrm{Pic}_{C/k}$.

Theorem 4.2.4. (St 0B9Z, St 0BA0) *Let C be a smooth projective curve of genus g over an algebraically closed field k .*

- (i) *The functor $\mathrm{Pic}_{C/k}$ is representable by a group scheme, denote it also by $\mathrm{Pic}_{C/k}$;*
- (ii) *There is the disjoint decomposition of g -dimensional smooth proper varieties*

$$\mathrm{Pic}_{C/k} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}_{C/k}^d;$$

- (iii) *The closed points of $\mathrm{Pic}_{C/k}^d$ correspond to invertible \mathcal{O}_C -modules of degree d ;*
- (iv) *$\mathrm{Pic}_{C/k}^0$ is an open and closed subgroup scheme.*

Sketch. (iv) follows from the fact (ii). (iii) is trivial by definition. For the disjoint decomposition in (ii), by St 0B9T (locally constant of Euler characteristic) that for all $d \in \mathbb{Z}$ there is an open subfunctor $\mathrm{Pic}_{C/k}^d \subset \mathrm{Pic}_{C/k}$ whose value on a scheme T over k consists of those $L \in \mathrm{Pic}_{C/k,\sigma}(T)$ such that $\chi(C_t, L_t) = d + 1 - g$ and moreover we have $\mathrm{Pic}_{C/k,\sigma} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}_{C/k,\sigma}^d$. For (i) and the smoothness and properness of $\mathrm{Pic}_{C/k}$ we omitted, we refer St 0BA0. \square

4.3 Basic fact of the determinantal varieties

We refer [9] Chapter II.

Let $M = M(m, n) := \mathbb{A}_{\mathbb{C}}^{mn}$ be the variety of $m \times n$ matrix. Let $M_k \subset M$ be a subvariety consist of matrixes at most rank k . This is called the generic determinantal variety.

We also can let $\widetilde{M}_k = \{(A, W) \in M \times \mathrm{Gr}(n - k, n) : AW = 0\}$ be a smooth connected subvariety of $M \times \mathrm{Gr}(n - k, n)$. If $\pi : \mathrm{Gr}(n - k, n) \times M \rightarrow M$, then $\pi(\widetilde{M}_k) = M_k$. Hence we can get M_k is irreducible of codimension $(m - k)(n - k)$.

Proposition 4.3.1. *We have $\mathrm{Sing}(M_k) = M_{k-1}$.*

Proof. This need some calculation, I omit it here. \square

Theorem 4.3.2 (The Second Fundamental Theorem of Invariant Theory). *The ideal of M_k in M generated by all $(k + 1) \times (k + 1)$ minors and is radical.*

Proof. The proof of this fact relies on a detailed analysis of the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(k, N)$ under the Plücker embedding. See [9] Page 71-76. \square

Theorem 4.3.3 (The First Fundamental Theorem of Invariant Theory). *Let $G = \mathrm{GL}(n, \mathbb{C})$ and act on $M(m, k) \times M(k, n)$ as $g(A, B) = (Ag^{-1}, gB)$. Let the multiplication $\mu : M(m, k) \times M(k, n) \rightarrow M(m, n)$, hence $\mathrm{Im}\mu = M_k$. If we let $M(m, k) \times M(k, n) = \mathrm{Spec}S$, then $M_k = \mathrm{Spec}S^G$. Moreover, M_k is normal.*

Proof. Just some linear algebra, see [9] Page 77-79. \square

Theorem 4.3.4. *M_k is Cohn-Macaulay.*

Proof. This proof is much complicated by showing the cone over a Schubert variety is Cohen-Macaulay. See [9] Page 80-82. \square

Now we consider the general case of the determinantal variety. If X be a scheme over \mathbb{C} and $\phi : E \rightarrow F$ be a morphism of vector bundles of rank n, m . Let open U be the trivialization, hence $\phi|_U$ be a $m \times n$ matrix. Hence this induce $f : U \rightarrow M(m, n)$. Let $U_k = f^{-1}(M_k)$ and glue it together, we get $X_k(\phi) \subset X$. We call this the k -th determinantal variety. (Similarly we can get $\tilde{X}_k(\phi)$). Also we have an intrinsical definition, see [9] Page 84.)

Hence $\text{codim}_X X_k(\phi) \leq (m - k)(n - k)$. Combining the previous local theorem and some algebra, we can get:

Proposition 4.3.5. *Let X be a smooth projective variety over \mathbb{C} with $\phi : E \rightarrow F$ be a morphism of vector bundles of rank n, m . If $\text{codim}_X X_k(\phi) = (m - k)(n - k)$, then $X_k(\phi)$ is Cohen-Macaulay.*

4.4 The varieties of special linear series on a curve

Let C be a smooth projective curve of genus g over \mathbb{C} . Let $C_d := C^d/S_d$ and easy to see that C_d be the set of effective divisors of degree d on C .

Fixed $p_0 \in C$ we first consider $C \rightarrow J(C)$ as $p \mapsto \int_{p_0}^p$, then this can extend to $\text{Div}(C) \rightarrow J(C)$ as

$$\sum_i p_i - \sum_j q_j \mapsto \sum_i \int_{p_0}^{p_i} - \sum_j \int_{p_0}^{q_j}.$$

Hence we have $\nu : \text{Div}^d(C) \rightarrow J(C)$. Hence we also can restrict to $\mu : C_d \rightarrow J(C)$.

Now using the Abel's theorem (see [9] Page 18), we have the following factorization

$$\begin{array}{ccc} C_d & \hookrightarrow & \text{Div}^d(C) \xrightarrow{-/\sim} \text{Pic}^d(C) \\ & \searrow \mu & \downarrow \nu \swarrow u \\ & & J(C) \end{array}$$

► **The variety C_d^r :** Now let

$$C_d^r = \{D \in C_d : \dim |D| \geq r\}$$

with the variety-structure by using Brill-Noether matrix (see [9] IV.1, omitted here).

► **The variety $W_d^r(C)$:** Roughly speaking, we can let

$$W_d^r(C) = \{\text{parametrizing complete series } |D| : \deg D = d, h^0(C, \mathcal{O}_C(D)) \geq r + 1\} \subset \text{Pic}_C^d.$$

Hence if we consider $f : C_d \rightarrow \text{Pic}_C^d$, then $f(C_d^r) = W_d^r(C)$.

The precise argument coming from the representability of the Picard variety Pic_C^d as follows. Let the degree d universal line bundle (or they called Poincaré line bundle) $\mathcal{L} = \mathcal{L}_{\text{univ}}$ on $C \times \text{Pic}_C^d$. Let $v : \text{Pic}_C^d \times C \rightarrow \text{Pic}_C^d$ be the projection.

Take E be an effective divisor on C such that $m := \deg E \geq 2g - d - 1$. Let $\Gamma = E \times \text{Pic}_C^d$ be a divisor on $C \times \text{Pic}_C^d$, by some sheaf-theoric argument (kind of flat base-change, see [9] IV.2.6), we have $R^1 v_* \mathcal{L}(\Gamma) = 0$ and $v_* \mathcal{L}(\Gamma)$ is locally free of rank $n = d + m - g + 1$. Hence we have

$$0 \rightarrow v_* \mathcal{L} \rightarrow K^0 := v_* \mathcal{L}(\Gamma) \xrightarrow{\gamma} K^1 := v_*(\mathcal{L}(\Gamma)/\mathcal{L}) \rightarrow R^1 v_* \mathcal{L} \rightarrow 0.$$

Hence $\ker \gamma = v_* \mathcal{L}$, $\text{coker} \gamma = R^1 v_* \mathcal{L}$ and $\text{rank} K^0 = n, \text{rank} K^1 = m$. Now we let

$$W_d^r(C) := X_{m+d-g-r}(\gamma) \text{ where } X = \text{Pic}_C^d.$$

Note that $W_d^r(C)$ is independent of the choice of E , see [9] Page 179.

By [9] Lemma IV.3.1, we get

Proposition 4.4.1. *The variety $W_d^r(C)$ represented the functor*

$$S \mapsto \left\{ L \in \text{Pic}^d(C \times S) \text{ such that the fitting rank of } R^1\phi_*L \text{ is at least } g - d + r \right\}.$$

(For $\mathcal{F} \in \text{Coh}(X)$ with presentation $\mathcal{O}_X^n \xrightarrow{\gamma} \mathcal{O}_X^m \rightarrow \mathcal{F}$ the fitting rank of \mathcal{F} is the largest integer h such that the ideal in \mathcal{O}_X generated by the $(m - h + 1) \times (m - h + 1)$ minors of A vanishes.)

Also by [9] Lemma IV.3.1, we get the set of all \mathbb{C} -valued points of $W_d^r(C)$ is just

$$\{L \in \text{Pic}^d(C) : h^0(C, L) \geq r + 1\}.$$

Proposition 4.4.2. *For $r \geq d - g$, the each component of $W_d^r(C)$ has dimension greater or equal to the Brill-Noether number*

$$\rho(g, r, d) = g - (r + 1)(g - d + r).$$

Proof. Trivial by the local analysis above. □

Proposition 4.4.3. *For $f : C_d \rightarrow \text{Pic}_C^d$, we get $f^{-1}(W_d^r(C)) = C_d^r$.*

Proof. See [9] Proposition IV.3.4. □

►**The variety $G_d^r(C)$:** Roughly speaking, we can let

$$G_d^r(C) = \{\text{parametrizing } \mathfrak{g}_d^r \text{ of degree } d \text{ and dimension } r\}.$$

For precise definition, we let

$$G_d^r(C) := \tilde{X}_{m+d-g-r}(\gamma)$$

as previous construction. For any \mathbb{C} -valued point (L, V) where $L \in \text{Pic}^d(C)$ and V be a $(r + 1)$ -dimensional subspace of $\ker \gamma_L$. By [9] Lemma IV.3.1 we get $\ker \gamma_L = H^0(C, L)$, we get

$$G_d^r(C)(\mathbb{C}) = \{(L, V) : L \in \text{Pic}^d(C), V \in \text{Gr}(r + 1, H^0(C, L))\}.$$

Hence parametrizing all \mathfrak{g}_d^r . Actually $G_d^r(C)$ can also defined as a representable functor, we refer [9] page 182-183 and omit it here.

Now we collect some conclusions of these three varieties in [9] section IV.4.

Proposition 4.4.4. (i) *Every component of $G_d^r(C)$ has dimension at least equal to the Brill-Noether number $\rho = g - (r + 1)(g - d + r)$;*

(ii) *Let $w = (L, W \subset H^0(C, L))$ be a point in $G_d^r(C)$ and consider the cup product*

$$\mu_{0,W} : W \otimes H^0(C, K \otimes L^{-1}) \rightarrow H^0(C, K) = H^1(C, \mathcal{O}_C)^\vee,$$

then $\dim T_w G_d^r(C) = \rho + \dim \ker \mu_{0,W}$. In particular, $G_d^r(C)$ is smooth at w of dimension ρ if and only if $\ker \mu_{0,W} = 0$;

(iii) *Let $L \in W_d^r(C) \setminus W_d^{r+1}(C)$ (hence $r \geq d - g$), then $T_L W_d^r(C) \cong (\text{Im } \mu_0)^\perp$ where $\mu_0 : H^0(C, L) \otimes H^0(C, K \otimes L^{-1}) \rightarrow H^0(C, K)$ be the cup product;*

(iv) *Let $L \in W_d^{r+1}(C)$, then $T_L W_d^r(C) = T_L \text{Pic}_C^d$;*

(v) *If $G_d^r(C)$ is smooth of dimension ρ , then $W_d^r(C)$ is Cohn-Macaulay, reduced and normal. If $d < g + r$ then $\text{Sing}(W_d^r(C)) = W_d^{r+1}(C)$.*

Proof. For (i),(ii), we refer [9] IV.4.1; for (iii),(iv), we refer [9] IV.4.2; for (v) we refer [9] IV.4.4. □

Chapter 5

Ramification and Plücker Formula

We will follow the sequences of exercises in [9] as Exercise I.C. Let C be a smooth projective curve of genus g over \mathbb{C} and we will describe the notion of ramification of a map $C \rightarrow \mathbb{P}^r$, or more generally, of a linear series on C , fixed as $L = (\mathcal{L}, V)$ be a \mathfrak{g}_d^r . We also fix a point $p \in C(\mathbb{C})$.

Lemma 5.0.1. *We have $\#\{\text{ord}_p \sigma : \sigma \in V \setminus \{0\}\} = r + 1$.*

Proof. There exists a basis for V consisting of sections with distinct orders of vanishing at p . To construct this basis, replace a pair of sections with the same vanishing order by two sections, one with the same order, and one with one higher order. \square

Definition 5.0.2. (i) *If we let these $r + 1$ numbers as $0 \leq a_0^L(p) < \dots < a_r^L(p) \leq d$, then the sequence $\{a_0^L(p), \dots, a_r^L(p)\}$ is called **vanishing sequence** of L at p ;*

(ii) *Let $\alpha_i^L(p) := a_i^L(p) - i$, then the sequence $\{\alpha_0^L(p), \dots, \alpha_r^L(p)\}$ is called **ramification sequence** of L at p . The weight $w^L(p)$ of p with respect to L is defined by*

$$w^L(p) = \sum_{i=0}^r \alpha_i^L(p) = \sum_{i=0}^r a_i^L(p) - \binom{r+1}{2};$$

(iii) *We say that L is unramified at p if $\{\alpha_0^L(p), \dots, \alpha_r^L(p)\} = \{0, \dots, 0\}$, else that p is a ramification point of L . If we consider the canonical series $(K_C, |K_C|)$, then the ramification points are called Weierstrass points.*

Remark 5.0.3. *These can be also defined over some singular curves and p be a smooth point.*

Lemma 5.0.4. *There are only finitely many ramification points of L on C .*

Proof. \square

Theorem 5.0.5 (Plücker Formula). *We have*

$$\sum_{p \in C} w^L(p) = (r+1)d + \binom{r+1}{2}(2g+2).$$

Proof. See [32] Proposition 1.1. \square

Part II

The basic theory of moduli space of curves

Chapter 6

\mathcal{M}_g be a Deligne-Mumford Stack for $g \neq 1$

Here we mainly consider the $g \neq 1$ curves.

Definition 6.0.1. Let \mathcal{M}_g be the fibered category over schemes with objects of form $(S, f : C \rightarrow S)$ where S be a scheme and f be a proper smooth morphism such that every geometric fiber of S is a connected genus g curve. The morphisms are base-change.

Our main result of this section is to prove that \mathcal{M}_g is a Deligne-Mumford stack for $g \geq 2$. For $g = 0$ we can run the same argument and when we just consider the stack over Sch/k where k be algebraically closed, we can get $\mathcal{M}_0 \cong BPGL_2$. Here we follows [1].

6.1 \mathcal{M}_g be a stack for $g \neq 1$

Lemma 6.1.1 (Descent for polarized schemes). Let $\mathcal{P}ol$ be the category whose objects are pairs $(f : X \rightarrow Y, L)$ where f is a proper flat morphism and L is a relatively ample invertible sheaf. The morphism are diagrams of cartesian with isomorphic pullback of line bundles. Consider the fibered category $\mathcal{P}ol \rightarrow (Sch)$, then it has effective fppf-descent.

Proof. See [75] 4.4.10. □

Theorem 6.1.2. For $g \neq 1$, the fibered category \mathcal{M}_g is a stack.

Proof. Consider $\mathcal{M}_g \rightarrow \mathcal{P}ol$ sends $C \rightarrow S$ to $(C \rightarrow S, \Omega_{C/S}^1)$ when $g \geq 2$ and $(C \rightarrow S, \Omega_{C/S}^{1, \otimes -1})$ when $g = 0$.

For a fppf covering $S' \rightarrow S$, then we get

$$\begin{array}{ccc} \mathcal{M}_g(S) & \longrightarrow & \mathcal{P}ol(S) \\ \downarrow & & \downarrow \cong \\ \mathcal{M}_g(S' \rightarrow S) & \longrightarrow & \mathcal{P}ol(S' \rightarrow S) \end{array}$$

Hence every object of $\mathcal{M}_g(S' \rightarrow S)$ is in the essential image of $\mathcal{M}_g(S)$. By the descent of sheaves used in $h_- \rightarrow h_+$ making it fully faithful. □

6.2 For $g \geq 2$, \mathcal{M}_g be a Deligne-Mumford stack

Now let $L_{C/S} = (\Omega_{C/S}^1)^{\otimes 3}$. By Lemma 2.1.1 (iv), for any family of smooth curves $p : D \rightarrow S$ we get a closed immersion $D \hookrightarrow \mathbb{P}(p_*L_{D/S})$ where $p_*L_{D/S}$ is locally free of rank $5g - 5$. Let $H = \underline{\text{Hilb}}_{\mathbb{P}^{5g-6}}^P$

where $P(t) = \deg(L_{C/S}^{\otimes t}) + 1 - g = (6g - 6)t + 1 - g$ be the Hilbert polynomial of $D_s \hookrightarrow \mathbb{P}_{\kappa(s)}^{5g-6}$. Let the universal closed subscheme:

$$\begin{array}{ccc} C & \hookrightarrow & H \times \mathbb{P}^{5g-6} \\ & \searrow \pi & \downarrow \\ & & H \end{array}$$

► **Claim 1.** There is a unique subscheme $H' \subset H$ consist of $h \in H$ such that

- (a) $C_h \rightarrow \text{Spec}(\kappa(h))$ is smooth and geometrically connected;
 - (b) $C_h \hookrightarrow \mathbb{P}_{\kappa(h)}^{5g-6}$ is embedded by complete linear system $|L_{C_h/\kappa(h)}|$;
 - (c) the line bundles $L_{C_{H'}/H'}$ and $\mathcal{O}_{C_{H'}}(1)$ differ by a pullback of a line bundle from H' (that is, there exists a line bundle N over H' such that $L_{C_{H'}/H'} \otimes p^*N = \mathcal{O}_{C_{H'}}(1)$).
- Moreover, if $T \rightarrow H$ be a morphism such that (a)-(c) hold for the family $C_T \rightarrow T$, then $T \rightarrow H$ factors through H' .

Since the condition that a fiber of a proper morphism (of finite presentation) is smooth is an open condition on the target, the condition on H that C_h is smooth is open. Consider the Stein factorization (St 03H0) $C \rightarrow \tilde{H} := \underline{\text{Spec}}_H \pi_* \mathcal{O}_C \rightarrow H$ where $C \rightarrow \tilde{H}$ is proper with geometrically connected fibres and $\tilde{H} \rightarrow H$ is finite. As $\mathcal{O}_H \rightarrow \pi_* \mathcal{O}_C$ is a morphism between coherent sheaves, then the kernel and cokernel of it have closed supports. Hence $\tilde{H} \rightarrow H$ is an isomorphism over an open subscheme of H , which is precisely where the fibers of $C \rightarrow H$ are geometrically connected. Hence the points satisfies (a) be a open subscheme of H , denoted by $H_1 \subset H$.

By Review A.1.2, there exists a locally closed subscheme $H_2 \subset H_1$ such that a morphism $T \rightarrow H_1$ factor through H_2 if and only if $L_{C_T/T}$ and $\mathcal{O}_{C_T}(1)$ differ by a pullback of a line bundle from T . In particular, (c) holds and for all $h \in H_2$, $L_{C_h/\kappa(h)} \cong \mathcal{O}_{C_h}(1)$.

For (b), let $\pi_2 : C_2 := C_{H_2} \rightarrow H_2$. Consider $\alpha : H^0(\mathbb{P}_{\mathbb{Z}}^{5g-6}, \mathcal{O}(1)) \otimes \mathcal{O}_{H_2} \rightarrow \pi_{2,*} \mathcal{O}_{C_2}(1)$ of vector bundles of rank $5g - 5$ on H_2 with fiber $\alpha_h : H^0(\mathbb{P}_{\kappa(h)}^{5g-6}, \mathcal{O}(1)) \rightarrow H^0(C_h, \mathcal{O}_{C_h}(1)) \cong H^0(C_h, L_{C_h/\kappa(h)})$. As they have the same rank, α_h is an isomorphism if and only if h is not in $\text{supp}(\text{coker}(\alpha))$. Let $H' = H_2 \setminus (\text{supp}(\text{coker}(\alpha)))$ and it satisfies (a)-(c) with that universal property.

► **Claim 2.** The group scheme $\text{PGL}_{5g-5} = \underline{\text{Aut}}(\mathbb{P}_{\mathbb{Z}}^{5g-6})$ act on H as: for $g \in \text{Aut}(\mathbb{P}_S^{5g-6})$ and $[D \subset \mathbb{P}_S^{5g-6}] \in H(S)$, we let $g \cdot [D \subset \mathbb{P}_S^{5g-6}] = [g(D) \subset \mathbb{P}_S^{5g-6}]$. As H' is PGL_{5g-5} -invariant, we claim that $\mathcal{M}_g \cong [H'/\text{PGL}_{5g-5}]$ be an algebraic stack. (See St 044O, St 04UV for quot stacks)

Consider $H' \rightarrow \mathcal{M}_g$ as $[D \subset \mathbb{P}_S^{5g-6}] \mapsto (D \rightarrow \mathbb{P}_S^{5g-6} \rightarrow S)$ is well defined by **Claim 1**. This morphism is PGL_{5g-5} -invariant, hence descends to $[H'/\text{PGL}_{5g-5}]^{\text{pre}} \rightarrow \mathcal{M}_g$ (**Why?**). We claim that this map is fully faithful. Indeed, for a family $p : D \rightarrow S$ in H' given by $D \subset \mathbb{P}_S^{5g-6}$, we get $\mathcal{O}_D(1) \cong L_{D/S} \otimes p^*M$ for some line bundle M on S . Use (b) we get

$$H^0(\mathbb{P}_{\mathbb{Z}}^{5g-6}, \mathcal{O}(1)) \otimes \mathcal{O}_S \rightarrow p_* \mathcal{O}_D(1) \cong p_*(L_{D/S} \otimes p^*M) \cong p_* L_{D/S} \otimes M$$

be an isomorphism. Then any automorphism of $D \rightarrow S$ induces an automorphism of $L_{D/S}$ and thus an automorphism of $p_* L_{D/S} \otimes M$, which induce an automorphism of \mathbb{P}_S^{5g-6} preserving D . By Theorem 6.1.2, \mathcal{M}_g be a stack, hence induce $[H'/\text{PGL}_{5g-5}] \rightarrow \mathcal{M}_g$ which is fully faithful since stackification is fully faithful. Finally we check that $[H'/\text{PGL}_{5g-5}] \rightarrow \mathcal{M}_g$ is essentially surjective. As these are all stacks, then they satisfied effective descent of étale covering. Hence we just need to show that for any $p : D \rightarrow S$, there exists an étale covering $\{S_i \rightarrow S\}$ such that each D_{S_i} is in the image of $H'(S_i) \rightarrow \mathcal{M}_g(S_i)$. Actually since $L_{D/S}$ defined $D \hookrightarrow \mathbb{P}(p_* L_{D/S})$ and $p_* L_{D/S}$ is locally free of rank $5g - 5$, we let $\{S_i\}$ be open (zariski, hence étale) covering of S such that $(p_* L_{D/S})|_{S_i}$ are all free. Well done.

► **Claim 3.** The algebraic stack \mathcal{M}_g is a Deligne-Mumford stack.

By Theorem C.1.1, we just need to show for any smooth connected proper curve C over a algebraic closed field k , the group scheme $G := \underline{\text{Aut}}_k(C) = \text{Aut}(C)$ is finite and reduced. We find that $T_{G,e}$ can be identified with the automorphism group of the trivial first order deformation of C . Hence by Proposition 9.1.6, we get $T_{G,e} = H^0(C, T_C) = 0$, well done.

6.3 First properties of \mathcal{M}_g for $g \geq 2$

Proposition 6.3.1. *As $\mathcal{M}_g \cong [H'/\mathrm{PGL}_{5g-5}]$ and H' is locally of finite type, then \mathcal{M}_g is locally of finite type over \mathbb{Z} . As H' is noetherian, so is \mathcal{M}_g . So it is finite type over \mathbb{Z} .*

Proposition 6.3.2. *\mathcal{M}_g have affine diagonal. Indeed, since we have $\mathcal{M}_g \cong [H'/\mathrm{PGL}_{5g-5}]$ which is an algebraic stack, then we have cartesian square*

$$\begin{array}{ccc} H' \times \mathrm{PGL}_{5g-5} & \longrightarrow & H' \times H' \\ \downarrow & & \downarrow \\ \mathcal{M}_g & \longrightarrow & \mathcal{M}_g \times \mathcal{M}_g \end{array}$$

As PGL_{5g-5} affine, then $\mathrm{PGL}_{5g-5} \times H' \rightarrow H' \times H' \rightarrow H'$ affine, so is $\mathrm{PGL}_{5g-5} \times H' \rightarrow H' \times H'$.

6.4 Smoothness and dimension of \mathcal{M}_g for $g \geq 2$

Proposition 6.4.1. *If C is a smooth connected projective curve of genus $g \geq 2$ over k , then $\dim T_{\mathcal{M}_g, [C]} = 3g - 3$.*

Proof. By Proposition 9.1.2, we get $T_{\mathcal{M}_g, [C]} = H^1(C, T_C)$. As $\deg T_C < 0$, we get $H^0(C, T_C) = 0$. So by Riemann-Roch we get

$$\dim T_{\mathcal{M}_g, [C]} = \dim H^1(C, T_C) = -\chi(T_C) = -\deg T_C + g - 1 = 3g - 3,$$

well done. □

Theorem 6.4.2. *For $g \geq 2$, the Deligne-Mumford stack \mathcal{M}_g is smooth over \mathbb{Z} of relative dimension $3g - 3$.*

Proof. Let a field k and a smooth projective connected curve $C \rightarrow \mathrm{Spec}(k)$. Consider the following 2-diagram:

$$\begin{array}{ccccc} & & [C] & & \\ & \searrow & \text{---} & \searrow & \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A_0) & \longrightarrow & \mathcal{M}_g \\ & & \downarrow & & \downarrow f \\ & & \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}) \end{array}$$

where $A \rightarrow A_0$ be a surjective maps of artinian local rings with residue field k with $k = \ker(A \rightarrow A_0)$. The map $\mathrm{Spec}(A_0) \rightarrow \mathcal{M}_g$ corresponds to a family of curves $C_0 \rightarrow \mathrm{Spec}(A_0)$ and a cartesian:

$$\begin{array}{ccccc} C & \longleftarrow & C_0 & \dashrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A_0) & \longleftarrow & \mathrm{Spec}(A) \end{array}$$

of solid arrows. So to find the lifting, we just need to find the dashed arrows, that is, deformation of C along A . By Proposition 9.1.6(iii), there exists a cohomology class $ob_C \in H^2(C, T_C)$ such that this happens if and only if $ob_C = 0$. Hence this is right as C be a curve. Hence \mathcal{M}_g is smooth. By Theorem C.1.2, we get

$$\dim_{[C]} \mathcal{M}_g = \dim T_{\mathcal{M}_g, [C]} - \dim \mathrm{Aut}(C).$$

By the final step of the proof of the DM-ness of \mathcal{M}_g , we get $\dim \mathrm{Aut}(C) = 0$. Hence $\dim_{[C]} \mathcal{M}_g = \dim T_{\mathcal{M}_g, [C]} = 3g - 3$, well done. □

6.5 For $g = 0$

By the same argument we can get \mathcal{M}_0 be a Deligne-Mumford stack.

Proposition 6.5.1. \mathcal{M}_0 may (not) be a stack over \mathbb{Z} isomorphic to $BPGL_2$.

Analysis. We should repeat the proof of the case $g \geq 2$. But when we consider $L_{C/S} = (\Omega_{C/S}^1)^{\otimes(-1)}$ for $f : C \rightarrow S$, we get $\deg(L_{C_s/\kappa(s)}) = 2$ and $f_*L_{C/S}$ is locally free of rank 3, which induce $C \hookrightarrow \mathbb{P}_S^2$. And with the Hilbert polynomial $p(t) = 2t + 1$. So we will use PGL_3 and get $\mathcal{M}_0 \cong [H'/PGL_3]$ for some subscheme $H' \subset \underline{\text{Hilb}}_{\mathbb{P}^2}^{p(t)}$!

Actually by St 0C6U, we can not identify C_s over some $\kappa(s)$ with $\mathbb{P}_{\kappa(s)}^1$ since we may have **no invertible sheaves of odd degree!** If we can assume this, we have M of degree 1. The Hilbert polynomial $p(t) = t + 1$ in \mathbb{P}^1 and can show that it is \mathbb{P}^1 by some tricks (actually this is right for all linear subspaces). Hence

$$\mathcal{M}_0 \cong [\underline{\text{Hilb}}_{\mathbb{P}^1}^{p(t)}/PGL_2] = [\text{Grass}(2, 2)/PGL_2] = BPGL_2,$$

well done. □

Corollary 6.5.2. When we consider the stack over Sch/k for any algebraically closed field k , we have $\mathcal{M}_0 \cong BPGL_2$.

Proof. Actually here we have invertible sheaves of degree 1. Hence any proper smooth curve of genus 0 be \mathbb{P}_k^1 . The Hilbert polynomial $p(t) = t + 1$ in \mathbb{P}_k^1 and can show that it is \mathbb{P}_k^1 by some tricks. Hence

$$\mathcal{M}_0 \cong [\underline{\text{Hilb}}_{\mathbb{P}_k^1}^{p(t)}/PGL_2] = [\text{Grass}_k(2, 2)/PGL_2] = BPGL_2,$$

well done. □

Chapter 7

Nodal curves

7.1 Basic facts of nodal curves

The the more details, see St 0C46.

Definition 7.1.1 (Nodes). *Let C be a curve over k .*

- (a) *If k algebraically closed, we say that $p \in C(k)$ is a node is we have $\widehat{\mathcal{O}}_{C,p} \cong k[[x, y]]/(xy)$;*
- (b) *If k need not be algebraically closed, we say a closed point $p \in C$ is a node if there exists a node $p' \in C_{\bar{k}}$ over p .*

We say C be a nodal curve if every closed point is either smooth or nodal.

Proposition 7.1.2. *Let C be a curve over k . Consider the following statements.*

- (a) *$p \in C$ is a node;*
- (b) *$\kappa(p)/k$ is separable, $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[[x, y]]/(ax^2 + bxy + cy^2)$ as a k -algebra where $ax^2 + bxy + cy^2$ is a nondegenerate quadratic form over $\kappa(p)$;*
- (c) *$\kappa(p)/k$ is separable and $\mathcal{O}_{C,p}$ is reduced, has δ -invariant 1.*

Then we have (a) \Leftrightarrow (b) \Rightarrow (c).

Proof. See St 0C49 and St 0C4D.

We assume (a) \Leftrightarrow (b). Here by Lemma 3.1.3, we just need to consider the case $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[[x, y]]/(ax^2 + bxy + cy^2)$ where $Q = ax^2 + bxy + cy^2$ is a nondegenerate quadratic form.

Case (I): If Q is split, we may let $\mathcal{O}_{C,p} \cong \kappa(p)[[x, y]]/(xy)$ after some coordinate transformation. Then we get

$$\widetilde{\mathcal{O}}_{C,p} \cong \kappa(p)[[x, y]]/(x) \times \kappa(p)[[x, y]]/(y);$$

Case (II): If not, in this case $c \neq 0$ and nondegenerate means $b^2 - 4ac \neq 0$. Hence $\kappa' = \kappa(p)[t]/(a + bt + ct^2)$ is a degree 2 separable extension of $\kappa(p)$. Then $t = y/x$ is integral over $\mathcal{O}_{C,p}$. and we conclude that

$$\widetilde{\mathcal{O}}_{C,p} = \kappa'[[x]]$$

with y mapping to tx on the right hand side.

In both cases one verifies by hand that the δ -invariant is 1, well done. □

Remark 7.1.3. (i) *As for a node $p \in C$ in a nodal curve C , we have $\kappa(p)/k$ is separable. As the two cases above, if p is of case (I), then $f^{-1}(p)$ has two points with residue fields $\kappa(p)$. If p is of case (II), then $f^{-1}(p)$ has only one point with residue field κ' , a degree 2 separable extension of $\kappa(p)$;*

(ii) *As in (i), all closed points of \widetilde{C} is regular with separable residue fields over k . Hence \widetilde{C} is smooth over k by St 00TV.*

Proposition 7.1.4. *If C is a curve over k and $p \in C$ be a node. Then exists a finite separable field extension K/k , a point $P \in C_K$ over p and $\widehat{\mathcal{O}}_{C_K,P} \cong K[[x, y]]/(xy)$.*

Proof. By Proposition 7.1.2(b), we get $\kappa(p)/k$ is separable, $\widehat{\mathcal{O}}_{C,p} \cong \kappa(p)[[x, y]]/(ax^2 + bxy + cy^2)$ as a k -algebra where $Q = ax^2 + bxy + cy^2$ is a nondegenerate quadratic form over $\kappa(p)$. If Q is split, well done. If not, let $K = k[t]/(at^2 + bt + c)$ be a separable extension over k with Q split, well done. □

7.2 Genus fomula

Let k be algebraically closed field now. Let C be a connected nodal projective curve over k . Let z_1, \dots, z_s be its nodes and C_1, \dots, C_t be its irreducible components.

By Proposition A.3.3(1) and (4), we get $\tilde{C} = \coprod_{i=1}^t \tilde{C}_i$ where \tilde{C}, \tilde{C}_i are normalizations. Let $f : \tilde{C} \rightarrow C$. By Proposition 3.1.4, we get a exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow f_* \mathcal{O}_{\tilde{C}} \rightarrow \bigoplus_{i=1}^s \mathcal{Q}_i \rightarrow 0$$

where \mathcal{Q}_i supported over z_i . Since by Proposition 7.1.2(c), we get $\mathcal{Q}_i = \kappa(z_i)$ as the δ -invariant are all 1, hence we get

$$0 \rightarrow \mathcal{O}_C \rightarrow f_* \mathcal{O}_{\tilde{C}} \rightarrow \bigoplus_{i=1}^s \kappa(z_i) \rightarrow 0.$$

Hence we get long exact sequence

$$0 \rightarrow \underbrace{H^0(C, \mathcal{O}_C)}_1 \rightarrow \underbrace{H^0(\tilde{C}, \mathcal{O}_{\tilde{C}})}_t \rightarrow \underbrace{\bigoplus_{i=1}^s \kappa(z_i)}_s \rightarrow \underbrace{H^1(C, \mathcal{O}_C)}_{g(C)} \rightarrow \underbrace{H^1(\tilde{C}, \mathcal{O}_{\tilde{C}})}_{\sum_{i=1}^t g(\tilde{C}_i)} \rightarrow 0$$

with the labels underneath indicating the dimensions.

Theorem 7.2.1 (Genus fomula). *With the situation as above, we get*

$$g(C) = \sum_{i=1}^t g(\tilde{C}_i) + s - t + 1.$$

Proof. Trivial by the argument above. □

7.3 The dualizing sheaf

We have three way to see this. Consider C be a fixed nodal curve over k .

7.3.1 The first way

We find that C is locally complete intersection as we can checking locally. As for a node $p \in C$, we have $\hat{\mathcal{O}}_{C,p} \cong \kappa(p)[[x, y]]/(ax^2 + bxy + cy^2)$ for some nondegenerate quadratic form. By [72] Theorem 21.2(iii), we get $\hat{\mathcal{O}}_{C,p}$ is a complete intersection over k . Hence by [58] Theorem III.7.11 (adjunction formula for l.c.i), if we embedding it into \mathbb{P}^N , then we have $\omega_C \cong \omega_{\mathbb{P}^N} \otimes \bigwedge^{N-1} (\mathcal{I}/\mathcal{I}^2)$ where \mathcal{I} be the ideal sheaf. As this is locally complete intersection, this is a line bundle.

7.3.2 The second way

This is an abstract way of duality theory, see St 0E31 for more details. As C is locally complete intersection, then by St 0BVA we get C is Gorenstein. By St 0BS2, C must have a dualizing complex ω_C^* . By 0BFQ, as C is Gorenstein, ω_C^* is invertible. By C Cohen-Macaulay, $\omega_C^* = \omega_C[0]$. Hence we win.

7.3.3 The third way

We can explicit ω_C precisely. Let Σ be the set of nodes of C and let $U = C \setminus \Sigma$. Let the normalization $f : \tilde{C} \rightarrow C$ and $\tilde{\Sigma} := f^{-1}(\Sigma), \tilde{U} := f^{-1}(U)$. Now \tilde{C} is smooth, then we have the dualizing sheaf (line bundle) $\Omega_{\tilde{C}}$. We get

$$0 \rightarrow \Omega_{\tilde{C}} \rightarrow \Omega_{\tilde{C}}(\tilde{\Sigma}) \rightarrow \mathcal{O}_{\tilde{\Sigma}} \rightarrow 0.$$

Actually the sections of $\Omega_{\tilde{C}}(\tilde{\Sigma})$ is the rational sections of $\Omega_{\tilde{C}}$ with at worst simple poles in $\tilde{\Sigma}$. Hence for any open $V \subset \tilde{C}$ and $y \in V \cap \tilde{\Sigma}$ we have the residue $\text{res}_y : \Gamma(V, \Omega_{\tilde{C}}(\tilde{\Sigma})) \rightarrow \kappa(y)$.

Definition 7.3.1. We define the subsheaf $\omega_C \subset f_*\Omega_{\tilde{C}}(\tilde{\Sigma})$ as for any open $V \subset C$ we have

$$\Gamma(V, \omega_C) = \left\{ \begin{array}{l} s \in \Gamma(f^{-1}(V), \Omega_{\tilde{C}}(\tilde{\Sigma})) : \text{for any } z_i \in V \cap \Sigma \\ \text{and } f^{-1}(z_i) = \{p_i, q_i\} \text{ with } \text{res}_{p_i}(s) + \text{res}_{q_i}(s) = 0 \end{array} \right\}.$$

Hence we get two exact sequences

$$\begin{array}{ccccccc} 0 & \rightarrow & \omega_C & \rightarrow & f_*\Omega_{\tilde{C}}(\tilde{\Sigma}) & \longrightarrow & \bigoplus_{z_i \in \Sigma} \kappa(z_i) \longrightarrow 0 \\ & & & & s \longmapsto & & (\text{res}_{p_i}(s) - \text{res}_{q_i}(s)) \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \rightarrow & f_*\Omega_{\tilde{C}} & \rightarrow & \omega_C & \rightarrow & \bigoplus_{z_i \in \Sigma} \kappa(z_i) \rightarrow 0 \\ & & & & s \longmapsto & & (\text{res}_{p_i}(s)) \end{array}$$

Proposition 7.3.2. Let C be a nodal curve C over k .

- (a) If $g : C' \rightarrow C$ be an étale morphism, then $g^*\omega_C \cong \omega_{C'}$;
- (b) Conclude that ω_C be a line bundle.

Proof. (a) As the normalization commutes with étale base change (see St 03GE), we have the cartesian with normalizations

$$\begin{array}{ccc} \tilde{C}' & \xrightarrow{g'} & \tilde{C} \\ \downarrow f' & & \downarrow f \\ C' & \xrightarrow{g} & C \end{array}$$

By flat base change, we have the process $g^*\omega_C \subset g^*f_*\Omega_{\tilde{C}}(\tilde{\Sigma}) \cong f'_*(g')^*\Omega_{\tilde{C}}(\tilde{\Sigma}) = f'_*\Omega_{\tilde{C}'}(\tilde{\Sigma}')$. By definition and this process, we get $g^*\omega_C \cong \omega_{C'}$.

(b) Use Corollary A.2.3 and Proposition 7.1.4, there exists a separable extension K/k such that we get the common étale neighborhood as

$$\begin{array}{ccc} & (U, u) & \\ F \swarrow & & \searrow G \\ (C, p) & & (\text{Spec}K[x, y]/(xy), 0) \end{array}$$

Let $D = \text{Spec}K[x, y]/(xy)$ and normalization $\tilde{D} \cong \mathbb{A}_K^1 \sqcup \mathbb{A}_K^1$. Then $\Gamma(\tilde{D}, \Omega_{\tilde{D}}) = \Gamma(\mathbb{A}_K^1, \omega_{\mathbb{A}_K^1}) \times \Gamma(\mathbb{A}_K^1, \omega_{\mathbb{A}_K^1})$ and $(\frac{dx}{x}, -\frac{dy}{y})$ be a section of ω_D . As any section is of form $(f(x)\frac{dx}{x}, -g(y)\frac{dy}{y})$ where $f(0) = g(0)$, which is precisely the condition for $(f, g) \in \Gamma(\tilde{D}, \mathcal{O}_{\tilde{D}})$ to descend to a global function on D . In other words, $\omega_D \cong \mathcal{O}_D$ with generator $(\frac{dx}{x}, -\frac{dy}{y})$. By (a), we get $\omega_U = G^*\omega_D$, hence ω_U is a line bundle. As $F^*\omega_C = \omega_U$ be a line bundle, we use the descent theory and we win. \square

Proposition 7.3.3. *Let C be a proper nodal curve C over k , then ω_C be the dualizing line bundle of C .*

Proof. (See [8]) We may assume that k is algebraic closed. Choose a divisor $D = r_1 + \dots + r_h$ consisting of distinct smooth points of C , with the property that any component of C contains at least one of the r_i 's. We first claim that $H^1(\omega_C(D)) = 0$. Indeed, we get an exact sequence

$$0 \rightarrow (f_*\omega_{\tilde{C}}) \otimes \mathcal{O}_C(D) = f_*(\omega_{\tilde{C}}(D)) \rightarrow \omega_C(D) \rightarrow \bigoplus_{\text{nodes}} k \rightarrow 0.$$

Hence deduce a surjection

$$H^1(C, f_*(\omega_{\tilde{C}}(D))) = H^1(\tilde{C}, \omega_{\tilde{C}}(D)) \rightarrow H^1(C, \omega_C(D)).$$

As \tilde{C} is smooth, we get for any irreducible components and Serre duality in smooth case, we get $H^1(\tilde{C}, \omega_{\tilde{C}}(D)) = 0$ as D meets every irreducible components. Hence $H^1(\omega_C(D)) = 0$.

Next we deduce an exact sequence by using the claim as

$$H^0(C, \omega_C(D)) \rightarrow H^0(C, \omega_C(D)/\omega_C) \rightarrow H^1(C, \omega_C) \rightarrow H^1(C, \omega_C(D)) = 0.$$

For any $\phi \in H^1(C, \omega_C)$ we have some lifts $\phi' \in H^0(C, \omega_C(D)/\omega_C)$. We define the trace map as

$$\text{tr}_C : H^1(C, \omega_C(D)) \rightarrow k, \phi \mapsto 2\pi\sqrt{-1} \sum_{i=1}^l \text{res}_{r_i} \phi'$$

and this is well defined by using residue theorem (of definition). Perfect pairing is omitted. \square

Proposition 7.3.4. *Let C be a nodal curve C over k and $T \subset C$ be an irreducible component and D_T be the union of the intersections of T and another irreducible components, then $\omega_C|_T = \omega_T(D_T)$.*

Proof. Trivial by definition of the dualizing sheaves. \square

7.4 Local structure of nodes

Theorem 7.4.1 (Local structure of nodes). *Let $\pi : C \rightarrow S$ be a flat and finitely presented morphism such that every geometric fiber is a curve. Let $p \in C$ be a node in C_s . Then we have a following diagram*

$$\begin{array}{ccccc} (C, p) & \xleftarrow{\text{\acute{e}t}} & (U, u) & \xrightarrow{\text{\acute{e}t}} & (\text{Spec}A[x, y]/(xy - f), 0) \\ & \downarrow & \downarrow & \swarrow & \\ (S, s) & \xleftarrow{\text{\acute{e}t}} & (\text{Spec}A, s') & & \end{array}$$

where each horizontal arrow is a residually-trivial pointed étale morphism and $f \in A$ is a function vanishing at s' .

Sketch. See [1] 5.2.12 or St 0CBY for more details.

Step 1. Reduce to S of finite type over \mathbb{Z} . Using noetherian approximation.

Step 2. Reduce to the case where $\hat{\mathcal{O}}_{C_s, p} \cong \kappa(s)[[x, y]]/(xy)$. Just need to use Proposition 7.1.4 and since separable, we can find a étale neighborhood (S', s') such that $\kappa(s') = K$.

Step 3. Show that $\hat{\mathcal{O}}_{C, p} \cong \hat{\mathcal{O}}_{S, s}[[x, y]]/(xy - f)$ where $f \in \hat{\mathfrak{m}}_s$. Using the Schlessinger's theorem in formal deformation theory to deduce a diagram similar as what we want at the completion level.

Step 4. Apply Artin approximation (Theorem A.2.2). Using Artin approximation to deduce our diagram from the completion level. \square

Corollary 7.4.2. *Let $\pi : C \rightarrow S$ be a flat and finitely presented morphism such that every geometric fiber is a curve, then the locus $C^{\leq \text{nod}} = \{p \in C : p \in C_{\pi(p)} \text{ either smooth or node}\} \subset C$ is open.*

Proof. First we know the smooth locus is open. If $p \in C_{\pi(p)}$ is a node, then by Theorem 7.4.1 we get an étale morphism $g : (U, u) \rightarrow (C, p)$. Then $p \in g(U) \subset C^{\leq nod}$ is open. \square

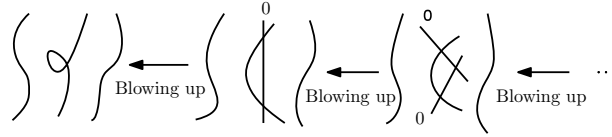
Corollary 7.4.3. *Let $\pi : C \rightarrow S$ be a proper flat and finitely presented morphism such that every geometric fiber is a curve, then the locus $S^{\leq nod} = \{s \in S : C_s \text{ is nodal}\} \subset S$ is open.*

Proof. As we find that

$$S^{\leq nod} = S \setminus \pi(C \setminus C^{\leq nod}).$$

By the previous Corollary and π is proper, then $S^{\leq nod}$ is open. \square

Remark 7.4.4. *Actually later we can prove that the stack $\mathcal{M}_g^{\leq nod}$ is an algebraic stack. But the main problem of $\mathcal{M}_g^{\leq nod}$ is that it is not separated and not of finite type. We can see the figure below for intuitive understanding:*



Corollary 7.4.5 (Comparison). *Let's compare $\omega_{C/Y}$ and $\Omega_{C/Y}^1$ where $\phi : C \rightarrow Y$ are a family of complex nodal curves. We will follow [8] X.2 and more general we can see [70] 6.4.2. Also, we will work on the complex topology.*

Pick a node p in some fiber, then by Theorem 7.4.1 we get near p we have the composition $\phi|_U : U \hookrightarrow \mathbb{C}^2 \times Y \rightarrow Y$ where U defined by $F := xy - f$. By adjunction formula we get the local generator of $\omega_{C/Y}$ is $F^{-1}dx \wedge dy \pmod{F}$. Using [70] Lemma 6.4.12, we get a homomorphism

$$\rho : \Omega_{C/Y}^1 \rightarrow \omega_{C/Y}$$

given by id if it near smooth points and $\rho(\alpha) = F^{-1}\alpha' \wedge dF \pmod{F}$ if near the nodes where α' is on $\mathbb{C}^2 \times Y \rightarrow Y$ restriction is α . Actually near nodes we have $\rho(dx) = xF^{-1}dx \wedge dy$ and $\rho(dy) = -yF^{-1}dx \wedge dy$. Now we consider

$$0 \rightarrow \ker \rho \rightarrow \Omega_{C/Y}^1 \xrightarrow{\rho} \omega_{C/Y} \rightarrow \text{coker} \rho \rightarrow 0.$$

• **Claim 1.** $\rho(\Omega_{C/Y}^1) = \mathcal{I}\omega_{C/Y}$ where \mathcal{I} be the ideal locally generated by x, y (locally ideal of that node).

Let S be the subspace correspond to \mathcal{I} , then for now $\text{coker} \rho = \omega_{C/Y} \otimes \mathcal{O}_S$. As locally near nodes we get $xy = f$ and $\rho(\Omega_{C/Y}^1)$ generated by $xF^{-1}dx \wedge dy$ and $yF^{-1}dx \wedge dy$, then \mathcal{I} be the ideal locally generated by x, y .

• **Claim 2.** *When Y be a single point, then $\ker \rho$ is the one-dimensional complex vector space generated by the class of $xdy = -ydx$.*

This is trivial by this construction.

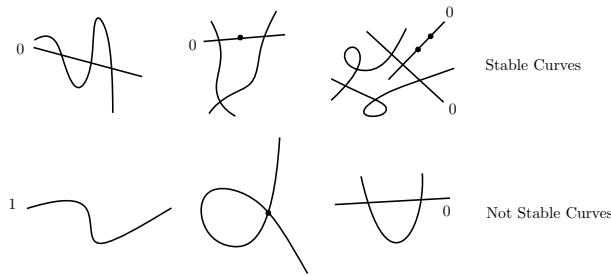
• **Claim 3.** *When Y is integral and generic fiber of ϕ is smooth, then ρ is injective.*

Chapter 8

Stable curves

8.1 Basic facts of stable curves

An n -pointed curve is a curve C over a field k together with an ordered collection of k -rational points $p_1, \dots, p_n \in C$ which we call the marked points. A point $q \in C$ of an n -pointed curve is called special if q is a node or a marked point.



Definition 8.1.1. A n -pointed curve (C, p_1, \dots, p_n) over k is prestable if it is a geometrically connected, nodal and projective curve, and $p_1, \dots, p_n \in C(k)$ are distinct smooth points.

A n -pointed curve (C, p_1, \dots, p_n) over k is semistable if

- (a) it is prestable;
- (b) every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 2 special points;
- (c) C is not of genus 1 without marked points.

A n -pointed curve (C, p_1, \dots, p_n) over k is stable if

- (a) it is semistable;
- (b) every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 3 special points.

Remark 8.1.2. (1) Note that there are no n -pointed stable curve of genus g if $2g - 2 + n \leq 0$ by Proposition 8.1.4. We will often impose the condition that $2g - 2 + n > 0$ in order to exclude these special cases;

(2) An automorphism of a stable curve (C, p_1, \dots, p_n) is an automorphism $\alpha : C \rightarrow C$ such that $\alpha(p_i) = p_i$. We denote by $\text{Aut}(C, p_1, \dots, p_n)$ the group of automorphisms;

(3) For some general Riemann Roch theorem (such as OBS6) and the fact that the prestable curves are proper geometrically connected and geometrically reduced, then $\deg(\omega_C) = 2g - 2$.

Proposition 8.1.3. Let (C, p_1, \dots, p_n) be an n -pointed nodal projective curve such that the points p_i are distinct and smooth. Let $\pi : \tilde{C} \rightarrow C$ be the normalization and $\tilde{p}_i \in \tilde{C}$ be the unique preimage of p_i and $\tilde{q}_1, \dots, \tilde{q}_m \in \tilde{C}$ be an ordering of the preimages of nodes. Then

- (a) (C, p_1, \dots, p_n) is stable if and only if every connected component of $(\tilde{C}, \{\tilde{p}_i\}, \{\tilde{q}_j\})$ is stable.
- (b) The group scheme $\underline{\text{Aut}}(C, \{p_i\})$ is an algebraic group.
- (c) $\underline{\text{Aut}}(C, \{p_i\})$ is naturally a closed scheme of $\underline{\text{Aut}}(\tilde{C}, \{\tilde{p}_i\}, \{\tilde{q}_j\})$ with the same connected component of identity.

Proof. (a) Easy to see that we just need to verify that every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 3 special points if and only if every connected component of $(\tilde{C}, \{\tilde{p}_i\}, \{\tilde{q}_j\})$ have the same property. This is also trivial as we just need to consider the rational component of $(\tilde{C}, \{\tilde{p}_i\}, \{\tilde{q}_j\})$ and using the genus formula.

- (b)
(c) □

Proposition 8.1.4. *Let (C, p_1, \dots, p_n) be an n -pointed prestable curve. The following are equivalent*

- (i) (C, p_1, \dots, p_n) is stable;
(ii) $\text{Aut}(C, p_1, \dots, p_n)$ is finite; and
(iii) $\omega_C(p_1 + \dots + p_n)$ is ample.

Proof. (i) \Leftrightarrow (ii). By the results in Section 1.2 we get for smooth connected projective curve, its automorphism group is finite if and only if $2g - 2 + n > 0$ for (g, n) . Now consider the normalization $f : (\tilde{C}, \{\tilde{p}_i\}_{i=1}^n, \{\tilde{q}_j\}_{j=1}^{2s}) \rightarrow (C, p_1, \dots, p_n)$ with $\tilde{C} = \coprod_{j=1}^t \tilde{C}_j$. By Proposition 8.1.3 (a), we have (i) \Leftrightarrow for all j , $(\tilde{C}_j, \{\tilde{p}_i \in \tilde{C}_j\}_{i=1}^n, \{\tilde{q}_k \in \tilde{C}_j\}_{k=1}^{2s})$ is stable. As all \tilde{C}_j have marked points and use Proposition 8.1.3 (c), (ii) \Leftrightarrow for all j , $\text{Aut}(\tilde{C}_j, \{\tilde{p}_i \in \tilde{C}_j\}_{i=1}^n, \{\tilde{q}_k \in \tilde{C}_j\}_{k=1}^{2s})$ are finite. Hence by the case of smooth case, we win.

(i) \Leftrightarrow (iii). By Proposition A.3.4, 7.3.4 and consider the normalization $\pi : \tilde{C} \rightarrow C$, we get $\omega_C(p_1 + \dots + p_n)$ is ample if and only if $\pi^* \omega_C(p_1 + \dots + p_n)$ is ample if and only if for any irreducible components $T \subset \tilde{C}$, $\omega_C(p_1 + \dots + p_n)|_T = \omega_T(\sum_{p_i \in T} p_i + D_T)$ is ample. This latter condition holds precisely if each $\mathbb{P}^1 \subset \tilde{C}$ contains at least three points that lie over nodes or marked points (using Theorem 1.1.2) and we win. □

8.2 Positivity of the dualizing sheaf

Theorem 8.2.1. *For any n -pointed stable curve (C, p_1, \dots, p_n) , the bundle $(\omega_C(p_1 + \dots + p_n))^{\otimes k}$ is very ample for $k \geq 3$.*

Proof. We just prove the case of k is algebraically closed and no marked points. In this case we just need to show that its sections separates points and tangent vectors. We just need to show

- (i) for all closed points $x \neq y$, we have surjection

$$H^0(C, \omega_C^{\otimes k}) \rightarrow H^0(C, (\omega_C^{\otimes k} \otimes \kappa(x)) \oplus (\omega_C^{\otimes k} \otimes \kappa(y)));$$

- (ii) for all closed point x , we have surjection

$$H^0(C, \omega_C^{\otimes k}) \rightarrow H^0(C, \omega_C^{\otimes k} \otimes \mathcal{O}_C/I_x^2).$$

Hence for $x, y \in C(k)$ (maybe the same points) and their ideal I_x, I_y , we have

$$0 \rightarrow \omega_C^{\otimes k} \otimes I_x I_y \rightarrow \omega_C^{\otimes k} \rightarrow \omega_C^{\otimes k} \otimes \mathcal{O}_C/I_x I_y \rightarrow 0.$$

So we just need to show that $H^1(C, \omega_C^{\otimes k} \otimes I_x I_y) = 0$. By Serre duality, we need to show

$$\begin{aligned} H^1(C, \omega_C^{\otimes k} \otimes I_x I_y) &= H^0(C, (\omega_C^{\otimes k} \otimes I_x I_y)^\vee \otimes \omega_C) \\ &= H^0(C, \mathcal{H}om(\omega_C^{\otimes k} \otimes I_x I_y, \omega_C)) = \text{Hom}(I_x I_y, \omega_C^{\otimes(1-k)}) = 0. \end{aligned}$$

We need a case analysis on whether x, y are smooth or nodal.

If $x \in C$ is smooth, then $I_x = \mathcal{O}_C(-x)$ is invertible. If $x \in C$ is a node, consider the blowing up $\pi : C' \rightarrow C$ along x with $\pi^{-1}(x) = \{x_1, x_2\}$. Then for any line bundle L on C we claim that

$$\text{Hom}(I_x, L) \cong H^0(C', \pi^* L), \text{Hom}(I_x^2, L) \cong H^0(C', \pi^* L(x_1 + x_2)).$$

We just prove the first statement, the second is similar. First we have

$$0 \rightarrow \mathcal{O}_C \rightarrow \pi_* \mathcal{O}_{C'} \rightarrow \kappa(x) \rightarrow 0,$$

tensoring L we get

$$0 \rightarrow L \rightarrow \pi_* \mathcal{O}_{C'} \otimes L = \pi_* \pi^* L \rightarrow L(x) \rightarrow 0.$$

Hence we have

$$0 \rightarrow \mathrm{Hom}(I_x, L) \rightarrow \mathrm{Hom}(\pi^* I_x, \pi^* L) \xrightarrow{f} \mathrm{Hom}(I_x, L(x)) = \mathrm{Hom}(I_x/I_x^2, L(x)).$$

On the other hand, we have a short exact sequence

$$0 \rightarrow \pi^* L \rightarrow \pi^* L(x_1 + x_2) \rightarrow \pi^* L(x_1) \oplus \pi^* L(x_2) \rightarrow 0$$

inducing

$$0 \rightarrow H^0(C', \pi^* L) \rightarrow H^0(C', \pi^* L(x_1 + x_2)) \xrightarrow{g} \pi^* L(x_1) \oplus \pi^* L(x_2).$$

Let $J = \mathcal{O}_{C'}(-x_1 - x_2) \subset \mathcal{O}_{C'}$, we get

$$0 \rightarrow K \rightarrow \pi^* I_x \rightarrow J \rightarrow 0$$

where $\mathrm{supp}(K) = \{x_1, x_2\}$ by checking locally. Since $\pi^* L$ is torsion free at x_1, x_2 , we have $\mathrm{Hom}(K, \pi^* L) = 0$, so this defines an isomorphism

$$\mathrm{Hom}(\pi^* I_x, \pi^* L) \cong \mathrm{Hom}(J, \pi^* L) \cong H^0(C', \pi^* L(x_1 + x_2)).$$

We also have isomorphisms $I_x/I_x^2 \cong \pi_*(J/J^2)$ with $\mathrm{Hom}(I_x/I_x^2, L(x)) \cong \mathrm{Hom}(\pi_*(J/J^2), L(x))$. Actually $\mathrm{Hom}(\pi_*(J/J^2), L(x)) \cong \pi^* L(x_1) \oplus \pi^* L(x_2)$ (**Why?**). Hence conclude these, we get

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(I_x, L) & \longrightarrow & \mathrm{Hom}(\pi^* I_x, \pi^* L) & \longrightarrow & \mathrm{Hom}(I_x/I_x^2, L(x)) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & H^0(C', \pi^* L) & \longrightarrow & H^0(C', \pi^* L(x_1 + x_2)) & \longrightarrow & \pi^* L(x_1) \oplus \pi^* L(x_2) \end{array}$$

Hence the claim is right.

Case (I). If x, y are all smooth points, then $\deg(\omega_C^{\otimes(1-k)}(x+y)) = (1-k)(2g-2) + 2 < 0$ for $k \geq 3$. Hence

$$\mathrm{Hom}(I_x I_y, \omega_C^{\otimes(1-k)}) = H^0(C, \omega_C^{\otimes(1-k)}(x+y)) = 0.$$

Case (II). If x is a node and y is a smooth point, then by the claim, we win.

Case (III.1). If $x = y$ is a node, then by the claim we get

$$\mathrm{Hom}(I_x^2, \omega_C^{\otimes(1-k)}) \cong H^0(C', \pi^* \omega_C^{\otimes(1-k)}(x_1 + x_2)).$$

Consider the normalization \tilde{C} of C (and C' , also), we consider an irreducible component $E \subset \tilde{C}$. Then $\pi^* \omega_C^{\otimes(1-k)}(x_1 + x_2)$ restrict to E has degree

$$(1-k)(2g_E - 2 + \#\{E \cap \tilde{\Sigma}\}) + \#\{x_1, x_2\} \cap E$$

is negative unless $k = 3$, $\{x_1, x_2\} \subset E$, E is a rational curve meeting the other components of C in exactly one other point. In this case the degree on E is zero. So this global section is determined by its value at the point of E meeting the other components of C . Since not every component of \tilde{C} , we win.

Case (III.2). If $x \neq y$ are all nodes, the blowing up $\varpi : C''' \rightarrow C$ along $\{x, y\}$. We can get similar conclusion

$$\mathrm{Hom}(I_x I_y, \omega_C^{\otimes(1-k)}) \cong H^0(C', \varpi^* \omega_C^{\otimes(1-k)}).$$

This is zero since in any irreducible of normalization has negative degree. \square

8.3 Families of stable curves

Definition 8.3.1. (1) A family of n -pointed nodal curves is a flat, proper and finitely presented morphism $C \rightarrow S$ of schemes with n sections $\sigma_1, \dots, \sigma_n : S \rightarrow C$ such that every geometric fiber C_s is a (reduced) connected nodal curve.

(2) A family of n -pointed stable curves (resp. semistable curves, prestable curves) is a family $C \rightarrow S$ of n -pointed nodal curves such that every geometric fiber $(C_s, \sigma_1(s), \dots, \sigma_n(s))$ is stable (resp. semistable, prestable).

Remark 8.3.2. (1) We can define the fibered category of groupoid $\overline{\mathcal{M}}_{g,n}$ as for any scheme S , define $\overline{\mathcal{M}}_{g,n}(S) = \{(C, \sigma_1, \dots, \sigma_n) \rightarrow S : \text{is a family of stable curves of genus } g\}$. Note also that since the geometric fibers are stable curves, the image of each σ_i is a divisor contained in the smooth locus and we can form the line bundle $\omega_{C/S}(\sum_i \sigma_i)$.

(2) We can define relative dualizing line bundle $\omega_{C/S}$ as $C \rightarrow S$ is l.c.i. By [57], we can get the following properties: (2.a) $\omega_{C/S}|_{C_s} = \omega_{C_s}$; (2.b) for any $f : T \rightarrow S$ we have $f^* \omega_{C/S} = \omega_{C \times_S T/T}$.

Proposition 8.3.3. Let $\pi : (C, \sigma_1, \dots, \sigma_n) \rightarrow S$ be a family of n -pointed stable curves of genus g . Let $L = \omega_{C/S}(\sum_i \sigma_i)$. If $k \geq 3$, then $L^{\otimes k}$ is relatively very ample and $\pi_* L^{\otimes k}$ is a vector bundle of rank $(2k-1)(g-1) + kn$.

Proof. Similar as the smooth case by using Riemann Roch and cohomology and base change. Omitted here. \square

Proposition 8.3.4 (Openness of stability). Let $\pi : (C, \sigma_1, \dots, \sigma_n) \rightarrow S$ be a family of n -pointed nodal curves. The locus of S such that $(C_s, \sigma_i(s))$ is stable is open.

Proof. As the locus such that $\sigma_i(s)$ is smooth is open, we just need to let this family is prestable. Using 8.1.4, we have two arguments:

Argument 1. Group scheme $\underline{\text{Aut}}(C/S, \sigma_i) \rightarrow S$ has upper semicontinuous dimension of fibers, then as stable locus is the locus such that it is dimension 0 locus. Hence open.

Argument 2. Using the openness of ample locus. \square

Proposition 8.3.5 (Openness of being nodal). Let $f : X \rightarrow S$ be a flat proper morphism of \mathbb{C} -schemes. Then the set of all $s \in S$ such that $X_s = f^{-1}(s)$ is not a connected nodal curve is closed in S . If, in addition, n sections σ_i of f are given, then the set of all $s \in S$ such that $(X_s; \sigma_i(s))$ is not a connected n -pointed nodal curve is closed in S .

Sketch. We will give a sketch and for the detailed proof see [8] Proposition XI.5.1. First we need to let the fibers of f has dimension 1 by flatness and properness.

• **Step 1. Reduce to the case that fibers are connected and have no embedded components.** Easy to see that $\dim H^0(X_s, \mathcal{O}_{X_s}) = 1$ for all $s \in S$ if X_s is connected and reduced. As this is the stalks of $f_* \mathcal{O}_X$, we consider the free resolution $K^0 \xrightarrow{\alpha} K^1 \rightarrow \dots$ at some open subset. Hence the locus of $\dim H^0(X_s, \mathcal{O}_{X_s}) > 1$ is the locus of $\text{rank}(\alpha) \leq \text{rank}(K^0) - 2$. Hence is closed.

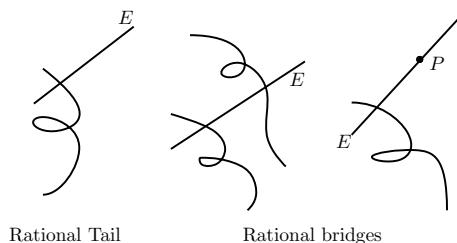
• **Step 2. Show that being neither nodal nor smooth is closed.** Here we need to represent nodes by some functions. Then we use some equivalent conditions (see [8] Lemma X.2.3) that if f be a function over 0 and $f(0) = 0$, then f defines the smooth point 0 if and only if the first-order partials of f not vanish at the origin; f defines the node 0 if and only if the first-order partials of f vanish and the Hessian not vanish. \square

8.4 Rational tails and bridges

Definition 8.4.1. Let (C, p_1, \dots, p_n) be a n -pointed prestable curve. We say a smooth rational subcurve $E \cong \mathbb{P}^1 \subset C$ is

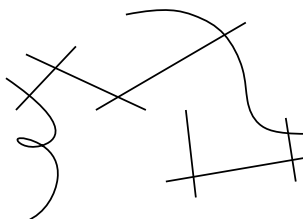
(i) a rational tail if E meets other irreducible components at exactly 1 time, and E contains no marked points;

(ii) a rational bridge if either E meets other irreducible components at exactly 2 time and contains no marked points, or E meets other irreducible components at exactly 1 time and contains exactly 1 marked point.



Remark 8.4.2. (1) C is stable if and only if it is prestable and has no rational tails and bridges;
 (2) C is semistable if and only if it is prestable and has no rational tails.

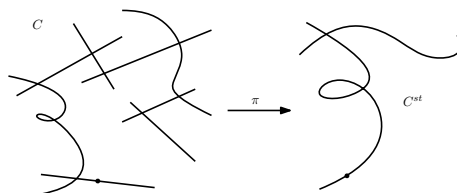
Or we can also have chain of rational tails and bridges like:



8.5 The stable model

8.5.1 The stable model of a single curve

Let (C, p_1, \dots, p_n) be a n -pointed prestable curve. Let $E_i \subset C$ are its rational tails and bridges. We let $C^{st} := C \setminus \bigcup_i E_i$ and let $\pi : C \rightarrow C^{st}$ be the induced map. Let $p'_i = \pi(p_i)$, then $(C^{st}, \{p'_i\})$ is a stable curve, which we call the stable model of $(C, \{p_i\})$ and $\pi : C \rightarrow C^{st}$ the stabilization morphism. Like this:



For the serious argument of contraction to the stable curves, we refer [11]: contracting rational tails (St 0E3G), contracting rational bridges (St 0E7M), contracting to a stable curve (St 0E7N). We omitted here.

8.5.2 The stable model of a family of curves

For a family of nodal curves, we also have the following conclusion.

Proposition 8.5.1. Let $(C \rightarrow S, \sigma_i)$ be a family of n -pointed prestable curves. Then there exists a unique (up to isomorphism) morphism $\pi : C \rightarrow C^{st}$ such that

- (a) $(C^{st} \rightarrow S, \{\sigma'_i\})$ is a n -pointed family of stable curves where $\sigma'_i = \pi \circ \sigma_i$;
- (b) for each $s \in S$, $(C_s, \{\sigma_i(s)\}) \rightarrow (C_s^{st}, \{\sigma'_i(s)\})$ is the stable model;
- (c) $\mathcal{O}_{C^{st}} = \pi_* \mathcal{O}_C$ and $R^1 \pi_* \mathcal{O}_C = 0$ and this remains true after base change by a morphism $S' \rightarrow S$ of schemes;
- (d) If $C \rightarrow S$ is a family of semistable curves, then $\omega_{C/S}(\sum_i \sigma_i)$ is the pullback of the relatively ample line bundle $\omega_{C^{st}/S}(\sum_i \sigma'_i)$.

Proof. See St 0E7B.

□

Chapter 9

Deformation theory of nodal and stable curves

After some basic results over arbitrary fields, we will focus on the curves over \mathbb{C} . We mainly follows [8] chapter XI (all results over \mathbb{C}) and some results over arbitrary fields we follows [1]. Some basic result and proofs we follows [78]. Here we let $k[\varepsilon] := k[x]/(x^2)$.

9.1 Elementary deformation theory and smooth objects

Definition 9.1.1. *Let X be a scheme over k . A first order deformation of X is a scheme \mathcal{X} flat over $k[\varepsilon] = k[\varepsilon]/(\varepsilon^2)$ with $X \cong \mathcal{X} \times_{k[\varepsilon]} k$.*

We say \mathcal{X} is trivial if \mathcal{X} is isomorphic as first deformations to $\mathcal{X} \times_k k[\varepsilon]$, and locally trivial if there exists a Zariski-cover $X = \bigcup_i U_i$ such that $\mathcal{X}|_{U_i}$ is a trivial first order deformation of U_i , that is, $U_i \times_k k[\varepsilon] \cong \mathcal{X}|_{U_i}$ where $\mathcal{X}|_{U_i} \subset \mathcal{X}$ be a open subscheme with the same topology of U_i .

We let $\text{Def}(X)$ be the isomorphism classes of first order deformations of X and $\text{Def}^{\text{lt}}(X)$ be the isomorphism classes of locally trivial first order deformations of X .

Proposition 9.1.2 (See [1] D.1.11). *For a scheme X of finite type over k with affine diagonal, there is a bijection*

$$\text{Def}^{\text{lt}}(X) \leftrightarrow H^1(X, T_X).$$

In particular, if X_0 is smooth, then we have bijection

$$\text{Def}(X) \leftrightarrow H^1(X, T_X),$$

as every first order deformations of smooth affine schemes is trivial.

Sketch. For a locally trivial first order deformation

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \text{Spec}k & \longleftarrow & \text{Spec}k[\varepsilon] \end{array}$$

let affine covering $\{U_i\}$ of X such that $\mathcal{X}|_{U_i}$ be a trivial first order deformation. Hence we get isomorphisms $\phi_i : U_i \times_k k[\varepsilon] \cong \mathcal{X}|_{U_i}$. Let $\phi_{ij} := \phi_j^{-1}|_{U_{ij} \times_k k[\varepsilon]} \circ \phi_i|_{U_{ij} \times_k k[\varepsilon]}$ are automorphisms of first order defs, hence we get $\phi_{ij} \in \text{Hom}_{\mathcal{O}_{U_{ij}}}(\Omega_{U_{ij}/k}, \mathcal{O}_{U_{ij}})$. As they satisfies cocycle condition, we get $\{\phi_{ij}\} \in H^1(X, T_X)$ by Čech theory (this is independent on the choice of covering, see [78] Proposition 1.2.9). Converse is trivial. \square

Remark 9.1.3. *For a locally trivial first order deformations ξ of X , we gives a class $\kappa(\xi) \in H^1(X, T_X)$ is called the Kodaira-Spencer class of ξ .*

Definition 9.1.4. Consider a family of deformation of a smooth algebraic variety X over k

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}_f \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}k & \xleftarrow{s} & S & \xleftarrow{f} & \mathrm{Spec}k[\varepsilon] \end{array}$$

\curvearrowright

hence we get

$$\kappa_{\mathcal{X}/S,s} : T_{S,s} \rightarrow H^1(X, T_X),$$

we called it Kodaira-Spencer map.

Definition 9.1.5. Let $A' \rightarrow A$ has square-free kernel and $X \rightarrow \mathrm{Spec}(A)$ is flat. A deformation of $X \rightarrow \mathrm{Spec}(A)$ over A' is $X' \rightarrow \mathrm{Spec}(A')$ with $X' \times_{A'} A \cong X$. A morphism of deformations over A' is a morphism of schemes over A' restricting to the identity on X .

Proposition 9.1.6. Let $A' \rightarrow A$ has square-free kernel J . If $X \rightarrow \mathrm{Spec}(A)$ is a smooth morphism of schemes where X has affine diagonal, then

- (a) the group of automorphisms of a deformation $X' \rightarrow \mathrm{Spec}(A')$ of $X \rightarrow \mathrm{Spec}(A)$ over A' is bijective to $H^0(X, T_{X/A} \otimes_A J)$;
- (b) If there exists a deformation of $X \rightarrow \mathrm{Spec}(A)$ over A' , then the set of isomorphism classes of all such deformations is a torsor under $H^1(X, T_{X/A} \otimes_A J)$;
- (c) There is an element $ob_X \in H^2(X, T_{X/A} \otimes_A J)$ with the property that there exists a deformation of $X \rightarrow \mathrm{Spec}(A)$ over A' if and only if $ob_X = 0$.

Proof. See [1] Proposition D.2.6. To add. □

Back to smooth curves over \mathbb{C} of genus g .

Theorem 9.1.7. Let $(C; q_1, \dots, q_n)$ be a n -pointed smooth genus g curve over \mathbb{C} .

(i) We have

$$\mathrm{Def}(C; q_1, \dots, q_n) \leftrightarrow H^1(C, T_C(-\sum_{i=1}^n q_i));$$

(ii) There exists a deformation

$$\phi : C \rightarrow (B, b_0), \sigma_i : B \rightarrow C \text{ such that } \chi : (C; q_1, \dots, q_n) \cong (\phi^{-1}(b_0), \sigma_i(b_0))$$

of $(C; q_1, \dots, q_n)$ such that the Kodaira-Spencer map

$$\kappa : T_{b_0}B \rightarrow H^1(C, T_C(-\sum_{i=1}^n q_i))$$

is an isomorphism and B is a polydisc of dimension $3g - 3 + n + h^0(C, T_C(-\sum_{i=1}^n q_i))$.

Proof. See [8] Theorem XI.2.12. To add. □

9.2 Elementary deformations of nodal and stable curves

Lemma 9.2.1. Let (C, p_1, \dots, p_n) be an n -pointed nodal, connected and projective curve over k with each $p_i \in C$ smooth. Let $\{q_1, \dots, q_s\}$ be the nodes of C . Let $(\tilde{C}, p_i, q'_j, q''_j)$ be the pointed normalization $\pi : \tilde{C} \rightarrow C$ and $\pi^{-1}(q_j) = \{q'_j, q''_j\}$. Then we have the spectral sequence

$$E_2^{p,q} = H^p(C, \mathcal{E}xt_{\mathcal{O}_C}^q(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)) \Rightarrow \mathrm{Ext}_{\mathcal{O}_C}^{p+q}(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$$

such that induce the following exact sequence

$$\begin{array}{ccc} 0 & \longrightarrow & H^1(C, \mathcal{H}om_{\mathcal{O}_C}(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)) \\ & & \downarrow \\ 0 & \longleftarrow & \bigoplus_j \text{Ext}_{\widehat{\mathcal{O}}_{C,q_j}}^1(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) \longleftarrow \text{Ext}_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C) \end{array}$$

Moreover, $\forall j$ we have $\text{Ext}_{\widehat{\mathcal{O}}_{C,q_j}}^1(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = k$ and $\text{Ext}_{\mathcal{O}_C}^2(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C) = 0$.

Proof. By Grothendieck spectral sequence, we have

$$E_2^{p,q} = H^p(C, \mathcal{E}xt_{\mathcal{O}_C}^q(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)) \Rightarrow \text{Ext}_{\mathcal{O}_C}^{p+q}(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C).$$

As C is a curve, $E_2^{p,q} = 0$ for $p \geq 2$.

By [40] Propostion 2.3, we get an exact sequence

$$0 \rightarrow E_2^{1,0} \rightarrow \text{Ext}_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C) \rightarrow E_2^{0,1} \rightarrow E_2^{2,0} \rightarrow 0.$$

As Ω_C is locally free away from nodes, $\mathcal{E}xt_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$ is zero-dimensional sheaf supported only at nodes. Hence $E_2^{1,1} = 0$ and

$$\begin{aligned} E_2^{0,1} &= H^0(C, \mathcal{E}xt_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)) \\ &= \bigoplus_j \text{Ext}_{\mathcal{O}_{C,q_j}}^1(\Omega_{C,q_j}, \mathcal{O}_{C,q_j}) = \bigoplus_j \text{Ext}_{\widehat{\mathcal{O}}_{C,q_j}}^1(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}). \end{aligned}$$

where $\widehat{\Omega}_{C,q_j} = \Omega_{\widehat{\mathcal{O}}_{C,q_j}}$. Hence we get that exact sequence.

Similarly $\mathcal{E}xt_{\mathcal{O}_C}^2(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$ is zero-dimensional sheaf supported only at nodes, then

$$E_2^{0,2} = H^0(C, \mathcal{E}xt_{\mathcal{O}_C}^2(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)) = \bigoplus_j \text{Ext}_{\widehat{\mathcal{O}}_{C,q_j}}^2(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}).$$

Write $\widehat{\mathcal{O}}_{C,q_j} = k[[x, y]]/(xy)$ and consider the locally free resolution

$$0 \longrightarrow \widehat{\mathcal{O}}_{C,q_j} \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} \widehat{\mathcal{O}}_{C,q_j}^{\oplus 2} \xrightarrow{(dx, dy)} \Omega_{\widehat{\mathcal{O}}_{C,q_j}} \longrightarrow 0$$

Hence we get $\text{Ext}_{\widehat{\mathcal{O}}_{C,q_j}}^1(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = k$ and $\text{Ext}_{\widehat{\mathcal{O}}_{C,q_j}}^2(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}, \widehat{\mathcal{O}}_{C,q_j}) = 0$. Hence $E_2^{0,2} = E_2^{1,1} = E_2^{2,0} = 0$ and $\text{Ext}_{\mathcal{O}_C}^2(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C) = 0$. \square

Proposition 9.2.2. *Let (C, p_1, \dots, p_n) be an n -pointed nodal, connected and projective curve over k with each $p_i \in C$ smooth. Let $\{q_1, \dots, q_s\}$ be the nodes of C . Let $(\widetilde{C}, p_i, q'_j, q''_j)$ be the pointed normalization $\pi : \widetilde{C} \rightarrow C$ and $\pi^{-1}(q_j) = \{q'_j, q''_j\}$. Then we have the following exact sequence*

$$0 \rightarrow \text{Def}^{dt}(C) \rightarrow \text{Def}(C) \rightarrow \bigoplus_j \text{Def}(\widehat{\mathcal{O}}_{C,q_j}) \rightarrow 0$$

and

$$\text{Def}^{dt}(C) \cong \text{Def}(\widetilde{C}, p_i, q'_j, q''_j) \cong H^1(\widetilde{C}, T_{\widetilde{C}}(-\sum_i p_i - \sum_j (q'_j + q''_j))),$$

$$\text{Def}(C) \cong \text{Ext}_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C),$$

$$\text{Def}(\widehat{\mathcal{O}}_{C,q_j}) \cong \text{Ext}_{\widehat{\mathcal{O}}_{C,q_j}}^1(\Omega_{\widehat{\mathcal{O}}_{C,q_j}}^1, \widehat{\mathcal{O}}_{C,q_j}) \cong k.$$

Under these identifications, this exact sequence corresponds to the exact sequence in the Lemma.

Sketch. WLOG again we let $n = 0$. If $\mathcal{C} \rightarrow \text{Speck}[\varepsilon]$ is a locally trivial first order deformation of C , each node q_j extend to a section $\tilde{q}_j : \text{Speck}[\varepsilon] \rightarrow \mathcal{C}$. The pointed normalization of \mathcal{C} along the sections \tilde{q}_j is a first order deformation of the (possible disconnected) pointed normalization $(\tilde{C}, p_i, q'_j, q''_j)$. This gives a map $\text{Def}^{\text{lt}}(C) \rightarrow \text{Def}(\tilde{C}, p_i, q'_j, q''_j)$. The inverse is provided by gluing the sections of a first order deformation of $(\tilde{C}, p_i, q'_j, q''_j)$ along nodes.

If $\mathcal{C} \rightarrow \text{Speck}[\varepsilon]$ is a first order deformation of C , then ideal sheaf I of $\mathcal{C} \rightarrow C$ is $I = I/I^2 \cong \mathcal{O}_C$. The right exact sequence

$$I/I^2 \rightarrow \Omega_{\mathcal{C}/k} \rightarrow \Omega_{C/k} \rightarrow 0$$

is left exact at every smooth point of C . As $C \rightarrow \text{Speck}$ is generically smooth and it follows that $\mathcal{O}_C \cong I/I^2 \rightarrow \Omega_{\mathcal{C}/k}$ is generically injective, hence injective. Hence this defines $\text{Ext}^1_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)$. This is bijective (one can see [8] section XI.3). \square

Remark 9.2.3. Hence we also have the Kodaira-Spencer map for some $\mathcal{C} \rightarrow (S, s)$ as

$$\kappa_{S,s} : T_{S,s} \rightarrow \text{Ext}^1_{\mathcal{O}_C}(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C).$$

Remark 9.2.4. Let C be a nodal curve over \mathbb{C} and let $p \in C$ be a node with normalization N and preimages $\{p_1, p_2\}$.

• **Claim 1.** $\text{Ext}^1(\Omega_{C,p}, \mathcal{O}_{C,p}) \cong \bigwedge^2(\mathfrak{m}_p/\mathfrak{m}_p^2) \otimes_{\mu_2} \tau$ where $\mu_2 = \{\pm 1\}$ and τ be the set consisting of the two possible orderings of the branches of C at p .

I omit this, see [8] page 180.

• **Claim 2.** $\text{Ext}^1(\Omega_{C,p}, \mathcal{O}_{C,p}) \cong T_{N,p_1} \otimes T_{N,p_2}$.

Trivial by Claim 1 and $\mathfrak{m}_p/\mathfrak{m}_p^2 = T_{C,p} = T_{N,p_1} \oplus T_{N,p_2}$ and $\bigwedge^2 T_{N,p_1} \oplus T_{N,p_2}$ identify with $T_{N,p_1} \otimes T_{N,p_2}$ depends on the choice of an ordering of the two summands, we win.

Remark 9.2.5. Let $(C; p_1, \dots, p_n)$ be an n -pointed nodal curve over \mathbb{C} for simplicity, and let $W = \{w_1, \dots, w_l\}$ be some set of nodes of C . Let $f : N \rightarrow C$ be the partial normalization at these nodes with $f^{-1}(w_i) = \{r_i, q_i\}$. Let $D = \sum_i p_i$ with inverse \tilde{D} and $E = \sum(r_i + q_i)$.

• **Claim 1.** $\mathcal{H}om(\Omega_C^1, \mathcal{O}_C(-D)) \cong f_* \mathcal{H}om(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E))$.

This is trivially true at points away from W , so we just need to consider the points in W . Pick any $w_i \in W$, we get $\text{Hom}(\Omega_{C,w_i}^1, \mathcal{O}_{C,w_i}) = \text{Hom}(\mathcal{I}_{w_i} \omega_{C,w_i}, \mathcal{O}_{C,w_i})$ by Corollary 7.4.5. As $\mathcal{I}_{w_i} \omega_{C,w_i} = \omega_{N,r_i} \oplus \omega_{N,q_i}$ and $\mathcal{I}_{w_i} = \mathcal{O}_{N,r_i}(-r_i) \oplus \mathcal{O}_{N,q_i}(-q_i)$ (Why?), we get

$$\text{Hom}(\Omega_{C,w_i}^1, \mathcal{O}_{C,w_i}) = \bigoplus_{p=r_i, q_i} \text{Hom}(\omega_{N,p}, \mathcal{O}_{N,p}(-p)),$$

hence $\mathcal{H}om(\Omega_C^1, \mathcal{O}_C(-D)) \cong f_* \mathcal{H}om(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E))$.

• **Claim 2.** We have

$$0 \rightarrow \text{Ext}^1(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E)) \rightarrow \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \rightarrow \bigoplus_{w_i \in W} \text{Ext}^1(\Omega_{C,w_i}^1, \mathcal{O}_{C,w_i}) \rightarrow 0.$$

By Claim 1, we get $H^1(N, \mathcal{H}om(\Omega_C^1, \mathcal{O}_C(-D))) \cong H^1(N, \mathcal{H}om(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E)))$. Hence by

Lemma 9.2.1, we get

$$\begin{array}{ccc}
0 & \longrightarrow & H^1(N, \mathcal{H}om(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E))) \longrightarrow \text{Ext}^1(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E)) \\
& & \parallel \qquad \qquad \qquad \downarrow \\
0 & \longrightarrow & H^1(N, \mathcal{H}om(\Omega_C^1, \mathcal{O}_C(-D))) \longrightarrow \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \\
& & \downarrow \\
& \longrightarrow & \bigoplus_{w \in \text{Sing}(C), w \notin W} \text{Ext}^1(\Omega_{C,w}^1, \mathcal{O}_{C,w}) \longrightarrow 0 \\
& & \downarrow \\
& \longrightarrow & \bigoplus_{w \in \text{Sing}(C)} \text{Ext}^1(\Omega_{C,w}^1, \mathcal{O}_{C,w}) \longrightarrow 0
\end{array}$$

hence we get

$$\begin{aligned}
0 & \rightarrow \text{Ext}^1(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - E)) \rightarrow \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \rightarrow \\
& \bigoplus_{w_i \in W} \text{Ext}^1(\Omega_{C,w_i}^1, \mathcal{O}_{C,w_i}) \rightarrow 0.
\end{aligned}$$

Note that the term on the left classifies first-order deformations which are locally trivial at the nodes belonging to W , and the one on the right classifies first-order smoothings of these nodes.

• **Claim 3.** $\bigoplus_{w_i \in W} \text{Ext}^1(\Omega_{C,w_i}^1, \mathcal{O}_{C,w_i}) = \bigoplus_{i=1}^l T_{N,r_i} \otimes T_{N,q_i}$.

By claims in Remark 9.2.4, this is trivial.

Here is a similar result as before over \mathbb{C} via analytic GAGA.

Theorem 9.2.6. Let $(C; p_1, \dots, p_n)$ be an n -pointed nodal curve of genus g over \mathbb{C} . There exists a deformation

$$\phi : \mathcal{C} \rightarrow (B, b_0), \sigma_i : B \rightarrow \mathcal{C} \text{ such that } \chi : (C; p_1, \dots, p_n) \cong (\phi^{-1}(b_0), \sigma_i(b_0))$$

of $(C; p_1, \dots, p_n)$ such that the Kodaira-Spencer map

$$\kappa : T_{b_0} B \rightarrow \text{Ext}_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$$

is an isomorphism and B is a polydisc of dimension $3g - 3 + n + \dim \text{Hom}(\Omega_C, \mathcal{O}_C)$.

Finally, if s is the number of nodes of C , one can choose coordinates t_1, \dots, t_s, \dots on B , vanishing at b_0 , in such a way that the locus parameterizing deformations which are locally trivial at the i -th node is $t_i = 0$; in particular, the locus parameterizing singular curves is $t_1 \cdots t_s = 0$.

Proof. See [8] Theorem XI.3.17. To add. □

Back to the general case.

Proposition 9.2.7. Let (C, p_1, \dots, p_n) be an n -pointed nodal, connected and projective curve over k with each $p_i \in C$ smooth. Let $A' \twoheadrightarrow A$ be a surjection of artinian local k -algebras with residue field k such that $J = \ker(A' \rightarrow A)$ satisfies $\mathfrak{m}_{A'} J = 0$. If $C_A \rightarrow \text{Spec}(A)$ be a family of nodal curves such that $C \cong C_A \times_A k$, then

(a) The group of automorphisms of a deformation $C_{A'} \rightarrow \text{Spec}(A')$ of $C_A \rightarrow \text{Spec}(A)$ over A' is bijective to $\text{Ext}_{\mathcal{O}_C}^0(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C \otimes_k J)$;

(b) If there exists a deformation of $C_A \rightarrow \text{Spec}(A)$ over A' , then the set of isomorphism classes of all such deformations is a torsor under $\text{Ext}_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C \otimes_k J)$;

(c) There is an element $ob_{C_A} \in \text{Ext}_{\mathcal{O}_C}^2(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C \otimes_k J)$ with the property that there exists a deformation of $C_A \rightarrow \text{Spec}(A)$ over A' if and only if $ob_{C_A} = 0$.

Proof. To add. □

Lemma 9.2.8 (St 0E68). *Let k be an algebraically closed field. Let X be an at-worst-nodal, proper, connected 1-dimensional scheme over k . Let $f : \tilde{X} \rightarrow X$ be the normalization. Let $S \subset \tilde{X}$ be the set of points where f is not an isomorphism.*

$$\mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) = \{D' \in \mathrm{Der}_k(\mathcal{O}_{\tilde{X}}, \mathcal{O}_{\tilde{X}}) : D' \text{ fixed every } x' \in S\}.$$

Proof. Let $x \in X$ be a node with the preimage $x', x'' \in \tilde{X}$. Pick two uniformizers u, v in $\mathcal{O}_{\tilde{X}, x'}$ and $\mathcal{O}_{\tilde{X}, x''}$, respectively. Hence we have

$$0 \rightarrow \mathcal{O}_{X, x} \rightarrow \mathcal{O}_{\tilde{X}, x'} \times \mathcal{O}_{\tilde{X}, x''} \rightarrow k \rightarrow 0,$$

thus we can view u, v as elements of $\mathcal{O}_{X, x}$ with $uv = 0$.

Since (u) is annihilator of v in $\mathcal{O}_{C, x}$ and vice versa, we see that $D(u) \in (u)$ and $D(v) \in (v)$. As $\mathcal{O}_{\tilde{C}, x'} = k + (u)$ we conclude that we can extend D to $\mathcal{O}_{\tilde{C}, x'}$ and moreover the extension fixes x' . This produces a D' in the right hand side of the equality. Conversely, given a D' fixing x' and x'' we find that D' preserves the subring $\mathcal{O}_{C, x} \subset \mathcal{O}_{\tilde{C}, x'} \times \mathcal{O}_{\tilde{C}, x''}$ and this is how we go from right to left in the equality. \square

Proposition 9.2.9. *Let (C, p_1, \dots, p_n) be an n -pointed stable curve of genus g over k . Then*

$$\dim_k \mathrm{Ext}_{\mathcal{O}_C}^i \left(\Omega_C \left(\sum_i p_i \right), \mathcal{O}_C \right) = \begin{cases} 0, & i = 0, 2; \\ 3g - 3 + n, & i = 1. \end{cases}$$

Proof. We let k is algebraically closed and has no marked point. Let $\pi : \tilde{C} \rightarrow C$ be a normalization and let $\Sigma \subset C$ be the set of nodes. Let $\tilde{\Sigma} = \pi^{-1}(\Sigma) \subset \tilde{C}$.

By Lemma 9.2.1, we get $\dim_k \mathrm{Ext}_{\mathcal{O}_C}^2(\Omega_C, \mathcal{O}_C) = 0$. For Ext^0 , we first claim that

$$\mathrm{Hom}_{\mathcal{O}_{\tilde{C}}}(\Omega_{\tilde{C}}(\tilde{\Sigma}), \mathcal{O}_{\tilde{C}}) \cong \mathrm{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C).$$

This is equivalent to show

$$\mathrm{Der}_k(\mathcal{O}_C, \mathcal{O}_C) \cong \{D' \in \mathrm{Der}_k(\mathcal{O}_{\tilde{C}}, \mathcal{O}_{\tilde{C}}) : D' \text{ fixes every points in } \tilde{\Sigma}\}.$$

Actually this is just Lemma 9.2.8. This finish the claim. Hence we get

$$\mathrm{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) \cong \mathrm{Hom}_{\mathcal{O}_{\tilde{C}}}(\Omega_{\tilde{C}}(\tilde{\Sigma}), \mathcal{O}_{\tilde{C}}) \cong H^0(\tilde{C}, T_{\tilde{C}}(-\tilde{\Sigma})) = 0.$$

For Ext^1 , by Lemma 9.2.1 we have

$$\begin{array}{ccc} 0 \rightarrow H^1(C, \mathcal{H}om_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C)) & \rightarrow & \mathrm{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C) \rightarrow \bigoplus_j \mathrm{Ext}_{\hat{\mathcal{O}}_{C, q_j}}^1(\Omega_{\hat{\mathcal{O}}_{C, q_j}}, \hat{\mathcal{O}}_{C, q_j}) \\ & \parallel & \downarrow \\ & H^1(\tilde{C}, T_{\tilde{C}}(-\tilde{\Sigma})) & 0 \end{array}$$

and $\mathrm{Ext}_{\hat{\mathcal{O}}_{C, q_j}}^1(\Omega_{\hat{\mathcal{O}}_{C, q_j}}, \hat{\mathcal{O}}_{C, q_j}) = k$. This equality in this exact sequence is because

$$H^1(C, \mathcal{H}om_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C))$$

be the set of locally trivial first order deformation of C preserving nodes and this is equivalent to the set of locally trivial first order deformation of \tilde{C} fixed $\tilde{\Sigma}$, which is $H^1(\tilde{C}, T_{\tilde{C}}(-\tilde{\Sigma}))$.

Now let $\tilde{C} = \coprod_{i=1}^t \tilde{C}_i$ are connected components and $\tilde{\Sigma}_i = \tilde{C}_i \cap \tilde{\Sigma}$. First we have

$$h^1(\tilde{C}_i, T_{\tilde{C}_i}(-\tilde{\Sigma}_i)) = h^0(\tilde{C}_i, \Omega_{\tilde{C}_i}^{\otimes 2}(\tilde{\Sigma}_i)) = 3g(\tilde{C}_i) - 3 + \#(\tilde{\Sigma}_i),$$

hence

$$\begin{aligned} \dim_k \text{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C) &= h^1(\tilde{C}_i, T_{\tilde{C}_i}(-\tilde{\Sigma}_i)) + \#(\Sigma) \\ &= \sum_{i=1}^t (3g(\tilde{C}_i) - 3 + \#(\tilde{\Sigma}_i)) + \#(\Sigma) = 3 \sum_{i=1}^t g(\tilde{C}_i) - 3t + 3\#(\Sigma) = 3g - 3 \end{aligned}$$

by the genus formula, we win. \square

9.3 Basic concept of Kuranishi family

We will work on analytic category over \mathbb{C} via Serre's GAGA.

Definition 9.3.1. Let $(C; p_1, \dots, p_n)$ be an n -pointed connected nodal curve of genus g over \mathbb{C} . A deformation $\phi : \mathcal{C} \rightarrow (B, b_0)$, $\sigma_i : B \rightarrow \mathcal{C}$ such that $\chi : (C; p_1, \dots, p_n) \cong (\phi^{-1}(b_0), \sigma_i(b_0))$ of $(C; p_1, \dots, p_n)$ is said to be a Kuranishi family for $(C; p_1, \dots, p_n)$ if it satisfies the following condition:

(Condition K). For any deformation $\psi : \mathcal{D} \rightarrow (E, e_0)$ of $(C; p_1, \dots, p_n)$ and for any sufficiently small connected neighborhood U of e_0 , there is a unique morphism of deformations of n -pointed curves

$$\begin{array}{ccc} \mathcal{D}|_U & \xrightarrow{F} & \mathcal{C} \\ \downarrow \psi|_U & & \downarrow \phi \\ (U, e_0) & \xrightarrow{f} & (B, b_0) \end{array}$$

Remark 9.3.2. In the algebraic case, the neighborhood U of e_0 taken étale locally. Actually this is the same as all analytic local and étale local here.

Remark 9.3.3 (Versal). If we just let the deformation satisfies (condition K) except for uniqueness and the Kodaira-Spencer map at the central fiber be an isomorphism, then we call it a versal deformation.

We will show, the Kuranishi family for $(C; p_1, \dots, p_n)$ exists if and only if $(C; p_1, \dots, p_n)$ is stable, at next two sections.

Corollary 9.3.4. The Kodaira-Spencer map of a Kuranishi family at the base point is an isomorphism.

Proof. This is trivial as the family $\mathcal{C}_\varepsilon \rightarrow \text{Spec} \mathbb{C}[\varepsilon]$ just has and has unique map to $\phi : \mathcal{C} \rightarrow (B, b_0)$. This defines a bijection between T_{B, b_0} and $\text{Def}(\mathcal{C}) = \text{Ext}_{\mathcal{O}_C}^1(\Omega_C(p_1 + \dots + p_n), \mathcal{O}_C)$ via κ_{B, b_0} . \square

Corollary 9.3.5. Let there be given a deformation of a stable n -pointed curve $(C; p_1, \dots, p_n)$ over the pointed analytic space (E, e_0) . Suppose that its Kodaira-Spencer map at e_0 is an isomorphism and that E is smooth at e_0 . Then the deformation is a Kuranishi family for $(C; p_1, \dots, p_n)$.

Proof. To add. \square

Corollary 9.3.6. The base of the Kuranishi family of a stable n -pointed curve $(C; p_1, \dots, p_n)$ of genus g is smooth of dimension $3g - 3 + n$.

Proof. By the previous corollary and Theorem 9.2.6 and the uniqueness of the Kuranishi family by the universal property. \square

Corollary 9.3.7. Let $X \rightarrow S$ be a family of stable n -pointed curves, and s_0 a point of S . If $X \rightarrow S$ is a Kuranishi family for X_{s_0} , then it is a Kuranishi family for X_s , for all s in an open neighborhood U of s_0 .

Proof. From the previous results that $X \rightarrow S$ is Kuranishi for X_s if and only if s is a smooth point of S and the Kodaira-Spencer map at s is an isomorphism. The first of these conditions is clearly open.

Since the dimension of $\text{Ext}^1(\Omega_{X_s}^1, \mathcal{O}_{X_s}(-\sum \sigma_i(s)))$ is independent of s , the second condition translates into a rank condition for a map between vector bundles and hence is open and We win. \square

9.4 The Hilbert scheme of ν -canonical curves

For any stable n -pointed genus g curve $(C; p_i)$, if we let $D = \sum_i p_i$, then by Proposition 8.3.3 that for all $\nu \geq 3$, the ν -log-canonical bundle $\omega_C(D)^{\otimes \nu}$ is very ample and embeds C into \mathbb{P}^{N-1} where $N = (2\nu - 1)(g - 1) + \nu n$. Let $P_\nu(t) = (2\nu t - 1)(g - 1) + \nu n t$, we consider the Hilbert scheme $\underline{\text{Hilb}}_{\mathbb{P}^{N-1}}^{P_\nu}$.

By Proposition 8.3.5, we get the nonempty subset $U \subset \underline{\text{Hilb}}_{\mathbb{P}^{N-1}}^{P_\nu}$ parameterizing connected n -pointed nodal curves is open. Let the $(\pi : \mathcal{Y} \rightarrow U, \sigma_i)$ be the restriction of the universal family. As the general points of U does not correspond to an n -pointed curve embedded by the ν -fold log-canonical sheaf, we need to define a new subscheme.

Definition 9.4.1. Let $F = (\pi^* \mathcal{O}_{\mathbb{P}^{N-1}}(1))^{-1} \otimes \omega_\pi(\sum_i \sigma_i)^{\otimes \nu}$. We define $H_{\nu, g, n} \subset U \subset \underline{\text{Hilb}}_{\mathbb{P}^{N-1}}^{P_\nu}$ as a subscheme by

$$H_{\nu, g, n}(X) := \left\{ \alpha : X \rightarrow U \left| \begin{array}{l} \mathcal{Y} \times_U X \xrightarrow{\beta} \mathcal{Y} \\ \text{correspond to} \quad \downarrow \eta \quad \quad \downarrow \pi \quad \text{such} \\ X \xrightarrow{\alpha} U \\ \text{that } \beta^* F \cong \eta^* G \text{ for some } G \in \text{Pic}(X) \end{array} \right. \right\}.$$

We call $H_{\nu, g, n}$ as the Hilbert scheme of ν -log-canonically embedded, stable, n -pointed, genus g curves.

Lemma 9.4.2. Let $h = (C \subset \mathbb{P}^m; p_1, \dots, p_n)$ be a nodal curve where p_1, \dots, p_n be distinct smooth points of C and $D = \sum_i p_i$. Let H be the Hilbert scheme parameterizing the $(n + 1)$ -tuples $(Y; q_1, \dots, q_n)$, where Y is a subscheme of $C \subset \mathbb{P}^m$ and q_1, \dots, q_n points on it, then we have the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathcal{O}_C}(\Omega_C^1, \mathcal{O}_C(-D)) &\rightarrow \text{Hom}_{\mathcal{O}_{\mathbb{P}^m}}(\Omega_{\mathbb{P}^m}^1, \mathcal{O}_C) \\ &\rightarrow T_h H \xrightarrow{\beta} \text{Ext}_{\mathcal{O}_C}^1(\Omega_C^1, \mathcal{O}_C(-D)) \end{aligned}$$

where β is just the Kodaira-Spencer map at h associated to the universal family over H .

Proof. Consider

$$\begin{array}{ccccccc} \mathcal{D}^* & & \mathcal{C}^* & & \mathcal{B}^* & & \mathcal{A}^* \\ \parallel & & \parallel & & \parallel & & \parallel \\ \mathcal{O}_C & \longrightarrow & \mathcal{I}_C / \mathcal{I}_C^2 & \longrightarrow & \Omega_{\mathbb{P}^m}^1 \otimes \mathcal{O}_C & \longrightarrow & \Omega_C^1 \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_D & \longrightarrow & \mathcal{I}_D / \mathcal{I}_D^2 & \longrightarrow & \Omega_{\mathbb{P}^m}^1 \otimes \mathcal{O}_D & \longrightarrow & 0 \end{array}$$

Hence we get

$$0 \rightarrow \text{Hom}_{\mathcal{O}_C}(\mathcal{A}^*, \mathcal{D}^*) \rightarrow \text{Hom}_{\mathcal{O}_C}(\mathcal{B}^*, \mathcal{D}^*) \rightarrow \text{Hom}_{\mathcal{O}_C}(\mathcal{C}^*, \mathcal{D}^*) \rightarrow \text{Ext}_{\mathcal{O}_C}(\mathcal{A}^*, \mathcal{D}^*).$$

As $\mathcal{A}^* \rightarrow \mathcal{D}^*$ is equivalent to $\Omega_C^1 \rightarrow \ker(\mathcal{O}_C \rightarrow \mathcal{O}_D) = \mathcal{O}_C(-D)$, hence $\text{Hom}_{\mathcal{O}_C}(\mathcal{A}^*, \mathcal{D}^*) = \text{Hom}(\Omega_C^1, \mathcal{O}_C(-D))$. As $\mathcal{B}^* \rightarrow \mathcal{D}^*$ determined by $\mathcal{B}^0 \rightarrow \mathcal{D}^0$, we get

$$\text{Hom}_{\mathcal{O}_C}(\mathcal{B}^*, \mathcal{D}^*) = \text{Hom}_{\mathcal{O}_C}(\Omega_{\mathbb{P}^m}^1 \otimes \mathcal{O}_C, \mathcal{O}_C) = \text{Hom}_{\mathcal{O}_{\mathbb{P}^m}}(\Omega_{\mathbb{P}^m}^1, \mathcal{O}_C).$$

It is trivial that $\text{Hom}_{\mathcal{O}_C}(\mathcal{C}^*, \mathcal{D}^*) \cong T_h H$. The final term is actually the isomorphism classes of first-order deformations of h , hence is $\text{Ext}_{\mathcal{O}_C}^1(\Omega_C^1, \mathcal{O}_C(-D))$ (see [8] XI.(3.11)). Hence we win. \square

Theorem 9.4.3. Let $2g - 2 + n > 0$ and $\nu \geq 3$ and $N = (2\nu - 1)(g - 1) + \nu n$. Then $H_{\nu, g, n}$ defined as above satisfied the following statements.

(i) Let $h = (C; p_1, \dots, p_n)$ be a stable curve in \mathbb{P}^{N-1} embedded by the ν -fold log-canonical system and $D = \sum_i p_i$. Then we have the exact sequence

$$0 \rightarrow H^0(C, \mathcal{O}_C(1))^{\oplus N} / H^0(C, \mathcal{O}_C) \rightarrow T_h(H_{\nu, g, n}) \xrightarrow{\lambda} \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \rightarrow 0$$

where λ is the Kodaira-Spencer map at h of the universal family on $H_{\nu, g, n}$. In particular,

$$\dim T_h H_{\nu, g, n} = 3g - 3 + n + N^2 - 1;$$

(ii) $H_{\nu, g, n}$ is smooth and quasi-projective of dimension $3g - 3 + n + N^2 - 1$.

Sketch. (i) By Euler sequence and Lemma 9.4.2, we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}(\Omega_C^1, \mathcal{O}_C(-D)) & & & & \\ & & \downarrow & & & & \\ 0 & \rightarrow & H^0(C, \mathcal{O}_C) & \rightarrow & H^0(C, \mathcal{O}_C(1))^{\oplus N} & \longrightarrow & \text{Hom}(\Omega_{\mathbb{P}^{N-1}}^1, \mathcal{O}_C) \xrightarrow{\delta} H^1(C, \mathcal{O}_C) \\ & & & & & & \downarrow \gamma \\ & & & & & & T_h H \xrightarrow{\beta} \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \end{array}$$

Now we will analyze several groups and morphisms above.

- **The map β associates to every first-order embedded deformation of h .** (Trivial)
 - **The elements of $\text{Hom}(\Omega_{\mathbb{P}^{N-1}}^1, \mathcal{O}_C)$ correspond to fiber space maps $j : C \times \text{Spec}\mathbb{C}[\varepsilon] \rightarrow \mathbb{P}^{N-1} \times \text{Spec}\mathbb{C}[\varepsilon]$.** (Omitted, see [8] page 200)
 - **The map δ associates to any such object the infinitesimal deformation of line bundles on C given by $j^*(\mathcal{O}_{\mathbb{P}^{N-1}}^1(1) \otimes \mathcal{O}_{\text{Spec}\mathbb{C}[\varepsilon]}) \otimes (\mathcal{O}_C^1(1) \otimes \mathcal{O}_{\text{Spec}\mathbb{C}[\varepsilon]})^{-1}$.** (Omitted, see [8] page 201)
 - **The elements of $H^0(C, \mathcal{O}_C(1))^{\oplus N} / H^0(C, \mathcal{O}_C)$ is the tangent space to $\text{PGL}(N)$.** (Omitted)
- Let $v = \Gamma(\alpha) \in T_h H$ tangent to $H_{\nu, g, n}$ where $\alpha \in \text{Hom}(\Omega_{\mathbb{P}^{N-1}}^1, \mathcal{O}_C)$. Hence v from a fiber space map $j : C \times \text{Spec}\mathbb{C}[\varepsilon] \rightarrow \mathbb{P}^{N-1} \times \text{Spec}\mathbb{C}[\varepsilon]$ such that

$$j^*(\mathcal{O}_{\mathbb{P}^{N-1}}^1(1) \otimes \mathcal{O}_{\text{Spec}\mathbb{C}[\varepsilon]}) = \omega_C(D)^{\otimes \nu} \otimes \mathcal{O}_{\text{Spec}\mathbb{C}[\varepsilon]}.$$

Then $\delta(\alpha) = 0$ and hence $\alpha \in H^0(C, \mathcal{O}_C(1))^{\oplus N} / H^0(C, \mathcal{O}_C)$. Conversely we find that the image of $H^0(C, \mathcal{O}_C(1))^{\oplus N} / H^0(C, \mathcal{O}_C)$ in $T_h H$ contained in $T_h H_{\nu, g, n}$. By Proposition 9.2.9 we get $\text{Hom}(\Omega_C^1, \mathcal{O}_C(-D)) = 0$, hence we have

$$0 \rightarrow H^0(C, \mathcal{O}_C(1))^{\oplus N} / H^0(C, \mathcal{O}_C) \rightarrow T_h(H_{\nu, g, n}) \xrightarrow{\lambda} \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)).$$

Actually λ is surjective since any infinitesimal deformations of h can be embedded via the ν -fold log-canonical system. Hence we win.

(ii) By the basic theory of Hilbert schemes, $H_{\nu, g, n}$ is quasi-projective by the trivial reason. We now will show that $H_{\nu, g, n}$ is smooth of dimension $3g - 3 + n + N^2 - 1$. By (i) we get $\dim T_h H_{\nu, g, n} = 3g - 3 + n + N^2 - 1$, hence $\dim H_{\nu, g, n} \leq 3g - 3 + n + N^2 - 1$. If we have showed that $\dim H_{\nu, g, n} \geq 3g - 3 + n + N^2 - 1$, then well done.

Here we just give a sketch, details see [8] Proposition XI.5.12. By Theorem 9.2.6, we get a $(3g - 3 + n)$ -dimensional deformation $\phi : \mathcal{C} \rightarrow (B, b_0)$. Let $\mathcal{C}_b = \phi^{-1}(b)$ and $D_b = \sum_i \sigma_i(b)$. Consider a principle $\text{PGL}(N)$ -bundle over B as

$$\mathcal{B} := \left\{ (b, F) \left| \begin{array}{l} b \in B \text{ and } F \text{ a basis of } H^0(\mathcal{C}_b, \omega_{\mathcal{C}_b}(D_b)^{\otimes \nu}), \\ \text{modulo homotheties} \end{array} \right. \right\}.$$

Take F_0 correspond to $C \subset \mathbb{P}^{N-1}$ and consider the family

$$\mathcal{X} := \mathcal{B} \times_B \mathcal{C} \xrightarrow{\psi} \mathcal{B}, \tau_i : \mathcal{B} \rightarrow \mathcal{X}.$$

Via some projective frame of $\psi_*(\omega_{\mathcal{X}/\mathcal{B}}(\sum \tau_i)^{\otimes \nu})$, we have $\mathcal{X} \rightarrow \mathbb{P}^{N-1} \times \mathcal{B}$, which induce $\xi : \mathcal{B} \rightarrow H_{\nu, g, n}$. Hence we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & T_e(G) & \longrightarrow & T_{(b_0, F_0)}\mathcal{B} & \longrightarrow & T_{b_0}B \longrightarrow 0 \\ & & \parallel & & \downarrow d\xi & & \downarrow \rho \\ 0 & \longrightarrow & H^0(C, \mathcal{O}_C(1))^{\oplus N}/H^0(C, \mathcal{O}_C) & \longrightarrow & T_h(H_{\nu, g, n}) & \longrightarrow & \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \longrightarrow 0 \end{array}$$

where ρ is Kodaira-Spencer map. As ρ is an isomorphism, we have $d\xi$ is also an isomorphism. Hence locally ξ is a local isomorphism at (b_0, F_0) . As $\dim \mathcal{B} = 3g - 3 + n + N^2 - 1$, well done. \square

9.5 Construction of Kuranishi families

Let $\nu \geq 3$ and $(C; p_1, \dots, p_n) \subset \mathbb{P}^{N-1}$ be a stable n -pointed genus g curve where $N = (2\nu - 1)(g - 1) + \nu n$, via ν -fold log-canonical system. We consider it as $x_0 \in H_{\nu, g, n}$. Fix the universal family $\mathcal{Y} \rightarrow H_{\nu, g, n}$ with sections $\sigma_i : H \rightarrow \mathcal{Y}$. Let

$$H_{\nu, g, n} \subset \text{Hilb}_{\mathbb{P}^{N-1}}^{F_\nu} \times (\mathbb{P}^{N-1})^n \subset \mathbb{P}^M \times (\mathbb{P}^{N-1})^n \subset \mathbb{P}^K$$

acted by $\mathbb{G} = \text{PGL}(N) \subset \text{PGL}(K + 1)$ and let $\text{Aut}(C; p_i) = \mathbb{G}_{x_0} \subset \mathbb{G} = \text{PGL}(N)$ be the stabilizer of x_0 .

Let the orbit $O(x_0) \subset H_{\nu, g, n}$ of x_0 under \mathbb{G} , which is a smooth subvariety of dimension $N^2 - 1$. (Here is not important. But we need to read here, and to add.)

Lemma 9.5.1.

Proof. \square

Theorem 9.5.2. *There is a locally closed $(3g - 3 + n)$ -dimensional smooth subscheme $X \subset H_{\nu, g, n}$ including x_0 such that the restriction of the universal family of $H_{\nu, g, n}$ over X is a Kuranishi family for all of its fibers.*

In addition, one can choose an X with the following properties:

- (i) X is affine and \mathbb{G}_{x_0} -invariant;
- (ii) For any $y \in X$, we have $\mathbb{G}_y \subset \mathbb{G}_{x_0}$;
- (iii) For any $y \in X$, there is a \mathbb{G}_y -invariant neighborhood $U \subset X$ of y such that $\mathbb{G}_y = \{\gamma \in \mathbb{G} : \gamma(U) \cap U \neq \emptyset\}$ in the analytic topology.

Proof. See [8] Theorem XI.6.5. To add. \square

Hence we get a Kuranishi family $(\pi : \mathcal{C} \rightarrow (X, x_0), \sigma_i)$.

Definition 9.5.3 (Standard algebraic Kuranishi family). *Let $(C; p_1, \dots, p_n)$ be a stable n -pointed genus g curve with $G = \text{Aut}(C; p_i)$. Let $(\pi : \mathcal{C} \rightarrow (X, x_0), \sigma_i)$ be the Kuranishi family in Theorem 9.5.2 and it is called a standard algebraic Kuranishi family if the following conditions are satisfied.*

- (a) X is affine and the family is a Kuranishi family for all of its fibers;
- (b) The action of G_{x_0} on the central fiber extends to compatible actions on \mathcal{C} and X ;
- (c) For any $y \in X$ we have $G_y := \text{Aut}(\mathcal{C}_y; \sigma_i(y)) \cong \text{stab}_{G_{x_0}}(y)$;
- (d) For any $y \in X$, there is a G_y -invariant analytic neighborhood U of y in X such that any isomorphism (of n -pointed curves) between fibers over U is induced by an element of G_y .

Definition 9.5.4 (Standard Kuranishi family). *Let $(C; p_1, \dots, p_n)$ be a stable n -pointed genus g curve with $G = \text{Aut}(C; p_i)$. We will say a Kuranishi family $\mathcal{X} \rightarrow (B, b_0), \tau_i : B \rightarrow \mathcal{X}$ of $(C; p_1, \dots, p_n)$ is called a standard Kuranishi family if the following conditions are satisfied.*

- (a) B is a connected complex manifold and the family is a Kuranishi family at every points of B ;
- (b) The action of G on the central fiber extends to compatible actions on \mathcal{X} and B ;
- (c) Any isomorphism (of n -pointed curves) between fibers is induced by an element of G .

Remark 9.5.5. *In fact, given any Kuranishi family, there is a neighborhood of the base point such that the restriction is standard. By the uniqueness of the Kuranishi family, it suffices to notice that this is true for a standard algebraic Kuranishi family. By Theorem 9.5.2 and we win.*

Corollary 9.5.6. *Any (maybe not stable) nodal curves $(C; p_1, \dots, p_n)$ has a versal deformation (is unique up to an isomorphism, which however need not be unique).*

Proof. Adding some smooth marked points such that it becomes a stabel curve. Then taking the Kuranishi family of it and ignore the added marked points. \square

Corollary 9.5.7. *Any family of nodal curves can be locally embedded in a family of nodal curves with a reduced, or even smooth, base.*

Proof. For any family of nodal curves $\eta : X \rightarrow S$ and let $s_0 \in S$. By the previous corollary we can get a versal deformation $\pi : \mathcal{X} \rightarrow (B, b_0)$ of $\eta^{-1}(s_0)$. After shrinking S (étale locally), we have a closed immersion $S \hookrightarrow T$ where T is smooth and a cartesian

$$\begin{array}{ccc} X & \xrightarrow{\beta} & \mathcal{X} \\ \downarrow \eta & & \downarrow \pi \\ S & \xrightarrow{\alpha} & B \end{array}$$

with $b_0 = \alpha(s_0)$. Hence we get cartesians

$$\begin{array}{ccccc} X & \xrightarrow{(\eta, \beta)} & S \times \mathcal{X} & \longrightarrow & T \times \mathcal{X} \\ \downarrow \eta & & \downarrow (\text{id}_S, \pi) & & \downarrow (\text{id}_T, \pi) \\ S & \xrightarrow{(\text{id}_S, \alpha)} & S \times B & \longrightarrow & T \times B \end{array}$$

Clearly, $T \times B$ is smooth, and $S \rightarrow T \times B$ is a closed immersion. \square

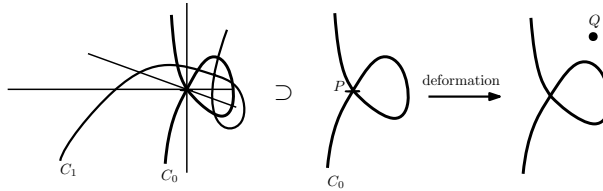
Chapter 10

The stack of all curves

10.1 Families of all arbitrary curves

Definition 10.1.1. Here we redefine a curve over k is a scheme C of finite type over k of dimension 1 (rather than pure dimension 1). The genus of C is defined as $g(C) = 1 - \chi(C, \mathcal{O}_C)$.

Remark 10.1.2. Why we not allow pure dimension 1? Since they may arise as deformations of connected pure one-dimensional curves; without this relaxation, the stack of all curves would fail to be algebraic. For example in [58] Example III.9.8.4, a flat family of rational curves defined by $\mathbb{P}^1 \rightarrow \mathbb{P}^3$ via $[x : y] \mapsto [x^3 : x^2y : xy^2 : ty^3]$ for any $t \neq 0$. As $t \rightarrow 0$, we may get a singular non-reduced curve C_0 with an embedded point at that node, but C_0 can deforms to the disjoint union of a plane nodal curve and a point in \mathbb{P}^3 .



Definition 10.1.3. (i) A family of curves over a scheme S is a flat, proper and finitely presented morphism $C \rightarrow S$ of algebraic spaces such that every fiber is a curve.

(ii) A family of n -pointed curves is a family of curves $C \rightarrow S$ together with n sections $\sigma_1, \dots, \sigma_n : S \rightarrow C$ (with no condition on whether they are distinct or land in the relative smooth locus of C over S).

Remark 10.1.4. (i) When we consider a family of stable curve, since the relative dualizing sheaf is ample, we can get it is projective, hence must be a scheme;

(ii) There are some examples such that C are not a scheme.

Proposition 10.1.5. If $C \rightarrow S$ is a family of curves over S , there exists an étale cover $S' \rightarrow S$ such that $C_{S'} \rightarrow S'$ is projective.

Sketch-Local to global. Consider cartesian

$$\begin{array}{ccccccc}
 C_0 := C_s & \longleftarrow & C_1 & \longleftarrow & \dots & \longleftarrow & \widehat{C} & \longrightarrow & C \\
 \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 S_0 := \text{Spec} \kappa(s) & \longleftarrow & S_1 := \text{Spec} \mathcal{O}_{S,s}/\mathfrak{m}_s^2 & \longleftarrow & \dots & \longrightarrow & \widehat{S} := \text{Spec} \widehat{\mathcal{O}}_{S,s} & \longrightarrow & S
 \end{array}$$

Step 1. $C_0 \rightarrow \text{Spec} \kappa(s)$. By St 0ADD, every separated algebraic space of dimension one is a scheme. Any one-dimensional proper $\kappa(s)$ -scheme is projective by St 0A26. In particular we get a ample line bundle L_0 on C_0 .

Step 2. $C_n \rightarrow S_n$. The obstruction to deforming a line bundle L_n on C_n to L_{n+1} on C_{n+1} lives in $H^2(C_0, \mathcal{O}_{C_0})$ and thus vanishes as $\dim C_0 = 1$. Thus there exists a compatible sequence of line bundles L_n on C_n . Since ampleness is an open condition in families and L_0 is ample, L_n is also ample.

Step 3. $\widehat{C} \rightarrow \widehat{S}$ with \widehat{S} noetherian. Use Grothendieck's Existence Theorem we get an equivalence $\text{Coh}(\widehat{C}) \rightarrow \varprojlim \text{Coh}(C_n)$. As $\widehat{C} \rightarrow \widehat{S}$ is proper, then by Chow's lemma there exists a projective birational morphism $C' \rightarrow \widehat{C}$ of algebraic spaces such that $C' \rightarrow S$ is projective. This allows one to reduce Grothendieck's Existence Theorem for $\widehat{C} \rightarrow \widehat{S}$ to $C' \rightarrow \widehat{S}$ using devissage. As a result, using again that ampleness is an open condition in families we can extend the sequence of line bundle L_n to a line bundle \widehat{L} on \widehat{C} which is ample.

Step 4. S is of finite type over \mathbb{Z} . For every closed point $s \in S$, apply Artin Approximation to the functor

$$(\text{Sch}/S) \rightarrow (\text{Sets}), (T \rightarrow S) \mapsto \text{Pic}(C_T)$$

to obtain an étale neighborhood $(S', s') \rightarrow (S, s)$ of s and a line bundle L' on $C_{S'}$ extending L_0 . By openness of ampleness, we can replace S' with an open neighborhood of s' such that L' is relatively ample over S' .

Step 5. General S . Use noetherian approximation. \square

10.2 Algebraicity of the stack of all curves

Definition 10.2.1. Let $\mathcal{M}_{g,n}^{\text{all}}$ denote the category over $\text{Sch}_{\text{ét}}$ whose objects over S consists of families of curves $C \rightarrow S$ and n sections $\sigma_i : S \rightarrow C$. A morphism $(C' \rightarrow S', \sigma'_i) \rightarrow (C \rightarrow S, \sigma_i)$ is the data of the cartesian

$$\begin{array}{ccc} C' & \xrightarrow{g} & C \\ \sigma'_i \uparrow \downarrow & & \sigma_i \uparrow \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

with $g \circ \sigma'_i \rightarrow \sigma_i \circ f$.

Lemma 10.2.2. $\mathcal{M}_{g,n}^{\text{all}}$ is a stack over $\text{Sch}_{\text{ét}}$.

Proof. Handle $n = 0$. Let $S' \rightarrow S$ be an étale cover with $C' \rightarrow S'$. And $\alpha : p_1^* C' \rightarrow p_2^* C'$ is an isomorphism over $S' \times_S S'$ satisfying the cocycle condition. The quotient of the étale equivalence relation

$$\begin{array}{ccccc} R := p_1^* C' & \xrightarrow[p_2 \circ \alpha]{p_1} & C' & \dashrightarrow & C := C'/R \\ \downarrow & & \downarrow & & \downarrow \\ S' \times_S S' & \xrightarrow[p_2]{p_1} & S' & \longrightarrow & S \end{array}$$

Well done. \square

Lemma 10.2.3. $\Delta : \mathcal{M}_{g,n}^{\text{all}} \rightarrow \mathcal{M}_{g,n}^{\text{all}} \times \mathcal{M}_{g,n}^{\text{all}}$ is representable.

Proof. Handle $n = 0$. Consider the cartesian

$$\begin{array}{ccc} \underline{\text{Isom}}_T(C_1, C_2) & \longrightarrow & T \\ \downarrow & & (C_1, C_2) \downarrow \\ \mathcal{M}_{g,n}^{\text{all}} & \xrightarrow{\Delta} & \mathcal{M}_{g,n}^{\text{all}} \times \mathcal{M}_{g,n}^{\text{all}} \end{array}$$

We need to show $\underline{\text{Isom}}_T(C_1, C_2)$ is an algebraic space. By Proposition 10.1.5, there exists an étale cover $T' \rightarrow T$ such that $C_{i,T'} \rightarrow T'$ is projective. Hence we may let C_1, C_2 are projective over T . Indeed, as

$$\underline{\text{Isom}}_T(C_1, C_2) \times_T T' = \underline{\text{Isom}}_{T'}(C_{1,T'}, C_{2,T'}),$$

we get $\text{Isom}_{T'}(C_{1,T'}, C_{2,T'}) \rightarrow \text{Isom}_T(C_1, C_2)$ is representable, surjective and étale. Hence if $\text{Isom}_{T'}(C_{1,T'}, C_{2,T'})$ is an algebraic space, so is $\text{Isom}_T(C_1, C_2)$.

Fact. (St 05XD) If $f : X \rightarrow Y$ are T -morphism such that X, Y are proper, flat and locally of finite presentation over T , then for any $U \rightarrow T$ such that $X_U \cong Y_U$ if and only if $U \rightarrow T$ factor through an open subscheme $T_0 \subset T$.

Now we get the inclusions

$$\underline{\text{Isom}}_T(C_1, C_2) \subset \underline{\text{Mor}}_T(C_1, C_2) \subset \underline{\text{Hilb}}(C_1 \times_T C_2/T)$$

where the second inclusion is $(g : C_1 \rightarrow C_2) \mapsto (\Gamma_g : C_1 \rightarrow C_1 \times_T C_2)$. The first inclusion is representable open immersion by the above fact. The second inclusion, we find that a subspace $[Z \subset C_1 \times_T C_2] \in \underline{\text{Hilb}}(C_1 \times_T C_2/T)$ is in the image of the inclusion if and only if $Z \rightarrow C_1 \times_T C_2 \rightarrow C_1$ is an isomorphism (and similarly for other valued points). Therefore by the above fact we win. \square

Theorem 10.2.4. $\mathcal{M}_{g,n}^{\text{all}}$ is an algebraic stack locally of finite type over \mathbb{Z} .

Sketch. Step 1. Reduce to $n = 0$. Since $\mathcal{M}_{g,n+1}^{\text{all}}$ is the universal family over $\mathcal{M}_{g,n}^{\text{all}}$, we can prove the conclusion at the case $\mathcal{M}_g^{\text{all}}$. (**Why?**)

Step 2. Look for possible smooth cover H' of $\mathcal{M}_g^{\text{all}}$. Let C_0 be any projective curves C_0 over k . Choosing an embedding $C_0 \subset \mathbb{P}_k^N$ such that $h^1(C_0, \mathcal{O}(1)) = 0$ by Serre's vanishing theorem. Let $P(t)$ be its Hilbert polynomial. Let $H := \underline{\text{Hilb}}_{\mathbb{P}_k^N/\mathbb{Z}}^P$ be the Hilbert scheme which is projective over \mathbb{Z} . Consider the universal family

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathbb{P}_H^N \\ \downarrow & \swarrow & \\ H & & \end{array}$$

there is a point $h_0 \in H(k)$ such that $\mathcal{C}_{h_0} = C_0$. By Review A.1.1 we can find an open neighborhood $H' \subset H$ of h_0 such that for any $s \in H'$ we have $h^1(\mathcal{C}_s, \mathcal{O}_{\mathcal{C}_s}(1)) = 0$. Now consider

$$H' \rightarrow \mathcal{M}_g^{\text{all}}, [C \hookrightarrow \mathbb{P}^N] \mapsto [C],$$

and by the representability of the diagonal, this map is representable as H' is a scheme.

Step 3. Show that $H' \rightarrow \mathcal{M}_g^{\text{all}}$ is smooth. Using Infinitesimal Lifting Criterion such that for all surjections $A \rightarrow A_0$ of artinian local rings with residue field k such that $k = \ker(A \rightarrow A_0)$ and for all diagrams

$$\begin{array}{ccc} & \text{Spec}k & \\ & \swarrow & \searrow \\ \text{Spec}A_0 & \xrightarrow{[\mathcal{C}_0 \subset \mathbb{P}_{A_0}^N]} & H' \\ \downarrow & \swarrow [\mathcal{C} \subset \mathbb{P}_A^N] & \downarrow \\ \text{Spec}A & \xrightarrow{\mathcal{C}} & \mathcal{M}_g^{\text{all}} \end{array}$$

We need to find that dotted arrow. This diagram is equivalent to

$$\begin{array}{ccccc} & & \mathbb{P}_k^N & & \mathbb{P}_{A_0}^N & & \mathbb{P}_A^N \\ & \nearrow & & \nearrow & & \nearrow & \\ \mathcal{C} & \hookrightarrow & \mathcal{C}_0 & \hookrightarrow & \mathcal{C} & \dashrightarrow & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Spec}k & \hookrightarrow & \text{Spec}A_0 & \hookrightarrow & \text{Spec}A & & \end{array}$$

For simplifying, we let C is of locally complete intersection (general case see [51] and [50]). Let \mathcal{J} be the ideal sheaf of $C \rightarrow \mathbb{P}_k^N$ generated by regular sequence locally and that $\mathcal{J}/\mathcal{J}^2$ is a vector bundle on C with

$$0 \rightarrow \mathcal{J}/\mathcal{J}^2 \rightarrow \Omega_{\mathbb{P}_k^N}|_C \rightarrow \Omega_C \rightarrow 0.$$

By long exact sequence we get

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_C) \rightarrow \mathrm{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C) \rightarrow \mathrm{Ext}_{\mathcal{O}_C}^1(\Omega_{\mathbb{P}_k^N}|_C, \mathcal{O}_C) = H^1(C, T_{\mathbb{P}_k^N}|_C).$$

Consider the canonical sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(1)^{\oplus N+1} \rightarrow T_{\mathbb{P}^N}|_C \rightarrow 0.$$

Since $H^2(C, \mathcal{O}_C) = 0$ and $H^1(C, \mathcal{O}_C(1)) = 0$ by $[C] \in H'$ we get $H^1(C, T_{\mathbb{P}_k^N}|_C) = 0$. Hence we get a surjection

$$\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_C) \rightarrow \mathrm{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C).$$

Use some deformation theory (**But I don't know! May be use something about cotangent complex which is the reason we let C is of locally complete intersection!**) we get $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_C)$ classifies embedded deformations of $C_0 \rightarrow \mathbb{P}_{A_0}^N$ to $C' \rightarrow \mathbb{P}_A^N$ and $\mathrm{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C)$ classifies deformations of C_0 over A_0 to C' over A . As the map is $[C' \rightarrow \mathbb{P}_A^N] \mapsto C'$ and is surjective, we win. \square

10.3 Algebraicity of several stacks and boundedness of stable curves

Proposition 10.3.1 (Several stacks). *We have inclusions*

$$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n} \subset \mathcal{M}_{g,n}^{ss} \subset \mathcal{M}_{g,n}^{\mathrm{pre}} \subset \mathcal{M}_{g,n}^{\leq \mathrm{nodal}} \subset \mathcal{M}_{g,n}^{\mathrm{all}}$$

of prestacks. Then all of these are open substacks, hence all of these are algebraic stacks locally of finite type over \mathbb{Z} .

Proof. • By Theorem 10.2.4, $\mathcal{M}_{g,n}^{\mathrm{all}}$ is an algebraic stack locally of finite type over \mathbb{Z} .

- $\mathcal{M}_{g,n}^{\leq \mathrm{nodal}} \subset \mathcal{M}_{g,n}^{\mathrm{all}}$ is an open substack. Actually by Corollary 7.4.3 we get the nodal locus is open when C is a scheme. In general for an étale cover $g : C' \rightarrow C$ by a scheme, we find that a point $p \in C'$ is a node in its fiber if and only if $g(p)$ is a node in its fiber. We win.
- $\mathcal{M}_{g,n}^{\mathrm{pre}} \subset \mathcal{M}_{g,n}^{\leq \mathrm{nodal}}$ is an open substack. This is because for a family $(C \rightarrow S, \{\sigma_i\})$ of nodal curves, the locus $\{s \in S : \sigma_i(s) \text{ are disjoint and smooth}\}$ is open.
- $\mathcal{M}_{g,n}^{ss} \subset \mathcal{M}_{g,n}^{\mathrm{pre}}$ is an open substack. This is because the nef locus is open (**But I don't know the relation between nefness and semistable!**).
- $\overline{\mathcal{M}}_{g,n} \subset \mathcal{M}_{g,n}^{ss}$ is an open substack. Indeed the stable locus is open by Proposition 8.3.4.
- $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is an open substack. Indeed this is by the fact that smooth locus is open. \square

Proposition 10.3.2. $\overline{\mathcal{M}}_{g,n}$ is a quasi-compact smooth Deligne-Mumford stack of dimension $3g - 3 + n$ over \mathbb{Z} .

Proof. • $\overline{\mathcal{M}}_{g,n}$ is quasi-compact. Let (C, p_1, \dots, p_n) be a n -pointed stable curve. By Theorem 8.2.1, we get $(\omega_C(p_1 + \dots + p_n))^{\otimes 3}$ is very ample, we get $C \hookrightarrow \mathbb{P}^N$ with Hilbert polynomial $P(t)$. This is independent of C . Hence consider closed subscheme $H \subset \mathrm{Hilb}_{\mathbb{P}_k^N/\mathbb{Z}}^P \times (\mathbb{P}^N)^n$ of embedded curve and n points $(C \hookrightarrow \mathbb{P}^N, p_i \in C)$. Consider a forgetful functor

$$H \rightarrow \mathcal{M}_{g,n}^{\mathrm{all}}, (C \hookrightarrow \mathbb{P}^N, p_i \in C) \mapsto (C, \{p_i\}).$$

Then the image of $|H| \rightarrow |\mathcal{M}_{g,n}^{\mathrm{all}}|$ contains $\overline{\mathcal{M}}_{g,n}$. As $\mathrm{Hilb}_{\mathbb{P}_k^N/\mathbb{Z}}^P$ is projective, then H is quasi-compact. Hence $\overline{\mathcal{M}}_{g,n}$ is quasi-compact.

- $\overline{\mathcal{M}}_{g,n}$ is Deligne-Mumford stack. By Proposition 9.2.9 for $i = 0$ and Proposition 9.2.7 (a), we get a n -pointed stable curve (C, p_1, \dots, p_n) has no infinitesimal automorphisms, i.e. that the Lie algebra $T_e \text{Aut}(C, p_1, \dots, p_n)$ is trivial. Since the automorphism group scheme $\text{Aut}(C, p_1, \dots, p_n)$ is of finite type, this implies that $\text{Aut}(C, p_1, \dots, p_n)$ is finite and discrete, hence $\overline{\mathcal{M}}_{g,n}$ is a quasi-compact Deligne-Mumford stack.
- $\overline{\mathcal{M}}_{g,n}$ is smooth over $\text{Spec}\mathbb{Z}$. Proposition 9.2.9 for $i = 2$ and Proposition 9.2.7 (c) implies that there are no obstructions to deforming C . As the algebraicity of $\overline{\mathcal{M}}_{g,n}$, this will allow us to invoke the Infinitesimal Lifting Criterion to establish that $\overline{\mathcal{M}}_{g,n}$ is smooth over $\text{Spec}\mathbb{Z}$.
- $\overline{\mathcal{M}}_{g,n}$ has relative dimension $3g - 3 + n$ over $\text{Spec}\mathbb{Z}$. Proposition 9.2.9 for $i = 1$ and Proposition 9.2.7 (b) implies that isomorphism classes of deformations of (C, p_1, \dots, p_n) , it is identified with the Zariski tangent space of $\overline{\mathcal{M}}_{g,n}$ at the point corresponding to (C, p_1, \dots, p_n) . This will allow us to conclude that $\overline{\mathcal{M}}_{g,n}$ has relative dimension $3g - 3 + n$ over \mathbb{Z} . \square

10.4 The family of elliptic curves $\mathcal{M}_{1,1}$

Proposition 10.4.1. \mathcal{M}_1 is not a stack.

Proof. See [75] Remark 8.4.15 for the References. \square

Remark 10.4.2. If we let \mathcal{M}'_1 as morphisms of algebraic spaces, then this will be a stack. This follows the Picard functor and the stack $\mathcal{M}_{1,1}$. In fact if we consider the universal elliptic curve $\mathcal{E} \rightarrow \mathcal{M}_{1,1}$, then Picard functor gives $\mathcal{M}'_1 \rightarrow \mathcal{M}_{1,1}$ which induce $\mathcal{M}'_1 \cong B\mathcal{E}$. We omitted here.

Proposition 10.4.3. $\mathcal{M}_{1,1}$ is a smooth Deligne-Mumford stack.

Proof. By Proposition 10.3.1 we get that $\mathcal{M}_{1,1}$ is an open substack of $\overline{\mathcal{M}}_{1,1}$. Hence it is a smooth Deligne-Mumford stack. \square

Proposition 10.4.4. $\mathcal{M}_{1,1}$ has a coarse moduli space $M_{1,1} \cong \mathbb{A}^1$.

Proof. \square

Chapter 11

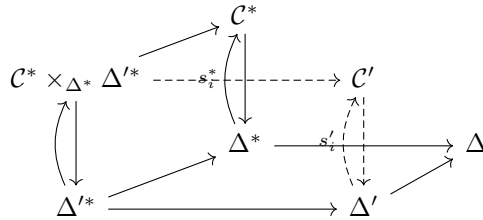
Stable reduction: why $\overline{\mathcal{M}}_{g,n}$ is proper?

In this section we will use the Valuative Criterion (Theorem C.1.3 (1)) to show that $\overline{\mathcal{M}}_{g,n}$ is proper. The existence of extension is called stable reduction, which is our main theorem:

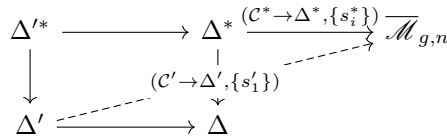
Lemma 11.0.1. *The diagonal of the stack $\mathcal{M}_{g,n}^{\text{all}}$ is separated. In particular, $\mathcal{M}_{g,n}^{\text{all}} \rightarrow \text{Spec}(\mathbb{Z})$ is quasi-separated.*

Proof. Omitted. See St 0DSQ. □

Theorem 11.0.2 (Stable reduction). *Let R be a DVR with fraction field K and $\Delta = \text{Spec}(R), \Delta^* = \text{Spec}(K)$. If $(\mathcal{C}^* \rightarrow \Delta^*, s_1^*, \dots, s_n^*)$ is a family of n -pointed stable curves of genus g , then there exists a finite cover $\Delta' \rightarrow \Delta$ of spectra of DVRs and a family $(\mathcal{C}' \rightarrow \Delta', s'_1, \dots, s'_n)$ of stable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \rightarrow \Delta'^*$. As*



given by



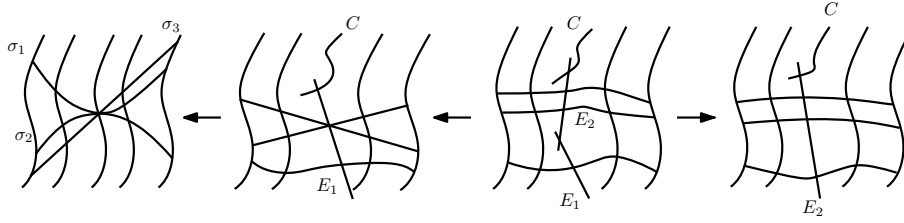
After proving this and the uniqueness, we can get the following conclusion:

Theorem 11.0.3. *If $2g - 2 + n > 0$, then $\overline{\mathcal{M}}_{g,n}$ is a proper smooth Deligne-Mumford stack of dimension $3g - 3 + n$ over \mathbb{Z} .*

By using the Keel-Mori Theorem, we get

Corollary 11.0.4. *If $2g - 2 + n > 0$, there exists a coarse moduli space $\overline{\mathcal{M}}_{g,n} \rightarrow \overline{M}_{g,n}$ where $\overline{M}_{g,n}$ is a proper algebraic space over \mathbb{Z} .*

Example 11.0.5. *Let $\Delta = \text{Spec}(R)$ where R be a DVR with uniformizer t . Let C be a smooth curve and consider $\mathcal{C} = C \times \Delta$ with sections $(\sigma_1, \sigma_2, \sigma_3) = (t^2, -t^2, 4t)$ as following diagram. The first two arrows are blowing up and the third is contracting E_1 .*



Remark 11.0.6. *Actually there are several methods to prove this. The first proof due to the original paper [30] by consider the Jacobians of curves and reduce the case into the semistable reduction of abelian varieties. Our method follows [55] by using some birational geometry of surfaces to prove the case of characteristic 0. There is another method can deal the positive or mixed characteristic case by [10], for this we refer St 0E8C.*

11.1 Proof of stable reduction in characteristic 0

Lemma 11.1.1. *Let R be a DVR with uniforming t and $0 := (t)$. Let $\mathcal{C} \rightarrow \Delta = \text{Spec}(R)$ be a flat, proper and finitely presented morphisms such that each geometric fiber is a curve. Assume that \mathcal{C} is regular. Let $p \in \mathcal{C}_0$.*

(a) *If p is a smooth point in the reduced fiber $(\mathcal{C}_0)_{\text{red}}$. Show that after possibly an extension of DVRs, there exists an étale neighborhood of p (defined over R)*

$$\text{Spec}R[x, y]/(x^a - t) \rightarrow \mathcal{C}.$$

(b) *If p is a node in the reduced fiber $(\mathcal{C}_0)_{\text{red}}$. Show that there exists an étale neighborhood of p (defined over R)*

$$\text{Spec}R[x, y]/(x^a y^b - t) \rightarrow \mathcal{C}.$$

Proof. □

Lemma 11.1.2. *Let a, b, m be positive integers such that both a and b divide m .*

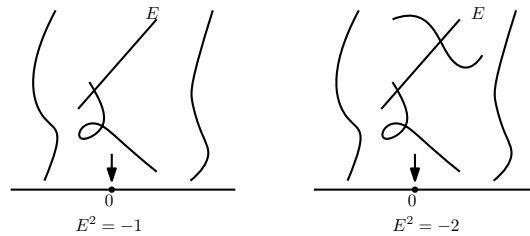
(a) *Let $X = \text{Spec}k[x, t]/(t^m - x^a)$ and normalization $\tilde{X} \rightarrow X$. Then each preimage of the origin is locally defined by $x = t^k$ for some k .*

(b) *Let $X = \text{Spec}k[x, y, t]/(t^m - x^a y^b)$ and normalization $\tilde{X} \rightarrow X$. Then each preimage of the origin is locally defined by $t^k = xy$. In particular is a reduced and nodal point in the fiber over $t = 0$.*

Proof. (a) We have $x^a - t^m = \prod_{i=0}^{a-1} (x - \zeta^i t^{m/a})$ where ζ be a primitive a -th root of unity. Hence the origin has a preimages in \tilde{X} locally defined by $x - \zeta^i t^{m/a}$, respectively.

(b) □

Lemma 11.1.3. *Let $\mathcal{C} \rightarrow \Delta = \text{Spec}(R)$ be a family of nodal curves where R be a DVR such that the general fiber \mathcal{C}^* is smooth. Then if E is a rational tail (rational bridge with out marked points) of \mathcal{C}_0 , then $E^2 = -1$ ($E^2 = -2$). As*



Proof. For any $E \cong \mathbb{P}^1 \subset \mathcal{C}_0$, then $0 = E \cdot \mathcal{C}_0 = E^2 + E \cdot E^c$. We win. □

For simplicity of notation, we assume that there are no marked points, i.e. $n = 0$. Fix a spectrum of DVR $\Delta = \text{Spec}(R)$, $\Delta^* = \text{Spec}(K)$ and $t \in R$ is the uniformizer, and $0 = (t) \in \text{Spec}R$ the unique closed point. Consider $\mathcal{C}^* \rightarrow \Delta^*$ be a family of stable curve.

STEP 1. Reduce to the case where $\mathcal{C}^* \rightarrow \Delta^*$ is smooth. If \mathcal{C}^* has k nodes, then after a finite extension of K we can arrange that each node is given by K -points $p_i \in \mathcal{C}^*(K)$. Let the pointed normalization $(\tilde{\mathcal{C}}^*, q_1, \dots, q_{2k})$ of it. By induction on the genus g , we perform stable reduction on each connected component and then take the nodal union along sections. After possibly an extension of K (and R), this produces a family of curves $\mathcal{C} \rightarrow \Delta$ extending $\mathcal{C}^* \rightarrow \Delta^*$.

STEP 2. Find some flat extension $\mathcal{C} \rightarrow \Delta$. As $\omega_{\mathcal{C}^*/\Delta^*}^{\otimes 3}$ is very ample, we can get an embedding as follows

$$\begin{array}{ccc} \mathbb{P}^{5g-6} \times \Delta^* & \hookrightarrow & \mathbb{P}^{5g-6} \times \Delta \\ \uparrow |\omega_{\mathcal{C}^*/\Delta^*}^{\otimes 3}| & & \uparrow \\ \mathcal{C}^* & \hookrightarrow & \mathcal{C} := \overline{\mathcal{C}^*} \\ \downarrow & & \downarrow f \\ \Delta^* & \hookrightarrow & \Delta \end{array}$$

where $\mathcal{C} := \overline{\mathcal{C}}$ be the scheme-theoretic image of $\mathcal{C}^* \hookrightarrow \mathbb{P}^{5g-6} \times \Delta$. Now we focus on f . Actually the scheme-theoretic closure does not bring more embedded points. Hence by Proposition 2.1.3 we get f is flat.

STEP 3. Use embedded resolutions to find a resolution of singularities $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ so that the reduced central fiber $(\tilde{\mathcal{C}}_0)_{red}$ is nodal. By Theorem B.0.2, there exists a finite sequence of blow-ups at closed points of \mathcal{C}_0 yielding a projective birational morphism

$$\begin{array}{ccc} \tilde{\mathcal{C}}_0 \subset \tilde{\mathcal{C}} & \longrightarrow & \dots \longrightarrow \mathcal{C} \supset \mathcal{C}_0 \\ & \searrow & \downarrow \\ & & \Delta \end{array}$$

such that $\tilde{\mathcal{C}}$ is regular flat family of curves and such that the reduced central fiber $(\tilde{\mathcal{C}}_0)_{red}$ is nodal. Now replace \mathcal{C} by $\tilde{\mathcal{C}}$.

STEP 4. Perform a base change $\Delta' \rightarrow \Delta$ such that the normalization of the total family $\mathcal{C} \times_{\Delta} \Delta'$ has a reduced nodal central fiber with many rational tails and bridges. By Lemma 11.1.1, we choose local coordinates x, y around $p \in \mathcal{C}_0$ (étale locally and formally locally) such that $\mathcal{C} \rightarrow \Delta$ can be described as follows:

- (i) If $p \in (\mathcal{C}_0)_{red}$ is a smooth point, then $(x, y) \mapsto x^a$ and the multiplicity of the irreducible component of \mathcal{C}_0 containing p is a ;
- (ii) If $p \in (\mathcal{C}_0)_{red}$ is a (separated) node, then $(x, y) \mapsto x^a y^b$ and the two components of \mathcal{C}_0 containing p have multiplicities a and b .

Let m be the least common multiple of the multiplicities of the irreducible components of \mathcal{C}_0 . Let the ramified morphism $\Delta' = \text{Spec}(R) \rightarrow \Delta$ given by $t \mapsto t^m$. Hence we get

$$\begin{array}{ccccc} \tilde{\mathcal{C}}' & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} \\ & & \downarrow & & \downarrow \\ & & \Delta' & \longrightarrow & \Delta \end{array}$$

where $\mathcal{C}' = \mathcal{C} \times_{\Delta} \Delta'$ and $\tilde{\mathcal{C}}' \rightarrow \mathcal{C}'$ be the normalization. Consider $p \in (\mathcal{C}_0)_{red}$.

(a) If p is a smooth point, then the unique preimage of p in \mathcal{C}' defined locally by $x^a - t^m$. By Lemma 11.1.2 (a), we get each preimage of p in $\tilde{\mathcal{C}}'$ is locally defined by $x = t^k$ which are the smooth points in $\tilde{\mathcal{C}}'_0$;

(b) If p is a node, then the unique preimage of p in \mathcal{C}' defined locally by $x^a y^b - t^m$. By Lemma 11.1.2 (a), we get each preimage of p in $\tilde{\mathcal{C}}'$ is locally defined by $xy = t^k$ which are reduced and nodal points in $\tilde{\mathcal{C}}'_0$. If $k > 1$, $\tilde{\mathcal{C}}'$ have A_{k-1} -singularity.

Hence now we replace \mathcal{C} by $\tilde{\mathcal{C}}'$, which has a reduced central fiber with many rational tails and bridges.

STEP 5. After taking the minimal model, contract all rational tails and bridges in the central fiber. Using Theorem B.0.1 we get a minimal resolution $\mathcal{C}' \rightarrow \mathcal{C}$ and we get a family of prestable curves $\mathcal{C}' \rightarrow \Delta$ where \mathcal{C}' is regular. By Lemma 11.1.3 and Corollary B.0.4, we can get a projective birational map $\mathcal{C}' \rightarrow \mathcal{C}'_{\min}$ where \mathcal{C}'_{\min} is semistable. So we replace \mathcal{C} by \mathcal{C}'_{\min} . (This is the semistable reduction!) Using Proposition 8.5.1, we can get a relative canonical stable model $\mathcal{C}' \rightarrow \mathcal{C}^{st}$.

11.2 Explicit stable reduction

Proposition 11.2.1. *Let $\mathcal{C} \rightarrow \Delta$ be a generically smooth, proper and flat family such that $(\mathcal{C}_0)_{red}$ is nodal. Let $\mathcal{C}_0 = \sum_i a_i D_i$ where a_i is the multiplicity of D_i . Let $\Delta' \rightarrow \Delta$ defined by $t \mapsto t^p$ where p prime and set $\mathcal{C}' = \mathcal{C} \times_{\Delta} \Delta'$. Then after taking normalization $\tilde{\mathcal{C}}' \rightarrow \mathcal{C}'$ is branched cover ramified over $\sum_i (a_i \pmod{p}) D_i$.*

Example 11.2.2 (Stable reduction of A_{2k+1} -singularity). *Let $\mathcal{C} \rightarrow \Delta = \text{Spec}(R)$ be a generically smooth family degenerating to a A_{2k+1} -singularity in the central fiber where have local equation around the singular point is $y^2 = x^{2k+1} + t$. Now we will work through the steps in the proof of stable reduction. The first two steps have already finished, now we start at step 3.*

► **STEP 3. Use embedded resolutions to find a resolution of singularities $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ so that the reduced central fiber $(\tilde{\mathcal{C}}_0)_{red}$ is nodal.** We consider two charts in blowing up with coordinates x', y' where the original coordinates are x, y , as:

$$\begin{array}{ccc} E|_{U_1} = V(x') \hookrightarrow U_1 \hookrightarrow \tilde{\mathcal{C}} = \text{Bl}_p \mathcal{C} & (x', y') & E|_{U_2} = V(y') \hookrightarrow U_2 \hookrightarrow \tilde{\mathcal{C}} = \text{Bl}_p \mathcal{C} & (x', y') \\ \downarrow \swarrow & \downarrow & \downarrow \swarrow & \downarrow \\ \mathcal{C} & (x', x'y') & \mathcal{C} & (x'y', y') \end{array}$$

• **The first blowing up.** In the first chart, the preimage of $y^2 - x^{2k+1}$ is $x'^2 y'^2 - x'^{2k+1} = x'^2 (y'^2 - x'^{2k-1})$; in the second chart it is $y'^2 - (x'y')^{2k+1} = y'^2 (1 - x'^{2k+1} y'^{2k-1})$. Hence the exceptional divisor E_1 has multiplicity 2.

• **The second blowing up.** In the first chart, the preimage of $x^2 (y^2 - x^{2k-1})$ is $x'^4 (y'^2 - x'^{2k-3})$; in the second chart it is $x'^2 y'^4 (1 - x'^{2k-1} y'^{2k-3})$. Hence the exceptional divisor E_2 has multiplicity 4.

• **After k blowing ups.** We get $x^{2k} (y^2 - x)$ with the exceptional divisors E_i has multiplicity $2i$.

• **One more blowing up.** We get the preimage of $x^{2k} (y^2 - x)$ in the second chart is

$$x'^{2k} y'^{2k+1} (y' - x')$$

with the exceptional divisor F has multiplicity $2k + 1$.

• **The final blowing up.** We get the preimage of $x^{2k} y^{2k+1} (y - x)$ in the first chart is

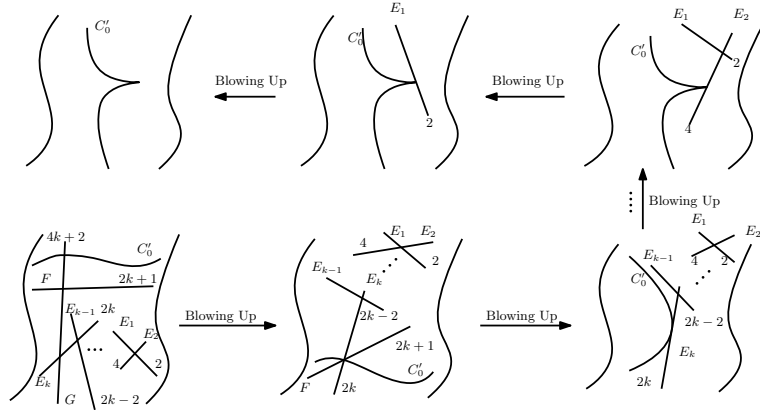
$$x'^{4k+2} y'^{2k+1} (y' - 1)$$

with the exceptional divisor G has multiplicity $4k + 2$.

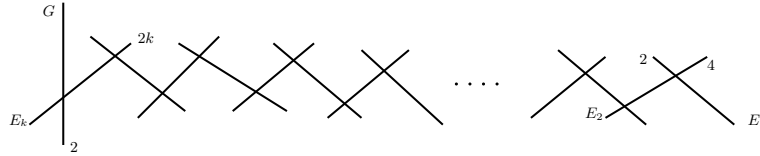
The process as follows:

► **STEP 4. Perform a base change $\Delta' \rightarrow \Delta$ such that the normalization of the total family $\mathcal{C} \times_{\Delta} \Delta'$ has a reduced nodal central fiber with many rational tails and bridges.**

• **The first base change.** First consider $\Delta' \rightarrow \Delta, t \mapsto t^{2k+1}$ and normalizing. After inductively apply to the prime factorization $2k + 1$ and normalization, we will use the proposition to analyze the preimage of these irreducible component. Actually we get this $2k + 1$ -degree cover ramified over $\mathcal{C}'_0 + \sum_i E_i$ and we just need to consider F, G . For G , its preimage G' ramified at two points (intersects E_k, \mathcal{C}'_0) with index $2k$. By Riemann-Hurwitz Theorem, we get $2g(G') - 2 = (2k + 1)(2g(G) - 2) + 4k = -2$. Hence $g(G') = 0$ and $G' \cong \mathbb{P}^1$. For F , its preimage F' is unramified at all points, hence $F' = \prod_{j=1}^{2k+1} F_j$ are copies of F . Hence replace Δ by Δ' , we get the central fiber as $\mathcal{C}_0 = \mathcal{C}'_0 + 2G' + \sum_j F_j + \sum_i 2iE_i$.



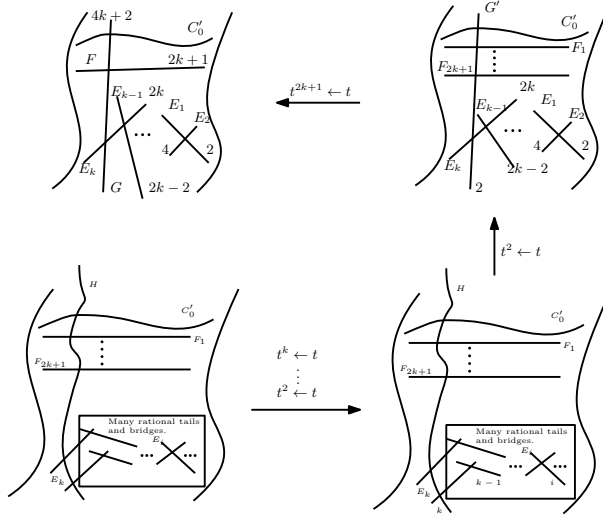
• **The second base change.** Consider $\Delta' \rightarrow \Delta, t \mapsto t^2$ and normalizing. Actually we get this 2-degree cover ramified over $C'_0 + \sum_j F_j$ and we just need to consider G', E_i . For G' , the preimage H ramified at $2k + 2$ points (C'_0, F_j). Hence we get $g(H) = k$ by Riemann-Hurwitz Theorem. For E_i , the things become more complicated as follows:



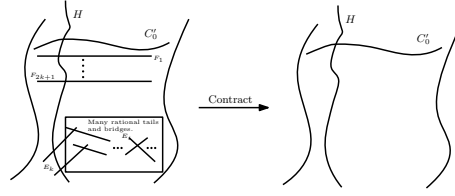
But we can easily see that after this process, these things are just plenty of rational bridges and rational tails.

• **The final base changes.** Here we just need to consider E_i and these have multiplicity i . Consider $t \mapsto t^k$, then (many) E_k has two ramified points, hence by Riemann-Hurwitz Theorem $g(E'_k) = 0$, hence rational. Then consider $t \mapsto t^{k-1}, \dots, t \mapsto t^2$, we have the same results. Hence we also get plenty of rational tails and bridges, which are all multiplicity 1.

The whole process as follows:



► **STEP 5. Contract all rational tails and bridges in the central fiber.** Now we kill all -1 -curves (many E_1 and all F_j), and then (every) E_2 become -1 -curves. Inductively, we kill all E_i and F_j and get a stable central fiber as follows and we win.



11.3 Separatedness of $\overline{\mathcal{M}}_{g,n}$

Proposition 11.3.1. *Let R be a DVR with fraction field K with $\Delta = \text{Spec}(R), \Delta^* = \text{Spec}(K)$. Let $(\mathcal{C} \rightarrow \Delta, \sigma_1^*, \dots, \sigma_n^*)$ and $(\mathcal{D} \rightarrow \Delta, \tau_1^*, \dots, \tau_n^*)$ are n -pointed stable curves, then for any $\alpha^* : \mathcal{C}^* \rightarrow \mathcal{D}^*$ with $\tau_i^* = \alpha^* \circ \sigma_i^*$ over generic fiber can extend to a unique isomorphism $\alpha : \mathcal{C} \rightarrow \mathcal{D}$ with $\tau_i = \alpha \circ \sigma_i$.*

$$\begin{array}{ccc}
 \mathcal{C}^* & \xrightarrow{\alpha^*} & \mathcal{D}^* \\
 \searrow & & \downarrow \\
 & & \Delta^*
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{C} & \xrightarrow{\alpha} & \mathcal{D} \\
 \searrow & & \downarrow \\
 & & \Delta
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & \Delta^* \xrightarrow{\quad} \Delta
 \end{array}$$

Proof. We only prove the case of $n = 0$ generically smooth curves. Let $\mathcal{C}' \rightarrow \mathcal{C}, \mathcal{D}' \rightarrow \mathcal{D}$ be the minimal resolutions and let $\Gamma \subset \mathcal{C}' \times_{\Delta} \mathcal{D}'$ be the closure of the graph of $\text{id} \times \alpha^* : \mathcal{C}^* \rightarrow \mathcal{C}^* \times_{\Delta^*} \mathcal{D}^*$. Let $\Gamma' \rightarrow \Gamma$ be the minimal resolution. Hence we get birational projective maps $\Gamma' \rightarrow \mathcal{C}'$ and $\Gamma' \rightarrow \mathcal{D}'$. By the same proof of [58] Theorem II.8.19, we get

$$\Gamma(\mathcal{C}', \omega_{\mathcal{C}'/\Delta}^{\otimes k}) \cong \Gamma(\Gamma', \omega_{\Gamma'/\Delta}^{\otimes k}) \cong \Gamma(\mathcal{D}', \omega_{\mathcal{D}'/\Delta}^{\otimes k})$$

for all $k \geq 0$. As the canonical bundle are ample, we get

$$\mathcal{C}' \cong \text{Proj} \bigoplus_k \Gamma(\mathcal{C}', \omega_{\mathcal{C}'/\Delta}^{\otimes k}) \cong \text{Proj} \bigoplus_k \Gamma(\mathcal{D}', \omega_{\mathcal{D}'/\Delta}^{\otimes k}) \cong \mathcal{D}'.$$

Furthermore, we know that \mathcal{C}, \mathcal{D} are stable models of $\mathcal{C}', \mathcal{D}'$, respectively. By the uniqueness of stable models, we get $\alpha : \mathcal{C} \cong \mathcal{D}$ extending α^* . \square

Chapter 12

Gluing and forgetful morphisms

We follow [35].

12.1 Gluing morphisms

Proposition 12.1.1. *There are finite morphisms of algebraic stacks*

$$F : \overline{\mathcal{M}}_{i,n} \times \overline{\mathcal{M}}_{g-i,m} \rightarrow \overline{\mathcal{M}}_{g,n+m-2}$$

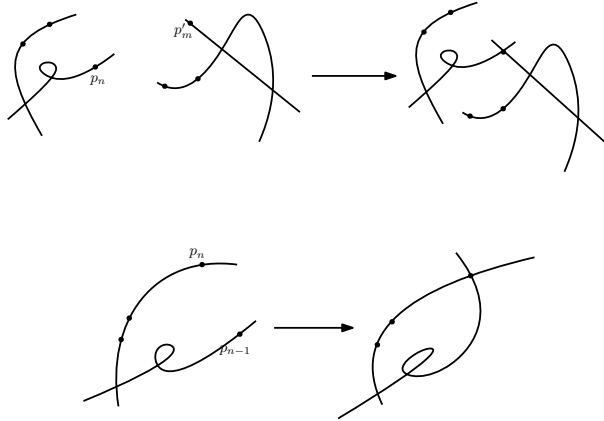
$$((C, p_1, \dots, p_n), (C', p'_1, \dots, p'_m)) \mapsto (C \cap C', p_1, \dots, p_{n-1}, p'_1, \dots, p'_m),$$

and

$$G : \overline{\mathcal{M}}_{g-1,n} \rightarrow \overline{\mathcal{M}}_{g,n-2}$$

$$(C, p_1, \dots, p_n) \mapsto (C/(p_{n-1} \sim p_n), p_1, \dots, p_{n-2}).$$

As follows:



Sketch-using pushout. By the stable reduction, these maps are of course representable and proper. As they have the finite fibers, these maps are now finite. Now for F we let $n = m = 1$ and for G we let $n = 2$.

For F : Let $(\pi : \mathcal{C} \rightarrow S, \sigma), (\pi' : \mathcal{C}' \rightarrow S, \sigma')$ are stable curves. As σ, σ' are closed immersions, we get the pushout exists by the theory of Ferrand (St 0ECH) and as we have the finite cover $\mathcal{C} \sqcup \mathcal{C}' \rightarrow \mathcal{C}$, we

get this pushout is proper and flat (omitted):

$$\begin{array}{ccccc}
 & \text{Spec}(A) & \xrightarrow{\quad} & \text{Spec}A[y] & \\
 & \swarrow & & \swarrow & \\
 \text{Spec}A[x] & \xrightarrow{\quad} & \text{Spec}A[x, y]/(xy) & & \\
 \downarrow & & \downarrow & & \downarrow \\
 \mathcal{C}' & \xrightarrow{\sigma'} & S & \xrightarrow{\sigma} & \mathcal{C} \\
 & \swarrow & & \swarrow & \\
 & & \tilde{\mathcal{C}} & &
 \end{array}$$

where $\text{Spec}A[x]$ is an étale neighborhood of $\sigma(s)$ which is the pulback of étale neighborhood $\text{Spec}(A)$ of any $s \in S$. Since an étale morphism from an affine scheme extend over closed immersions, there is an étale neighborhood $\text{Spec}A[y]$ is an étale neighborhood of $\sigma'(s)$. Then the pushout can be easy to compute as $\text{Spec}(A[x] \times_A A[x]) \cong \text{Spec}A[x, y]/(xy)$. By some results of pushout (in St 0D2G), we get $\text{Spec}A[x, y]/(xy) \rightarrow \tilde{\mathcal{C}}$ is an étale neighborhood of s . Hence $\tilde{\mathcal{C}} \rightarrow S$ is nodal along S . Checking fibers we get $\tilde{\mathcal{C}}_s$ is stable.

For G : Let $(\mathcal{C} \rightarrow S, \sigma_1, \sigma_2)$ are stable curve. Here we consider the pushout:

$$\begin{array}{ccc}
 S \sqcup S & \xrightarrow{\sigma_1 \sqcup \sigma_2} & \mathcal{C} \\
 \downarrow & & \downarrow \\
 S & \longrightarrow & \tilde{\mathcal{C}}
 \end{array}$$

which is étale locally like

$$\begin{array}{ccc}
 \text{Spec}(A \times A) & \xrightarrow{(0,1)} & \text{Spec}A[t] \\
 \downarrow & & \downarrow \\
 \text{Spec}A & \longrightarrow & \text{Spec}A[x, y]/(y^2 - x^2(x+1))
 \end{array}$$

where we find that $x := t^2 - 1, y = t^3 - t$ generate $A \times_{A \times A} A[t]$, then well done. \square

12.2 Boundary divisors of $\overline{\mathcal{M}}_g$

Consider the closed substacks

$$\begin{aligned}
 \delta_0 &= \text{Im}(\overline{\mathcal{M}}_{g-1,2} \rightarrow \overline{\mathcal{M}}_g) \\
 \delta_i &= \text{Im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \rightarrow \overline{\mathcal{M}}_g)
 \end{aligned}$$

where $i = 1, \dots, \lfloor g/2 \rfloor$.

As these maps are finite, we get $\dim \delta_0 = \dim \overline{\mathcal{M}}_{g-1,2} = 3(g-1) - 3 + 2 = 3g - 4$ and similar $\dim \delta_i = 3g - 4$. Hence these are divisors of $\overline{\mathcal{M}}_g$.

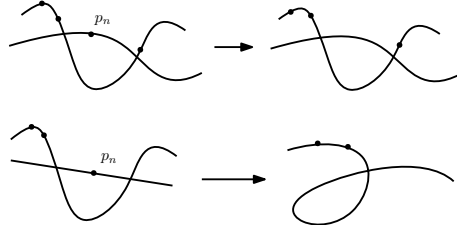
(By analyzing the formal deformation space of a stable curve, one can show that $\delta = \bigcup_{j=0}^{\lfloor g/2 \rfloor} \delta_j$ is a normal crossings divisor.)

12.3 Forgetful morphisms

Proposition 12.3.1. *By Proposition 8.5.1, there is a morphism of algebraic stacks*

$$\overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n-1}, (C, p_1, \dots, p_n) \mapsto (C^{st}, p_1, \dots, p_{n-1}).$$

As



12.4 Universal family $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$

This section we follow [35] and [36]. We consider the universal family $\mathcal{U}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ of $\overline{\mathcal{M}}_{g,n}$. Actually the definition of universal family as for any family of stable curves $(\mathcal{C} \rightarrow S, \{\sigma_i\})$, we have the following universal property of cartesian

$$\begin{array}{ccc} \mathcal{C} & \dashrightarrow & \mathcal{U}_{g,n} \\ \downarrow & & \downarrow \\ S & \dashrightarrow^{\exists!} & \overline{\mathcal{M}}_{g,n} \end{array}$$

The existence given by 2-Yoneda's Lemma and some descent theory (omitted). Here we express this family as follows:

Lemma 12.4.1 (See [36]). *$\mathcal{U}_{g,n}(S)$ to be the set of families of curves $(\mathcal{C} \rightarrow S, \sigma_1, \dots, \sigma_n, \sigma)$ where $(\mathcal{C} \rightarrow S, \sigma_1, \dots, \sigma_n) \in \overline{\mathcal{M}}_{g,n}(S)$ and σ is an extra section without smooth condition.*

Proof. Fix $\pi : \mathcal{C} \rightarrow S$. We first let $\Sigma(S) : \mathcal{U}_{g,n}(S) \rightarrow \overline{\mathcal{M}}_{g,n}(S)$ as $(\pi, \sigma_i, \sigma) \mapsto (\pi, \sigma_i)$ be the canonical map and $\Sigma_i(S) : \overline{\mathcal{M}}_{g,n}(S) \rightarrow \mathcal{U}_{g,n}(S)$ as $(\pi, \sigma_1, \dots, \sigma_n) \mapsto (\pi, \sigma_1, \dots, \sigma_n; \sigma_i)$. Finally we need to define $\mathcal{C} \rightarrow \mathcal{U}_{g,n}$ as $(\text{pr}_2 : \mathcal{C} \times_S \mathcal{C} \rightarrow \mathcal{C}, s_i, \Delta)$ where $s_i = (\sigma_i \circ \pi, \text{id}_{\mathcal{C}})$ and $\Delta = (\text{id}_{\mathcal{C}}, \text{id}_{\mathcal{C}})$. Hence we get the following cartesian diagram of fibered categories

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{U}_{g,n} \\ \sigma_1, \dots, \sigma_n \left(\begin{array}{c} \uparrow \\ \pi \\ \downarrow \end{array} \right) & & \left(\begin{array}{c} \uparrow \\ \Sigma \\ \downarrow \end{array} \right) \Sigma_1, \dots, \Sigma_n \\ S & \longrightarrow & \overline{\mathcal{M}}_{g,n} \end{array}$$

Well done. □

Now we consider

$$\overline{\mathcal{M}}_{g,n+1} \rightarrow \mathcal{U}_{g,n}, (\mathcal{C} \rightarrow S) \mapsto (\mathcal{C}^{\text{st}} \rightarrow S, \sigma'_1, \dots, \sigma'_n, \sigma')$$

where this stabilization aiming to make $(\mathcal{C}^{\text{st}} \rightarrow S, \sigma'_1, \dots, \sigma'_n)$ in $\overline{\mathcal{M}}_{g,n}(S)$.

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n+1} & \longrightarrow & \mathcal{U}_{g,n} \\ & \searrow & \downarrow \\ & & \overline{\mathcal{M}}_{g,n} \end{array}$$

Remark 12.4.2 (More explicit construction). *Fix $f : X \rightarrow S$ in $\overline{\mathcal{M}}_{g,n+1}(S)$ and hence we get*

$$X = \underline{\text{Proj}}_S \left(\bigoplus_{m \geq 0} f_* \omega_{X/S} \left(\sum_{i=1}^{n+1} \sigma_i \right)^{\otimes m} \right).$$

Now we let

$$c(X) := \underline{\text{Proj}}_S \left(\bigoplus_{m \geq 0} f_* \omega_{X/S} \left(\sum_{i=1}^n \sigma_i \right)^{\otimes m} \right).$$

with $\sigma'_i : S \xrightarrow{\sigma_i} X \rightarrow c(X)$. Hence $(c(X); \sigma'_1, \dots, \sigma'_n)$ be a family of n -pointed stable curves. Hence we get $\overline{\mathcal{M}}_{g,n+1} \rightarrow \mathcal{U}_{g,n}$. Here we follows the proof before chapter 8 in [23].

Proposition 12.4.3. *The morphism $\overline{\mathcal{M}}_{g,n+1} \rightarrow \mathcal{U}_{g,n}$ is an isomorphism over $\overline{\mathcal{M}}_{g,n}$.*

Sketch. Now we construct an inverse map $\mathcal{U}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n+1}$.

Step 1. Construct that family of curves. Let $(\mathcal{C} \rightarrow S, \sigma_1, \dots, \sigma_n, \sigma)$ be an element in $\mathcal{U}_{g,n}(S)$. As σ is a closed immersion, it defined by an ideal sheaf $i : \mathcal{I}_\sigma \hookrightarrow \mathcal{O}_\mathcal{C}$. Define the coherent sheaf K by the exact sequence

$$0 \rightarrow \mathcal{O}_\mathcal{C} \xrightarrow{\delta} \mathcal{I}_\sigma^\vee \oplus \mathcal{O}_\mathcal{C}(\sigma_1 + \dots + \sigma_n) \rightarrow K \rightarrow 0$$

where $\delta = (i^\vee, j)$ where j is also an embedding. Now consider $\mathcal{C}' = \underline{\text{Proj}}\text{Sym}(K) \xrightarrow{p} \mathcal{C} \rightarrow S$.

Step 2. Construct the section. In [35], Knudsen introduce a notion called stably reflexive module. Knudsen separate the two cases of σ as this is local on S : (I) σ meets a non-smooth point in the fiber; (II) σ is a divisor meets one of these sections σ_i .

In both cases we find the surjections form as $\sigma^* K \rightarrow \sigma^*(-)$ or $\sigma_i^* K \rightarrow \sigma_i^*(-)$ to getting lifts where showed that all $\sigma^*(-)$ are line bundles (may using stably reflexive module). The picture when $S = \text{Spec}(k)$ as follows:



Hence we omitted all details and get $(\mathcal{C}' \rightarrow S, \sigma'_1, \dots, \sigma'_n, \sigma') \in \overline{\mathcal{M}}_{g,n+1}(S)$. For this detailed proof, we refer the original paper [35] Theorem 2.4 or the new paper [36]. One can also see [8] X.8 for more detailed proof over \mathbb{C} . \square

Chapter 13

Irreducibility

As $\overline{\mathcal{M}}_{g,n}$ is a smooth Deligne-Mumford stack, its irreducibility if and only if connectedness. As $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ be a universal family, it has connected fibers. Hence by induction, we can reduce the case of $\overline{\mathcal{M}}_g$. Moreover, by Keel-Mori theorem we get the coarse moduli space $\overline{\mathcal{M}}_g \rightarrow \overline{M}_g$ which induce the homeomorphism $|\overline{\mathcal{M}}_g| \cong |\overline{M}_g|$. Hence we can reduce the case of \overline{M}_g . Hence we have the following relations:

$$\begin{aligned} \overline{\mathcal{M}}_{g,n} \text{ irreducible} &\Leftrightarrow \overline{\mathcal{M}}_{g,n} \text{ connected} \Leftrightarrow \overline{\mathcal{M}}_g \text{ connected (or irreducible)} \\ &(\Leftrightarrow \mathcal{M}_g \text{ connected and dense in } \overline{\mathcal{M}}_g) \Leftrightarrow \overline{M}_g \text{ connected.} \end{aligned}$$

Here the denceness of \mathcal{M}_g in the proper Deligne-Mumford stack $\overline{\mathcal{M}}_g$ is called **Deligne-Mumford compactification**.

Remark 13.0.1 (Some historical remarks). (i) In 19th century, Clebsch and Hurwitz establishing irreducibility of M_g in characteristic 0 by using the classical topological argument;

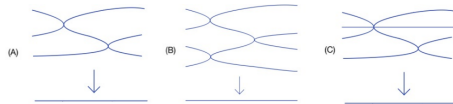
(ii) In the appendix of the paper [56] by Fulton in [42] gives a completely algebraic proof for this in characteristic 0 in 1982;

(iii) In the paper [30], Deligne and Mumford give two arguments of irreducibility of $\overline{M}_{g,n}$ in characteristic p (by reduction to characteristic 0) in 1969;

(iv) In paper [41], Fulton established the irreducibility of $\overline{M}_{g,n}$ in characteristic p where $p > g + 1$ in 1969.

13.1 Preliminaries–Branched coverings

Definition 13.1.1. Let C be a connected smooth curve on k . A branched covering of \mathbb{P}_k^1 is a separable finite morphism $f : C \rightarrow \mathbb{P}_k^1$. We say f is simply branched if for any branched point $x \in \mathbb{P}_k^1$, there is at most one ramification point in the fiber $f^{-1}(x)$ and such a point has index 2.



Here (A) is simply branched but (B),(C) are not.

Lemma 13.1.2. Let C be a smooth, connected and projective curve of genus g over an algebraically closed field k of characteristic 0. If L is a line bundle of degree $d \geq g + 1$ (*I think we may let $d \gg 0$*), then for a subspace $V \subset H^0(C, L)$ of dimension 2 we get $C \rightarrow \mathbb{P}^1$ a simply branched.

Proof. (This proof need to re-think.) As $h^0(C, L) = d+1-g$, we get $\dim \text{Gr}(2, H^0(C, L)) = 2(d-g-1)$. Here $\text{char}(k) = 0$, the map $C \rightarrow \mathbb{P}^1$ is finite separable. So $C \rightarrow \mathbb{P}^1$ is not a simply branched covering if and only if one of the following conditions holds

- (a) V has a base point;
- (b) there exists a ramification point with index > 2 ;
- (c) there exists 2 ramification points in the same fiber.

For (a), then there exists $p \in C$ such that for all $s \in V$ vanishing at p , that is, $s \in H^0(C, L(-p))$. The dimension of $V \in \text{Gr}(2, H^0(C, L))$ have this property is

$$\dim \text{Gr}(2, H^0(C, L(-p))) = 2d - 2g - 4.$$

For (b), then there exists $s \in V$ vanishing to order 3 at a point p , that is, $s \in H^0(C, L(-3p))$. The dimension of $V \in \text{Gr}(2, H^0(C, L))$ have this property is

$$\dim \mathbb{P}H^0(C, L(-3p)) + \dim \mathbb{P}(H^0(C, L)/(s)) = 2d - 2g - 4.$$

Hence varying $p \in C$, the locus of $\text{Gr}(2, H^0(C, L))$ failing (b) has dimension $\dim \text{Gr}(2, H^0(C, L)) - 1$;

For (c), then there exists independent $s_1, s_2 \in V$ vanishing to order 2 at a point p , that is, $s_1, s_2 \in H^0(C, L(-2p))$. The dimension of $V \in \text{Gr}(2, H^0(C, L))$ have this property is

$$\dim \text{Gr}(2, H^0(C, L(-2p))) = 2d - 2g - 6.$$

Hence varying $p \in C$, the locus of $\text{Gr}(2, H^0(C, L))$ failing (b) has dimension $---$ □

Lemma 13.1.3. *If $C \rightarrow \mathbb{P}^1$ is a simply branched cover of degree $d > 2$ in characteristic 0, then $\text{Aut}(C/\mathbb{P}^1)$ is trivial.*

Proof. Any $\alpha \in \text{Aut}(C/\mathbb{P}^1)$ must fix the $2g + 2d - 2$ branched points by Riemann-Hurwitz Theorem and simplicity. By Proposition 1.2.7, there are no non-trivial automorphisms of a smooth curve fixing more than $2g + 2$ points. Hence as $d > 2$, $\text{Aut}(C/\mathbb{P}^1)$ is trivial. □

Remark 13.1.4. *Here we give some notes for the proof of Clebsch and Hurwitz in 19th to show that M_g is connected over \mathbb{C} . We define*

$$H_{d,b} = \{C \rightarrow \mathbb{P}^1 \text{ simply branched covering of degree } d \text{ over } b \text{ points}\}$$

where $b = 2g + 2d - 2$. By the previous lemma, $H_{d,b}$ is an algebraic space or a topological space (if $k = \mathbb{C}$, *why?*). Let $\text{Sym}^b \mathbb{P}^1 \setminus \Delta$ as the variety of b unordered distinct points in \mathbb{P}^1 (which can also be written as the complement $\mathbb{P}^b \setminus \Delta$ of the discriminant hypersurface), we have a diagram

$$\begin{array}{ccc} & H_{d,b} & \\ & \swarrow \quad \searrow & \\ \mathcal{M}_g & & \text{Sym}^b \mathbb{P}^1 \setminus \Delta \end{array}$$

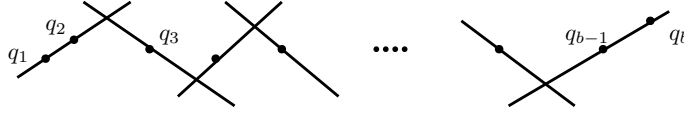
with the canonical maps. Then they showed that $H_{d,b} \rightarrow \text{Sym}^b \mathbb{P}^1 \setminus \Delta$ is finite étale (actually this can be showed by using deformation theory pure algebraically, see[1] Lemma 5.7.9. We omitted here). By Lemma 13.1.2, we get $H_{d,b} \rightarrow \mathcal{M}_g$ is surjective. Hence we need to show that $H_{d,b}$ is connected. Combining these and some properties of monodromy theory, they proved this. For more detail, see [1] subsection 5.7.2.

13.2 Irreducibility over characteristic 0 using admissible covers

In this section we will use the method of admissible covers to gives a completely algebraic proof for the irreducibility in characteristic 0, which appears in the appendix of the paper [56] by Fulton in [42].

Proposition 13.2.1. *Let C be a smooth, connected and projective curve of genus g over an algebraically closed field k of characteristic 0. There exists a connected curve T with points $t_1, t_2 \in T$ and a family $\mathcal{C} \rightarrow T$ of stable curves such that $\mathcal{C}_{t_1} \cong C$ and \mathcal{C}_{t_2} is a singular stable curve.*

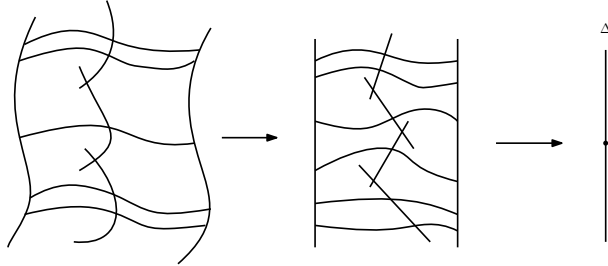
Proof. By Lemma 13.1.2 we get for $d \gg 0$ there exists a finite cover $C \rightarrow \mathbb{P}^1$ of degree d simply branched over $b = 2g + 2d - 2$ distinct points p_1, \dots, p_b in \mathbb{P}^1 . This gives a b -pointed stable curve $G = [\mathbb{P}^1, \{p_i\}] \in M_{0,b}$. By Remark 13.1.4 ($H_{d,b} \rightarrow \text{Sym}^b \mathbb{P}^1 \setminus \Delta$ is finite étale), we get $G \in M_{0,b}$ in general (WTF?). Then G can degenerate to (D_0, q_1, \dots, q_b) as the following picture



In the other words, there is a DVR R and fraction field K with $\Delta = \text{Spec}(R) \rightarrow \overline{M}_{0,n}$ be a stable curve $(\mathcal{D} \rightarrow \Delta, \sigma_i)$ with generic fiber (\mathbb{P}^1, p_i) and special fiber (D_0, q_1, \dots, q_b) . Hence we have a simply branched covering $\mathcal{C}^* \rightarrow \Delta^*$ and extend to $\mathcal{C} \rightarrow \mathcal{D}$ by taking \mathcal{C} as the integral closure of $\mathcal{O}_{\mathcal{D}}$ in $K(\mathcal{C}^*)$ as

$$\begin{array}{ccccc} \mathcal{C} & \dashrightarrow & \mathcal{D} & \longrightarrow & \Delta \\ \uparrow \downarrow & & \uparrow & & \uparrow \\ \mathcal{C}^* & \longrightarrow & \mathcal{D}^* & \longrightarrow & \Delta^* \end{array}$$

Hence we get this diagram:



Now we just need to make \mathcal{C} be a singular stable curve. Purity of the branch locus (What's this?) implies the central fiber $\mathcal{C}_0 \rightarrow \mathcal{D}_0$ is ramified at $\sigma_i(0)$. By $\Delta' \rightarrow \Delta, t \mapsto t^m$ we can replace \mathcal{C} such that $\mathcal{C}_0 \rightarrow \mathcal{D}_0$ is ramified only over $\sigma_i(0)$ and possibly over nodes of \mathcal{D}_0 . By an analysis of possible extensions $\mathcal{C} \rightarrow \mathcal{D}$, one can show that \mathcal{C}_0 is a nodal curve (missing details). Therefore $\mathcal{C} \rightarrow \Delta$ is a family of nodal curves.

Now we take $\mathcal{C} \rightarrow \mathcal{C}^{st}$ and just need to check \mathcal{C}_0^{st} is singular. For any irreducible component $T \subset \mathcal{C}_0^{st}$, apply Riemann-Hurwitz to $T \rightarrow \mathbb{P}^1 \subset \mathcal{D}_0$ we get $2g(T) - 2 = -2d + R$. If \mathbb{P}^1 is the middle one, we get $R \leq 2 + d - 1$; if \mathbb{P}^1 is the boundary one, we get $R \leq 1 + 2d - 2$. Hence $R \leq 2d - 1$ and $g(T) = 0$. Hence T is rational. Hence \mathcal{C}_0^{st} is singular. \square

Proposition 13.2.2. *If $\overline{\mathcal{M}}_{g',n'}$ is irreducible for all $g' < g$, then $\delta = \overline{\mathcal{M}}_{g,2} \setminus \mathcal{M}_{g,2}$ is connected.*

Proof. Let $\delta = \delta_0 \cup \delta_1 \cup \dots \cup \delta_{\lfloor g/2 \rfloor}$ where

$$\delta_0 = \text{Im}(\overline{\mathcal{M}}_{g-1,2} \rightarrow \overline{\mathcal{M}}_g)$$

$$\delta_i = \text{Im}(\overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1} \rightarrow \overline{\mathcal{M}}_g)$$

where $i = 1, \dots, \lfloor g/2 \rfloor$. Hence δ_0, δ_i are connected by hypotheses. Easy to see that these divisors intersect as the points of $|\overline{\mathcal{M}}_g|$: \square

Theorem 13.2.3. *$\overline{\mathcal{M}}_{g,n}$ is irreducible.*

$$\left[\begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ \text{---} \end{array} \right] \in \delta_i \cap \delta_j \qquad \left[\begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \\ | \quad | \\ \text{---} \end{array} \right] \in \delta_0 \cap \delta_i$$

Proof. By the argument at beginning, we just need to show $\overline{\mathcal{M}}_g$ is connected. By Proposition 13.2.1 every smooth curve degenerates to a stable singular curve in the boundary $\delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$. By induction on g and Proposition 13.2.2 we get δ is connected, so is $\overline{\mathcal{M}}_g$. \square

Remark 13.2.4. For the irreducibility in positive characteristic, we omitted and we refer the original [30], [41]. For the sketch, we refer subsection 5.7.4 in [1].

Chapter 14

Projectivity

We will prove the coarse moduli space $\overline{M}_{g,n}$ is projective follows [65] and [81].

Remark 14.0.1. *Some generalizations of the projectivities:*

- (a) In [64] shows the moduli of stable varieties in any dimension is projective;
- (b) In [22] and [83] shows the moduli of K -polystable Fano varieties is projective.

Let the universal family $\pi : \mathcal{U}_g \rightarrow \overline{\mathcal{M}}_g$ and we define k -th pluri-canonical bundle as the vector bundle $\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$. Indeed, $\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$ is a coherent sheaf on the stack $\overline{\mathcal{M}}_g$ by the coherence theorem. We need to check that it is a vector bundle. By definition of the vector bundle over Deligne-Mumford stack, we need to show for any $S \rightarrow \overline{\mathcal{M}}_g$ the sheaf $(\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}))|_S$ is a vector bundle over S . As $S \rightarrow \overline{\mathcal{M}}_g$ correspond to $\pi_S : \mathcal{C} \rightarrow S$, we get

$$(\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}))|_S \cong \pi_{S,*}(\omega_{\mathcal{C}/S}^{\otimes k}).$$

By some argument with Review A.1.1 we can show that $\pi_{S,*}(\omega_{\mathcal{C}/S}^{\otimes k})$ is a vector bundle.

Moreover, we get use the Riemann-Roch Theorem to deduce that

$$\text{rank}(\pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})) = \begin{cases} g, & k = 1; \\ (2k - 1)(g - 1), & k > 1. \end{cases}$$

Now we consider the line bundle over $\overline{\mathcal{M}}_g$

$$\lambda_k := \det \pi_*(\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k}).$$

We will show that for $k \gg 0$, the line bundle λ_k descends to an ample line bundle on \overline{M}_g , then we get \overline{M}_g is a projective scheme.

14.1 Kollár's Criteria

Lemma 14.1.1. *Let \mathcal{X} be a proper Deligne-Mumford stack with coarse moduli space $\mathcal{X} \rightarrow X$. Suppose L line bundle over \mathcal{X} with*

- (a) L is semiample (i.e. L^N is basepoint-free for some $N > 0$);
- (b) for every proper integral curve T and map $f : T \rightarrow \mathcal{X}$ such that $f(T) \subset |\mathcal{X}|$ is not a single point, $\deg L|_T > 0$.

Then for some $N > 0$, $L^{\otimes N}$ descends to an ample line bundle over X .

Proof. This is the stack-version of the Corollary 1.2.15 in [69]. Actually we consider the following diagram which come from (a) and the universal property of coarse moduli space:

$$\begin{array}{ccc} \mathcal{X} & & \\ \downarrow \pi & \searrow f & \\ X & \xrightarrow{g} & \mathbb{P}(H^0(\mathcal{X}, L^{\otimes N})) \end{array}$$

By (b), f doesn't contract curves, so is g . Hence g is quasi-finite and proper, hence finite by Zariski main theorem. Hence $M := g^*\mathcal{O}(1)$ is ample; moreover, $\pi^*M = L^{\otimes N}$, we win. \square

Theorem 14.1.2 (Nakai-Moishezon Criterion). *If X is a proper algebraic space, a line bundle L is ample if and only if for all irreducible closed subvarieties $Z \subset X$,*

$$L^{\dim Z} \cdot Z > 0.$$

Proof. This is the algebraic space-version of the Theorem 1.21 in [28]. By Le Lemme de Gabber (Theorem C.2.1), there exists a finite surjection $f : X' \rightarrow X$ and by the algebraic space version St 0GFB we get L is ample if and only if f^*L is ample. Hence by the scheme-version of Nakai-Moishezon Criterion ([28] Theorem 1.21), we win. \square

Let X be a proper algebraic space over k . Let $W \rightarrow Q$ be a surjection of vector bundles of rank w and q . Suppose that W has structure group $G \rightarrow \mathrm{GL}_w$. There is a classifying map

$$X \rightarrow [\mathrm{Gr}(q, w)/G], x \mapsto [W \otimes \kappa(x) \rightarrow Q \otimes \kappa(x)]$$

which is well defined because these killed by G .

Here we state our main theorem in this section. For simplicity, we only state it in characteristic 0. The criteria first appears in [65] and more general case we refer [64].

Theorem 14.1.3 (Kollár's Criterion). *Let X be a proper algebraic space over a field k of characteristic 0. Let $W \rightarrow Q$ be a surjection of vector bundles of rank w and q , where W has structure group $G \rightarrow \mathrm{GL}_w$. Suppose that*

- (a) *The classifying map $X(k) \rightarrow \mathrm{Gr}(q, w)(k)/G(k)$ has finite fibers;*
- (b) *W is nef.*

Then $\det Q$ is ample.

Proof. By Nakai-Moishezon criterion, for any irreducible subvariety $Z \subset X$ we need to verify $\det(Q)|_Z$ is big. As (a),(b) can restrict to Z , we can let X is an integral scheme and show that $\det Q$ is big.

By Le Lemme de Gabber (Theorem C.2.1), there exists a finite projective surjection $f : Y \rightarrow X$ of schemes. Hence we have $\det(f^*Q)^{\dim Y} = \deg(f) \det(Q)^{\dim X}$ and $\det Q$ is big if and only if $\det(f^*Q)$ is big. By taking the normalization, we can assume Y is normal and integral. So by Lemma 14.1.4 we win. \square

Lemma 14.1.4. *Let Y be a normal projective integral scheme over a field k of characteristic 0. Let $W \rightarrow Q$ be a surjection of vector bundles of rank w and q , where W has structure group $G \rightarrow \mathrm{GL}_w$. Suppose that*

- (a) *The classifying map $Y(k) \rightarrow \mathrm{Gr}(q, w)(k)/G(k)$ generically has finite fibers;*
- (b) *W is nef.*

Then $\det Q$ is big.

Sketch. To add. See Proposition 5.8.9 in [1]. \square

14.2 Nefness of pluri-canonical bundles

Theorem 14.2.1. *Let $\pi : \mathcal{C} \rightarrow T$ be a family of stable curves over a smooth curve T over k , then $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq 2$.*

Proof. We following several steps:

•**Step 1. Reduction to characteristic p .** Now we let k is of characteristic 0. Since \mathcal{C} and T are finite type over k , their defining equations only involve finitely many coefficients of k . Thus there exists a finitely generated \mathbb{Z} -subalgebra $A \subset k$ and a cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \tilde{\mathcal{C}} \\ \downarrow & & \downarrow \\ T & \longrightarrow & \tilde{T} \\ \downarrow & & \downarrow \\ \text{Spec}k & \longrightarrow & \text{Spec}A \end{array}$$

where $\tilde{\mathcal{C}}, \tilde{T}$ are schemes of finite type over A . By possibly enlarging A , we can arrange that $\tilde{T} \rightarrow \text{Spec}(A)$ is smooth and projective family and $\tilde{\mathcal{C}} \rightarrow \tilde{T}$ is a family of stable curves.

(Need to re-think, omitted here.)

•**Step 2. Second reductions.** We reduce to the case that

- (a) \mathcal{C} is a smooth and minimal surface;
- (b) $\mathcal{C} \rightarrow T$ is generically smooth;
- (c) The genus of T is at least 2.

(To add.) These implies \mathcal{C} is of general type.

•**Step 3. Positive characteristic case.** Let $p = \text{char}(k)$. If $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is not nef, then there exists a quotient line bundle $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee$ where $d = \deg(M) > 0$. Consider the absolute Frobenius

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{Frob}_{\mathcal{C}}} & \mathcal{C} \\ \downarrow & & \downarrow \\ T & \xrightarrow{\text{Frob}_T} & T \end{array}$$

By the property of the dualizing sheaf, we get $\text{Frob}_T^* \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \cong \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$. And $\deg \text{Frob}_T^* M = pd$, we can let $d \gg 0$. Hence we can let $M = \omega_T^{\otimes k} \otimes L$ where L is very ample.

The surjection $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow (\omega_T^{\otimes k} \otimes L)^\vee$ induce

$$\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L \twoheadrightarrow \mathcal{O}_T.$$

As $h^1(T, \mathcal{O}_T) \geq 2$, we have $h^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L) \geq 2$. Use the Leray spectral sequence

$$H^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L) \Rightarrow H^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L),$$

hence $h^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L) \geq 2$ by some calculation. By Lemma 14.2.2, we win. \square

Lemma 14.2.2 (Bombieri-Ekedahl). *Let S be a smooth projective surface over an algebraically closed field k which is minimal and of general type. Let D be an effective divisor with $D^2 = 0$. If $\text{char}(k) \neq 2$, then $H^1(S, \omega_S^{\otimes n}(D)) = 0$ for all $n \geq 2$. If $\text{char}(k) = 2$, then $h^1(S, \omega_S^{\otimes n}(D)) \leq 1$ for all $n \geq 2$.*

14.3 Positivity via positivity theory

For a morphism $S \rightarrow \overline{\mathcal{M}}_g$ correspond to $\mathcal{C} \rightarrow S$. Consider an integral d , we have

$$\mathrm{Sym}^d \pi_* (\omega_{\mathcal{C}/S}^{\otimes k}) \rightarrow \pi_* (\omega_{\mathcal{C}/S}^{\otimes dk}).$$

When $S = \mathrm{Spec}(K)$, $\mathcal{C} = C$, we get

$$\mathrm{Sym}^d H^0(C, \omega_C^{\otimes k}) \rightarrow H^0(C, \omega_C^{\otimes dk})$$

with kernel consists of degree d equations cutting out the image of $|\omega_C^{\otimes k}| : C \rightarrow \mathbb{P}^{r(k)-1}$. If $k \geq 3$, $\omega_C^{\otimes k}$ is very ample and thus $\mathcal{C} \rightarrow S$ can be recovered from the kernel of the multiplication map.

Proposition 14.3.1. *For $k \gg 0$ and N sufficiently divisible, then $\lambda_k = \det \pi_* (\omega_{\mathcal{U}_g/\overline{\mathcal{M}}_g}^{\otimes k})$ descends to an ample line bundle on $\overline{\mathcal{M}}_g$.*

Proof. Consider $\mathcal{C} = \mathcal{U}_g, S = \overline{\mathcal{M}}_g$. Choose k, d such that

- (a) $\omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and $R^1 \pi_* \omega_{\mathcal{C}/S}^{\otimes k} = 0$;
- (b) Every curve $|\omega_C^{\otimes k}| : C \hookrightarrow \mathbb{P}^{r(k)-1}$ is cut out by equations degree d ;
- (c) $\pi_* (\omega_{\mathcal{C}/S}^{\otimes k})$ is nef (by Theorem 14.2.1).

These implies surjection

$$W := \mathrm{Sym}^d \pi_* (\omega_{\mathcal{C}/S}^{\otimes k}) \rightarrow \pi_* (\omega_{\mathcal{C}/S}^{\otimes dk}) =: Q.$$

Let w, q be the rank of W, Q , respectively. Let W has structure group $G \rightarrow \mathrm{GL}_w$. Consider the classifying map

$$\overline{\mathcal{M}}_g \rightarrow [\mathrm{Gr}(q, w)/G], x \mapsto \underbrace{[\mathrm{Sym}^d H^0(C, \omega_C^{\otimes k})]}_{\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))} \rightarrow \underbrace{[H^0(C, \omega_C^{\otimes dk})]}_{\Gamma(C, \mathcal{O}(d))}$$

is injective as the conditions on d and k imply that the kernel of the multiplication map uniquely determines C .

By Le Lemme de Gabber we get a finite cover $X \rightarrow \overline{\mathcal{M}}_g$. By Kollár's Criterion (Theorem 14.1.3), we get the pullback of λ_k to X is ample for $k \gg 0$. By Proposition C.2.2, we get for N sufficiently divisible, $\lambda_k^{\otimes N}$ descends to a line bundle L on $\overline{\mathcal{M}}_g$. Since the pullback of L under the finite morphism $X \rightarrow \overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{M}}_g$, by St 0GFB we get the conclusion that L is ample. \square

Theorem 14.3.2. *If $2g - 2 + n > 0$, then $\overline{\mathcal{M}}_{g,n}$ is projective.*

Proof. The universal family $\overline{\mathcal{M}}_{g,n+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ is projective by Proposition 8.3.3. Hence we just consider $n = 0$. This is right directly by the previous proposition. \square

Remark 14.3.3. *If we consider $\omega_{\mathcal{U}_{g,n}/\overline{\mathcal{M}}_{g,n}}^{\otimes k} (\Sigma_1 + \dots + \Sigma_n)$ from begining, we can prove the projectivity of $\overline{\mathcal{M}}_{g,n}$ directly.*

14.4 Projectivity via GIT, a sketch

By our old way, we have $\overline{\mathcal{M}}_g \cong [H^1/\mathrm{PGL}_{r(k)}]$ for some locally closed $\mathrm{PGL}_{r(k)}$ -invariant subscheme of $\underline{\mathrm{Hilb}}_{\mathbb{P}^{r(k)-1}}^P$ where $P(t) = \chi(C, \omega_C^{\otimes kt})$ and $r(k) = (2k - 1)(g - 1)$.

Remark 14.4.1. *In fact we have $\overline{\mathcal{M}}_{g,n} \cong [H_{\nu,g,n}/\mathrm{PGL}(N)]$ where $N = (2\nu - 1)(g - 1) + \nu n$ and $H_{\nu,g,n} \subset \underline{\mathrm{Hilb}}_{\mathbb{P}^{N-1}}^{\nu}$ be the Hilbert scheme of ν -log-canonically embedded n -pointed stable curves of genus g where $P_{\nu}(t) = (2\nu t - 1)(g - 1) + \nu n t$ for $\nu \geq 3$. See [8] Theorem XII.5.6 for the proof.*

Let H be the closure of H' in $\underline{\text{Hilb}}_{\mathbb{P}^{r(k)-1}}^P$. By the proof of the representability of the quotient scheme, we get a closed immersion for $d \gg 0$:

$$\begin{array}{ccc} H & \hookrightarrow & \underline{\text{Hilb}}_{\mathbb{P}^{r(k)-1}}^P \hookrightarrow \text{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))) \\ [C \hookrightarrow \mathbb{P}^{r(k)-1}] & \longmapsto & [\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \Gamma(C, \mathcal{O}(d))] \end{array}$$

Next consider the Plücker embedding

$$\begin{array}{ccc} \text{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d))) & \hookrightarrow & \mathbb{P} \left(\bigwedge^{P(d)} \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \right) \\ [\Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \Gamma(C, \mathcal{O}(d))] & \mapsto & [\bigwedge^{P(d)} \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)) \twoheadrightarrow \bigwedge^{P(d)} \Gamma(C, \mathcal{O}(d))] \end{array}$$

we can get $L_d := \mathcal{O}_{\text{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathcal{O}(d)))}(1)|_H$ be the very ample line bundle over H . All these morphisms are $\text{PGL}_{r(k)}$ -equivariant, hence L_d inherits a $\text{PGL}_{r(k)}$ -linearization. Hence L_d can be defined on $[H/\text{PGL}_{r(k)}]$.

Using the theory of Hilbert-Mumford criteria, we can prove the following difficult result.

Theorem 14.4.2. *Let $k \geq 5$ and $d \gg 0$. For $h = [C \hookrightarrow \mathbb{P}^{r(k)-1}] \in H$, the curve C is stable if and only if $h \in H$ is GIT semistable with respect to L_d , that is, there exists an equivariant section $s \in \Gamma(H, L_d^{\otimes N})^{\text{PGL}_{r(k)}}$ with $N > 0$ such that $s(h) \neq 0$. Moreover, we have*

$$\overline{M}_g \cong \text{Proj} \left(\Gamma(H, L_d^{\otimes N})^{\text{PGL}_{r(k)}} \right),$$

hence projective.

Part III

More theory of the moduli of curves

Chapter 15

Preliminaries

We now consider $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ as the groupoid over the category ($Sch/\text{Spec}\mathbb{C}$). Then by the same arguments in the previous part, we can get $\overline{\mathcal{M}}_{g,n}$ is also a proper smooth Deligne-Mumford stack of dimension $3g - 3 + n$ over \mathbb{C} with a coarse moduli space $\overline{M}_{g,n}$ which is a projective variety over \mathbb{C} . Similarly for $\mathcal{M}_{g,n}$ and $M_{g,n}$. We will refer [8].

15.1 Boundary geometry I. Graphs and dual graphs

We can associate a graph to a nodal curve with marked points.

Definition 15.1.1. A graph Γ is the datum of:

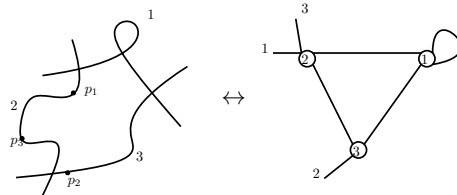
- (a) a finite nonempty set $V = V(\Gamma)$ (the set of vertices);
- (b) a finite set $L = L(\Gamma)$ (the set of half-edges);
- (c) an involution ι of L ;
- (d) a partition of L indexed by V , that is, the assignment to each $v \in V$ of a (possibly empty) subset L_v of L such that $L = \bigcup_{v \in V} L_v$ with $L_v \cap L_w = \emptyset$ when $v \neq w$.

A pair of distinct elements of L interchanged by the involution is called an edge of the graph. A fixed point of the involution is called a leg of the graph. The set of edges of Γ is denoted by $E(\Gamma)$. A dual graph is the datum of a graph together with the assignment of a nonnegative integer weight g_v to each vertex v . The genus of a dual graph Γ is defined to be

$$g_\Gamma = \sum_{v \in V(\Gamma)} g_v + 1 - \chi(\Gamma).$$

A graph (or a dual graph) endowed with a one-to-one correspondence between a finite set P and the set of its legs will be said to be P -marked, or numbered if P is of the form $\{1, \dots, n\}$ for some nonnegative integer n .

Let C be a nodal curve with a finite set D of smooth points of C . Let the vertex be each component of the normalization of C , and its weight is the genus of the component. The half-edges from a vertex are the points of the corresponding component which are nodes of C or marked points. Easy to see that the edges of the graph are nodes of C ; the legs are the marked points. This graph we denote it $\text{Graph}(C; D)$. (Easy to see by Theorem 7.2.1, we get the genus of the dual graph associated to $(C; D)$ is equal to the genus of C !) For example:

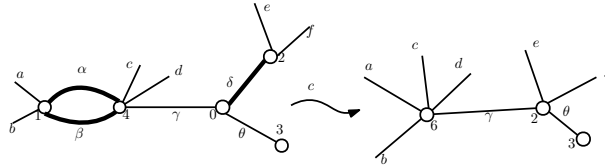


Definition 15.1.2. A curve is tree-like if, after deleting edges leading from a node to itself, the dual graph becomes a tree; it is of compact type if the dual graph actually is a tree.

Remark 15.1.3. Moreover, we also have some kind of localization. For $(C; D)$ we fixed a set S of nodes of C . We let a graph $\text{Graph}^S(C; D)$: The vertices are the connected components of the partial normalization C^S of C at S , the weight g_v is the genus of corresponding component of C^S , the edges correspond to the nodes in S , and the half-edges are the marked points or the points of C^S mapping to nodes in S .

Definition 15.1.4. Let Γ be a P -marked dual graph and let I be a subgraph without legs and having all the vertices of Γ . Let Γ_I be the graph that contracting each connected component of I to a point (one can speak this seriously, see [8] page 313). Hence we have a continuous map $c_I : \Gamma \rightarrow \Gamma_I$. There is a bijection between vertices of Γ_I and the connected components of I , we can let $g_w(\Gamma_I) = g(I_w)$ where w be a vertex of Γ_I and I_w be its connected component.

A P -marked dual graph Γ' is said to be a specialization of Γ if Γ is isomorphic to Γ'_I for some $I \subset \Gamma'$. We call $c : \Gamma' \rightarrow \Gamma \cong \Gamma'_I$ an I -contraction or simply a contraction. For example:

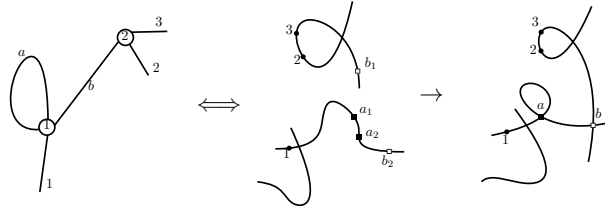


15.2 Boundary geometry II. More on gluing morphisms

15.2.1 Gluing via graphs

► **Gluing of curves.**

Fix a P -marked dual graph Γ and for any $v \in V$ we give a L_v -pointed nodal curve C_v of genus g_v . Let $C' = \coprod_{v \in V} C_v$ and let $C = C' / \sim$ where \sim means two points need to be glued together if and only if they are marked points labeled by the two halves of an edge of Γ . Hence $C' \rightarrow C$ is actually a partial normalization. For example:



Here we need to note that this graph here is kind of partial diagram.

► **Gluing of families of curves.**

Fix a P -marked dual graph Γ and for any $v \in V$ we give a family of stable L_v -pointed genus g_v curves $F_v = (f_v : X_v \rightarrow S, \sigma_l \text{ where } l \in L_v)$. Let $X' = \coprod_v X_v$ and we get $F' = (f' : X \rightarrow S, \sigma_i)$ a family of L -pointed nodal curves.

For any $m \in L$, by taking residue along σ_m we get a surjection

$$\omega_{f'}^k(k \sum \sigma_l) \rightarrow \mathcal{O}_{\sigma_m(S)}.$$

Hence we get

$$R_l^{(k)} : f_* \left(\omega_{f'}^k(k \sum \sigma_l) \right) \rightarrow \mathcal{O}_S$$

by some kind of positivity (see [8] Lemma X.6.1(i)) it is surjective for all $k > 1$. Consider

$$R^{(k)} : f_* \left(\omega_{f'}^k(k \sum \sigma_l) \right) \rightarrow \mathcal{O}_S^E$$

indexed by pairs of edges $\{l, l'\}$ with components $R_l^{(k)} + (-1)^{k-1}R_{l'}^{(k)}$. The kernel of its fiber at $s \in S$ is $H^0(X_s, \omega_{f'}^k(k \sum_{p \in P} \sigma_{l_p}(s)))$ where X_s be the gluing of X'_s via Γ , hence its dimension is independent of s . Hence the kernel of $R^{(k)}$, which we denote by \mathcal{S}_k , is locally free. It is locally finitely generated (see [8] Corollary X.6.4). Let

$$X = \underline{\text{Proj}}_S \bigoplus_{k \geq 0} \mathcal{S}_k$$

and hence the fibers of $X \rightarrow S$ is gluing via Γ . Let $\sigma'_p : S \rightarrow X$ is the composition of σ_{l_p} and $X' \rightarrow X$. Hence we get $F = (X \rightarrow S, \sigma'_p)_{p \in P}$.

15.2.2 Gluing functors

Fix a P -pointed dual genus g dual graph Γ and consider a Deligne-Mumford stack

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V} \overline{\mathcal{M}}_{g_v, L_v}.$$

Fixed S and we let $\eta = (\eta_v)_{v \in V} \in \overline{\mathcal{M}}_\Gamma(S)$ where $\eta_v : X_v \rightarrow S$ be a family of stable L_v -pointed curves of genus g_v . The morphisms are isomorphisms between these families Hence we get a gluing map via Γ to get $\xi_\Gamma(\eta) : X \rightarrow S$, a family of stable P -pointed genus g curves. Hence we get the gluing morphism of stacks

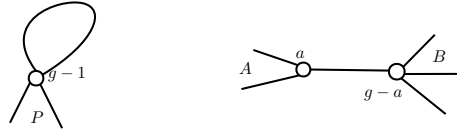
$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g, P}.$$

Let $\mathcal{D}_\Gamma \subset \overline{\mathcal{M}}_{g, P}$ be a closed substack as

$$\mathcal{D}_\Gamma(S) = \left\{ \begin{array}{l} \sigma : X \rightarrow S \text{ families of } P\text{-pointed stable curves of genus } g : \\ \text{fibers have dual graphs which are specializations of } \Gamma \end{array} \right\}$$

(as the image of ξ_Γ). It is also a Deligne-Mumford stack with a coarse moduli space $\Delta_\Gamma \subset \overline{M}_{g, P}$ as a closed subvariety. We often refer to the \mathcal{D}_Γ (or the Δ_Γ) as the boundary strata of $\overline{\mathcal{M}}_{g, P}$ (or of $\overline{M}_{g, P}$).

The simplest boundary strata are those of codimension 1 (as a divisor as before), which correspond to the stable graphs with a single edge. If we consider the following graphs:



then for the first we let Γ_{irr} and the second $\Gamma_{a, A}$ (or $\Gamma_{\mathcal{P}}$ if $\mathcal{P} = \{(a, A), (b = g - a, B)\}$ be a stable bipartition). Hence we can also define $\mathcal{D}_{irr} := \mathcal{D}_{\Gamma_{irr}}$ and $\mathcal{D}_{a, A} := \mathcal{D}_{\Gamma_{a, A}}$ (or $\mathcal{D}_{\mathcal{P}}$). The coarse case are the same $\Delta_{irr}, \Delta_{a, A}, \Delta_{\mathcal{P}}$. Moreover, in the case we get the old gluing way:

$$\xi_{irr} : \overline{\mathcal{M}}_{g-1, P \cup \{x, y\}} \rightarrow \overline{\mathcal{M}}_{g, P}, \xi_{a, A} : \overline{\mathcal{M}}_{a, A \cup \{x\}} \times \overline{\mathcal{M}}_{g-a, A^c \cup \{y\}} \rightarrow \overline{\mathcal{M}}_{g, P}.$$

Definition 15.2.1 (Weak Γ -marking). *Consider a family of stable P -pointed genus g curves $(\pi : \mathcal{C} \rightarrow S, \tau_p)$. Let subvariety $\Sigma \subset \text{Sing}(\mathcal{C})$ proper and étale over S , then for any $s \in S$, the fiber Σ_s be a finite set of nodes. Hence we can consider $\text{Graph}^{\Sigma_s}(\mathcal{C}_s)$.*

Fix a P -marked graph Γ of genus g , if $\text{Graph}^{\Sigma_s}(\mathcal{C}_s) \cong \Gamma$ for any s , then we call Σ is a weak Γ -marking. Hence we can define a stack \mathcal{E}_Γ as

$$\mathcal{E}_\Gamma(S) = \left\{ \begin{array}{l} \pi : \mathcal{C} \rightarrow S \text{ families of } P\text{-pointed stable curves} \\ \text{of genus } g : \text{ endowed with a weak } \Gamma\text{-marking} \end{array} \right\}$$

Definition 15.2.2 (Γ -marking). If $\mathcal{C} \rightarrow S$ coming from $(X \rightarrow S) \in \overline{\mathcal{M}}_\Gamma(S)$ by gluing via Γ with Σ the locus of nodes produced by gluing. As Σ be a union of sections on \mathcal{C} , so is the preimage over X (partial normalization). Hence we can get $\text{Graph}^\Sigma(\mathcal{C})$ with a family. Moreover $\Gamma \cong \text{Graph}^\Sigma(\mathcal{C})$. If these data exist for $\mathcal{C} \rightarrow S$, we called it endowed a Γ -marking.

Hence we can see that in this case we can do it conversely, hence we have $\overline{\mathcal{M}}'_\Gamma$ as

$$\overline{\mathcal{M}}_\Gamma(S) \Leftrightarrow \left\{ \begin{array}{l} \pi : \mathcal{C} \rightarrow S \text{ families of } P\text{-pointed stable curves} \\ \text{of genus } g : \text{ endowed with a } \Gamma\text{-marking} \end{array} \right\} := \overline{\mathcal{M}}'_\Gamma(S)$$

Hence we can find that the gluing map can be composited as

$$\begin{array}{ccccc} \overline{\mathcal{M}}_\Gamma & & & & \\ \downarrow \cong & \searrow \xi_\Gamma & & & \\ \overline{\mathcal{M}}'_\Gamma & \xrightarrow{F} \mathcal{E}_\Gamma & \xrightarrow{F'} \mathcal{D}_\Gamma & \hookrightarrow & \overline{\mathcal{M}}_{g,P} \end{array}$$

where F, F' are forgetful maps.

Proposition 15.2.3. (i) \mathcal{E}_Γ be the normalization of substack $\mathcal{D}_\Gamma \subset \overline{\mathcal{M}}_{g,P}$;
(ii) The morphism $\overline{\mathcal{M}}_\Gamma \rightarrow \mathcal{E}_\Gamma$ can be identified with $\overline{\mathcal{M}}_\Gamma \rightarrow [\overline{\mathcal{M}}_\Gamma/\text{Aut}(\Gamma)]$.

Proof. See [8] Proposition XII.10.11. □

Corollary 15.2.4. We can seen $\text{Im}(\xi_\Gamma) = \mathcal{D}_\Gamma$ as before.

Proof. Trivial by the Proposition. □

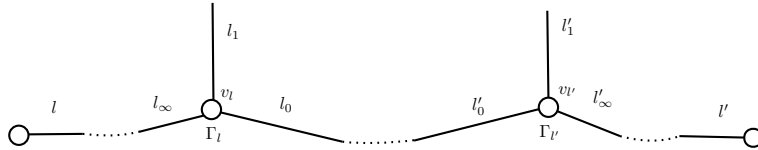
Corollary 15.2.5. Let Γ be a stable P -marked dual graph of genus g . Assume that $\text{Aut}(\Gamma) = \{\text{id}_\Gamma\}$. Furthermore, assume that, for every graph Γ' which is a specialization of Γ , all the elements in $\text{Aut}(\Gamma')$ are specializations of id_Γ . Then $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,P}$ is a closed immersion.

Proof. See [8] Corollary XII.10.22. □

Theorem 15.2.6. The map $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,P}$ is representable.

Proof. ► **Step 1. Construct a new graph $\widehat{\Gamma}$ from Γ .**

Fix an edge $\ell = \{l, l'\} \in E(\Gamma)$, consider the following graph $\Gamma_l, \Gamma_{l'}$ and splitting ℓ into l, l' and joint $\Gamma_l, \Gamma_{l'}$:



Repeat this operation for all edge of Γ , we get $\widehat{\Gamma}$. Hence $\widehat{\Gamma}$ is $P \cup H$ -marked where H the set of half-edges of Γ which are not legs.

► **Step 2. Decomposite ξ_Γ into closed immersion and projection.**

Consider maps

$$\begin{aligned} \iota_\Gamma : \overline{\mathcal{M}}_\Gamma &= \prod_{v \in V} \overline{\mathcal{M}}_{g_v, L_v} \rightarrow \\ &\overline{\mathcal{M}}_{\widehat{\Gamma}} = \prod_{v \in V} \overline{\mathcal{M}}_{g_v, L_v} \times \prod_{\{l, l'\} \in E} (\overline{\mathcal{M}}_{0, \{l_0, l_1, l_\infty\}} \times \overline{\mathcal{M}}_{0, \{l'_0, l'_1, l'_\infty\}}) \end{aligned}$$

and

$$\xi_{\widehat{\Gamma}} : \overline{\mathcal{M}}_{\widehat{\Gamma}} = \prod_{v \in V} \overline{\mathcal{M}}_{g_v, L_v} \times \prod_{\{l, l'\} \in E} (\overline{\mathcal{M}}_{0, \{l_0, l_1, l_\infty\}} \times \overline{\mathcal{M}}_{0, \{l'_0, l'_1, l'_\infty\}}) \rightarrow \overline{\mathcal{M}}_{g, P \cup H}$$

and $\pi_H : \overline{\mathcal{M}}_{g, P \cup H} \rightarrow \overline{\mathcal{M}}_{g, P}$ be the natural projection. Then

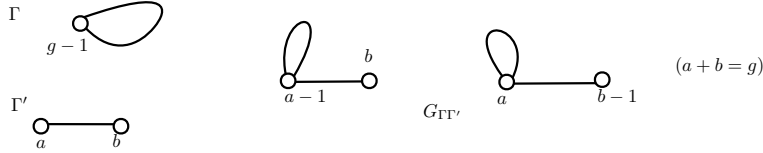
$$\xi_{\Gamma} = \pi_H \circ \xi_{\widehat{\Gamma}} \circ \iota_{\Gamma}.$$

As ι_{Γ} is isomorphism (Why?) and π_H is representable (as a universal family) and $\xi_{\widehat{\Gamma}}$ is a closed immersion by Corollary 15.2.5 and $\text{Aut}(\widehat{\Gamma}) = \{\text{id}_{\widehat{\Gamma}}\}$ (Why?). \square

It is important to describe how the various boundary strata intersect. Let Γ, Γ' are two P -marked dual graph of genus g . Consider

$$G_{\Gamma\Gamma'} = \left\{ \begin{array}{l} (\Lambda, c, c') / \cong: \Lambda \text{ be a } P\text{-marked dual graph of genus } g, \\ c : \Lambda \rightarrow \Gamma, c' : \Lambda \rightarrow \Gamma' \text{ are contractions with the} \\ \text{property that } E(\Lambda) = c^{-1}(E(\Gamma)) \cup c'^{-1}(E(\Gamma')) \end{array} \right\}.$$

For example:



Proposition 15.2.7. *If we let $\overline{\mathcal{M}}_{\Gamma\Gamma'} := \overline{\mathcal{M}}_{\Gamma} \times_{\overline{\mathcal{M}}_{g, P}} \overline{\mathcal{M}}_{\Gamma'}$, then*

$$\overline{\mathcal{M}}_{\Gamma\Gamma'} = \prod_{\Lambda \in G_{\Gamma\Gamma'}} \overline{\mathcal{M}}_{\Lambda}.$$

Proof. Fix a scheme T .

First we let $\xi : \mathcal{C} \rightarrow T$ in $\overline{\mathcal{M}}_{\Lambda} = \overline{\mathcal{M}}'_{\Lambda}$, then we are given a subvariety $\Sigma \subset \text{Sing}(\mathcal{C})$, proper and étale over T , whose inverse image in the partial normalization along Σ itself is a union of sections, plus an isomorphism $\gamma : \text{Graph}^{\Sigma}(\mathcal{C}) \cong \Lambda$. Let contractions $c : \Lambda \rightarrow \Gamma, c' : \Lambda \rightarrow \Gamma'$ and

$$\Sigma_1 = (c \circ \gamma)^{-1}(E(\Gamma)), \Sigma_2 = (c' \circ \gamma)^{-1}(E(\Gamma'))$$

such that $\Sigma = \Sigma_1 \cup \Sigma_2$ with isomorphisms $\gamma_1 : \text{Graph}^{\Sigma_1}(\mathcal{C}) \cong \Gamma$ and $\gamma_2 : \text{Graph}^{\Sigma_2}(\mathcal{C}) \cong \Gamma'$. Hence ξ is both in $\overline{\mathcal{M}}_{\Gamma}(T)$ and $\overline{\mathcal{M}}_{\Gamma'}(T)$. Hence in $\overline{\mathcal{M}}_{\Gamma\Gamma'}(T)$

Conversely, as we have

$$\overline{\mathcal{M}}_{\Gamma\Gamma'}(T) = \left\{ \begin{array}{l} (\xi, \xi', \phi) : \xi, \xi' \text{ are families of } \Gamma, \Gamma'\text{-marking stable} \\ P\text{-pointed genus } g \text{ curves over } T \text{ with } \phi : \xi \rightarrow \xi' \\ \text{a } T\text{-isomorphism} \end{array} \right\}.$$

Then let $(\xi, \xi', \phi) \in \overline{\mathcal{M}}_{\Gamma\Gamma'}(T)$, hence we have $\gamma : \text{Graph}^{\Sigma_1}(\mathcal{C}) \cong \Gamma$ and $\gamma' : \text{Graph}^{\Sigma_2}(\mathcal{C}) \cong \Gamma'$. Hence we get contractions $c, c' : \text{Graph}^{\Sigma_1 \cup \Sigma_2}(\mathcal{C}) \rightarrow \Gamma, \Gamma'$. Hence we get $(\text{Graph}^{\Sigma_1 \cup \Sigma_2}(\mathcal{C}), c, c') \in G_{\Gamma\Gamma'}$, hence we win. \square

15.3 Local structure of $\overline{\mathcal{M}}_{g,n}$ and $\overline{M}_{g,n}$

We also consider the case over \mathbb{C} . We will use the Kuranishi family and ν -log canonical Hilbert scheme to describe the local structure of the moduli stack and (coarse) space of the stable curves.

Recall that we have the local structure of the Deligne-Mumford stack and its coarse moduli space, that is, the Theorem C.1.6 and Theorem C.1.5 as follows.

Theorem A. Let \mathcal{X} be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S . Let $\pi : \mathcal{X} \rightarrow X$ be its coarse moduli space. For any closed point $x \in |X|$ with geometric stabilizer G_x , we have an étale neighborhood $\text{Spec} A^{G_x} \rightarrow X$ of $\pi(x) \in |X|$.

Theorem B. Let \mathcal{X} be a separated Deligne-Mumford stack and $x \in \mathcal{X}(k)$ be a geometric point with stabilizer G_x . Then exists an affine and étale map

$$f : ([\text{Spec} A / G_x], w) \rightarrow (\mathcal{X}, x)$$

where $w \in (\text{Spec} A)(k)$ such that f induces an isomorphism of the stabilizer groups at w . Moreover, it can be arranged that $f^{-1}(BG_x) \cong BG_w$.

But now we will get a more coarse (but useful) local structure by using the Kuranishi family as follows. Actually as a set, $\overline{M}_{g,n}$ is a set of isomorphism class of the n -pointed stable curves. Hence by Definition 9.5.3, for a n -pointed stable curve we have a standard Kuranishi family $\xi : \mathcal{C} \rightarrow (X_0, x_0) \subset H_{\nu,g,n}$. Hence we have a natural map

$$\psi : X_0 / G_{x_0} \rightarrow \overline{M}_{g,n}.$$

Recall some properties of X_0 in Definition 9.5.3:

- For any $y \in X_0$ we have $G_y := \text{Aut}(\mathcal{C}_y; \sigma_i(y)) \cong \text{stab}_{G_{x_0}}(y)$;
- For any $y \in X_0$, there is a G_y -invariant analytic neighborhood U of y in X such that any isomorphism (of n -pointed curves) between fibers over U is induced by an element of G_y .

Theorem 15.3.1. *The map $\psi : X_0 / G_{x_0} \rightarrow \overline{M}_{g,n}$ is étale. Moreover there are finite many such X_i and G_i covers $H_{\nu,g,n}$ such that the map*

$$\phi : Y := \coprod_i X_i / G_i \rightarrow \overline{M}_{g,n}$$

is étale and surjective.

Proof. See [8] Proposition XII.3.5. To add. □

Theorem 15.3.2. *The canonical map*

$$\alpha : X := \coprod_i X_i \rightarrow \overline{\mathcal{M}}_{g,n}$$

is étale and surjective where X_i are Kuranishi families as before covers $H_{\nu,g,n}$.

Proof. See [8] Theorem XII.8.3. To add. □

Chapter 16

Line bundles and Picard groups of the moduli of curves

We will refer [8] chapter XIII and [5].

16.1 Line bundles on the moduli stack of stable curves

Example 16.1.1 (Hodge bundle). *For any $S \rightarrow \overline{\mathcal{M}}_{g,n}$ which correspond to $\xi = (\pi : \mathcal{C} \rightarrow S)$, we let $\mathbb{E}_\xi := \pi_* \omega_\pi$. Hence induce a sheaf \mathbb{E} over $\overline{\mathcal{M}}_{g,n}$ called the Hodge bundle. As the relative dualizing sheaf is functorial with respect to morphisms of families, this is a quasi-coherent sheaf. By the cohomology and base change, it is actually a vector bundle of rank g as before. Let $\det \mathbb{E} = \bigwedge^g \mathbb{E}$ and we call it the Hodge line bundle. Usually we denote $\lambda := [\bigwedge^g \mathbb{E}] \in \text{Pic}(\overline{\mathcal{M}}_{g,n})$.*

Remark 16.1.2. *For the canonical map $\overline{\mathcal{M}}_{g,n} \rightarrow \overline{M}_{g,n}$ there are plenty of quasi-coherent sheaves on $\overline{\mathcal{M}}_{g,n}$ which do not come by pullback from the quasi-coherent sheaves on $\overline{M}_{g,n}$. For example, Hodge bundle as follows (see Remark 16.6.8 for more about Hodge bundle).*

Proposition 16.1.3. *The Hodge bundle and its determinant do not descend to coherent sheaves on the moduli space $\overline{M}_{g,n}$ except in genus zero.*

Proof. Consider a point $\xi = (C; p_i)$, then $\mathbb{E}_\xi = H^0(C, \omega_C)$. If $\det \mathbb{E}$ comes from $\overline{M}_{g,n}$, then any automorphism of ξ will act trivially on $\bigwedge^g H^0(C, \omega_C)$ by the basic theory of the coarse moduli space (Keel-Mori theory). Now we will find a curve and an automorphism of it which acts nontrivially.

For g odd, we let C be any hyperelliptic curve and consider hyperelliptic involution which acts as multiplication by -1 on $H^0(C, \omega_C)$, hence nontrivial over $\bigwedge^g H^0(C, \omega_C)$;

For g even, we let C be a ramified double covering of an elliptic curve and as $\{p_i\}$ any set of n points which is invariant under the covering involution. The eigenvalues of the covering involution acting on $H^0(C, \omega_C)$ are 1 with multiplicity 1 and -1 with multiplicity $g - 1$ (Why?), hence the covering involution acts as -1 on $\bigwedge^g H^0(C, \omega_C)$. \square

Example 16.1.4 (Generalization of the Hodge line bundle). *For any $S \rightarrow \overline{\mathcal{M}}_{g,n}$ which correspond to $\xi = (\pi : \mathcal{C} \rightarrow S)$ and any $\nu \in \mathbb{Z}$, we let*

$$\Lambda(\nu)_\xi := \left(\bigwedge^{\max} R^1 \pi_* \omega_\pi^{\otimes \nu} \right)^{-1} \otimes \bigwedge^{\max} \pi_* \omega_\pi^{\otimes \nu}.$$

Hence induce a line bundle $\Lambda(\nu)$ over $\overline{\mathcal{M}}_{g,n}$. Usually we denote $\lambda(\nu) := [\Lambda(\nu)] \in \text{Pic}(\overline{\mathcal{M}}_{g,n})$.

Actually when $\nu = 1$, by the same arguments in Lemma 2.1.1 we can show that $R^1 \pi_ \omega_\pi \cong \mathcal{O}_S$. So we have $\Lambda(1)_\xi \cong \bigwedge^g \mathbb{E}_\xi$ canonically, hence $\Lambda(1) \cong \bigwedge^g \mathbb{E}$.*

Example 16.1.5 (Point bundles). For $n > 0$ and for any $S \rightarrow \overline{\mathcal{M}}_{g,n}$ which correspond to $\xi = (\pi : \mathcal{C} \rightarrow S; \sigma_i)$, we let $(\mathcal{L}_i)_\xi := \sigma_i^* \omega_\pi$. Hence we get \mathcal{L}_i be the line bundles over $\overline{\mathcal{M}}_{g,n}$. We usually set

$$\psi_i = [\mathcal{L}_i] \in \text{Pic}(\overline{\mathcal{M}}_{g,n}), \psi = \sum_i \psi_i.$$

Remark 16.1.6. As the Hodge bundle, in general, \mathcal{L}_i can't descend to a line bundle on $\overline{M}_{g,n}$.

Example 16.1.7 (Boundary divisors and bundles). As before, we have

$$\partial \overline{\mathcal{M}}_{g,n} =: \mathcal{D} = \mathcal{D}_{irr} + \sum_{\mathcal{P}} \mathcal{D}_{\mathcal{P}},$$

where the sum runs through all stable bipartitions of $(g, \{1, \dots, n\})$. We denote

$$\delta_{irr} = [\mathcal{O}(\mathcal{D}_{irr})] \in \text{Pic}(\overline{\mathcal{M}}_{g,n}), \delta_{\mathcal{P}} = [\mathcal{D}_{\mathcal{P}}] \in \text{Pic}(\overline{\mathcal{M}}_{g,n}).$$

16.2 Tangent bundle, cotangent bundle and normal bundle

Proposition 16.2.1. Consider the moduli stack $\overline{\mathcal{M}}_{g,P}$, then tangent bundle $\mathcal{T} = T_{\overline{\mathcal{M}}_{g,P}}$ can be described as: for any $F = (f : X \rightarrow S, \{\sigma_p\}_{p \in P}) \in \overline{\mathcal{M}}_{g,P}(S)$, we have

$$\mathcal{T}_F = f_*(\Omega_f^1 \otimes \omega_f(D))^\vee$$

where $D = \sum \sigma_p(S)$.

Proof. By Theorem 15.3.2, the Kuranishi families formed an étale covering. Hence consider a stable P -pointed curve $\{C; x_p\}$ and its Kuranishi family (see Theorem 9.2.6) $\mathcal{X} \rightarrow (U, u_0)$, then we have

$$T_{u_0}U \cong \text{Ext}_{\mathcal{O}_C}^1(\Omega_C^1, \mathcal{O}_C(-\sum_p x_p)) \cong H^0(C, \Omega_C^1 \otimes \omega_C(\sum_p x_p))^\vee.$$

Hence we get the conclusion. \square

Example 16.2.2 (Canonical bundle). Hence the cotangent bundle \mathcal{T}^\vee given by $\mathcal{T}_F^\vee = f_*(\Omega_f^1 \otimes \omega_f(D))$. Hence we get the class of the canonical line bundle

$$K_{\overline{\mathcal{M}}_{g,P}} := \left[\bigwedge^{\max} \mathcal{T}^\vee \right] \in \text{Pic}(\overline{\mathcal{M}}_{g,P}).$$

Now we consider the normal bundle of $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,P}$ where Γ be a stable graph (For a map of smooth schemes $f : X \rightarrow Y$, we let $N_f = f^*T_Y/T_X$).

Example 16.2.3 (Single curves). Let N be a point of $\overline{\mathcal{M}}_\Gamma$ with image C in $\overline{\mathcal{M}}_{g,P}$. One can consider N as a partial normalization of C at some y_e . By Claim 2 in Remark 9.2.5 we get

$$\begin{aligned} 0 \rightarrow \text{Ext}^1(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - R)) \rightarrow \text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D)) \rightarrow \\ \bigoplus_{e \in E(\Gamma)} \text{Ext}^1(\Omega_{C, y_e}^1, \mathcal{O}_{C, y_e}) \rightarrow 0. \end{aligned}$$

where $D = \sum x_p$ with preimage \tilde{D} and R be the preimage of these y_e .

Easy to see that $\text{Ext}^1(\Omega_N^1, \mathcal{O}_N(-\tilde{D} - R))$ be the tangent space of $\overline{\mathcal{M}}_\Gamma$ at N and $\text{Ext}^1(\Omega_C^1, \mathcal{O}_C(-D))$ be the tangent space of $\overline{\mathcal{M}}_{g,P}$ at C , hence the normal space to ξ_Γ at N is

$$\bigoplus_{e \in E(\Gamma)} \text{Ext}^1(\Omega_{C, y_e}^1, \mathcal{O}_{C, y_e}) = \bigoplus_{e \in E(\Gamma)} T_{N, y'_e} \otimes T_{N, y''_e}$$

by Claim 2 in Remark 9.2.4 (or Claim 3 in Remark 9.2.5).

Proposition 16.2.4. *The normal bundle of $\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,P}$ can be expressed as*

$$N_{\xi_\Gamma} = \bigoplus_{\{l,l'\} \in E(\Gamma)} \eta_{v(l)}^* \mathcal{L}_l^\vee \otimes \eta_{v(l')}^* \mathcal{L}_{l'}^\vee$$

where $\eta_v : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g_v, L_v}$ be the projections.

Proof. Let F in $\overline{\mathcal{M}}_\Gamma$ is the datum of a family $X_v \rightarrow S$ of stable L_v -pointed curves of genus g_v for each vertex v of Γ . We let $X = \coprod_v X_v$. For each $l \in L(E)$, we denote by σ_l the corresponding section of $X \rightarrow S$. The gluing construction yields a family $X' \rightarrow S$ of stable P -pointed genus g curves. Then the normal

$$N_{\xi_\Gamma, F} = \bigoplus_{\{l,l'\} \in E(\Gamma)} \sigma_l^* T_{X/S} \otimes \sigma_{l'}^* T_{X/S} = \bigoplus_{\{l,l'\} \in E(\Gamma)} \mathcal{L}_{l,F}^\vee \otimes \mathcal{L}_{l',F}^\vee$$

where \mathcal{L}_i are point bundles. Hence we win. \square

Remark 16.2.5 (Excess intersection bundle). *By Proposition 15.2.7, we consider*

$$\begin{array}{ccc} \overline{\mathcal{M}}_{\Gamma'} & = \coprod_{\Lambda \in G_{\Gamma'}} \overline{\mathcal{M}}_\Lambda & \xrightarrow{\coprod \xi_{\Lambda\Gamma}} \overline{\mathcal{M}}_\Gamma \\ \downarrow & & \downarrow \xi_\Gamma \\ \overline{\mathcal{M}}_{\Gamma'} & \xrightarrow{\xi_{\Gamma'}} & \overline{\mathcal{M}}_{g,P} \end{array}$$

Then the excess intersection bundle is

$$F_{\Gamma'} = \bigoplus_{\Lambda \in G_{\Gamma'}} F_{\Lambda\Gamma'} := \bigoplus_{\Lambda \in G_{\Gamma'}} \xi_\Gamma^*(N_{\xi_{\Gamma'}})/N_{\xi_{\Lambda\Gamma}}.$$

We can show that (as [8] XIII.(3.8))

$$F_{\Gamma'} = \bigoplus_{\Lambda \in G_{\Gamma'}} \bigoplus_{\{l,l'\} \in c^{-1}(E(\Gamma)) \cap c'^{-1}(E(\Gamma'))} \eta_{v(l)}^* \mathcal{L}_l^\vee \otimes \eta_{v(l')}^* \mathcal{L}_{l'}^\vee.$$

Corollary 16.2.6. *We have*

$$\left[\bigwedge^{\max} N_{\xi_\Gamma} \right] = - \sum_{l \in H(\Gamma)} \eta_{v(l)}^* \psi_l$$

where $H(\Gamma)$ be the set of those half-edges of Γ which are not legs.

16.3 Determinant

16.3.1 Basic linear algebra

Definition 16.3.1. *A $\mathbb{Z}/2$ -graded line bundle is a pair (L, r) where L be a line bundle over a scheme X and $r \in \{0, 1\}$. We define the determinant of a finite vector bundle F over X is a $\mathbb{Z}/2$ -graded line bundle*

$$\det F := \left(\bigwedge^{\max} F, \text{rank} F \pmod{2} \right).$$

We say (L, r) is even/odd if r is even/odd. We define the tensor product of $\mathbb{Z}/2$ -graded line bundles as $(L, r) \otimes (T, s) := (L \otimes T, r + s)$. Let $A := (L, r)$, $B := (T, s)$ and define the canonical isomorphism

$$\tau_{A,B} : A \otimes B \rightarrow B \otimes A, l \otimes m \mapsto (-1)^{rs} m \otimes l.$$

Proposition 16.3.2. (i) For $\mathbb{Z}/2$ -graded line bundles A, B, C we have

$$\tau_{A \otimes B, C} = (\tau_{A, C} \otimes \text{id}) \circ (\text{id} \otimes \tau_{B, C});$$

(ii) For an exact sequence $\mathcal{E} : 0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$ of vector bundles, we have a canonical isomorphism $\phi_{\mathcal{E}} : \det E \otimes \det G \rightarrow \det F$;

(iii) Define $\mathbf{1}_X := (\mathcal{O}_X, 0)$ and $A^{-1} = (L^\vee, a)$ for a $\mathbb{Z}/2$ -graded line bundle $A = (L, a)$. Then $A \otimes A^{-1} \cong \mathbf{1}_X$, $\alpha, \phi \mapsto \phi(\alpha)$ and

$$S_{A, B} : B^{-1} \otimes A^{-1} \cong (A \otimes B)^{-1}, \phi \otimes \psi \mapsto (\chi : \alpha \otimes \beta \mapsto \phi(\alpha)\psi(\beta));$$

(iv) We have $\tau_{B, A}^\vee \circ S_{A, B} = S_{B, A} \circ \tau_{A^{-1}, B^{-1}}$.

Proof. Trivial by some easy linear algebra and calculation. \square

Definition 16.3.3. Let a finite complexes F^* of vector bundles on X , we define

$$\det F^* := \bigotimes_{q \in \mathbb{Z}} (\det F^q)^{(-1)^q}.$$

Proposition 16.3.4. (i) For a exact sequence of complexes $\mathcal{E} : 0 \rightarrow E^* \rightarrow F^* \rightarrow G^* \rightarrow 0$, we also have isomorphism

$$\phi_{\mathcal{E}} : \det E^* \otimes \det G^* \cong \det F^*;$$

(ii) The determinant and $\phi_{\mathcal{E}}$ are functorial in the base space X ;

(iii) Consider

$$\begin{array}{ccccccc} & & \mathcal{E}_1 & & \mathcal{E}_2 & & \mathcal{E}_3 \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_1 : & 0 \longrightarrow & A^* & \longrightarrow & B^* & \longrightarrow & C^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_2 : & 0 \longrightarrow & A'^* & \longrightarrow & B'^* & \longrightarrow & C'^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{R}_3 : & 0 \longrightarrow & A''^* & \longrightarrow & B''^* & \longrightarrow & C''^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

then we have $\phi_{\mathcal{E}_2} \circ (\phi_{\mathcal{R}_1} \otimes \phi_{\mathcal{R}_3}) = \phi_{\mathcal{R}_2} \circ (\phi_{\mathcal{E}_1} \otimes \phi_{\mathcal{E}_3}) \circ (\text{id} \otimes \tau_{C^*, A''^*} \otimes \text{id})$;

(iv) If A^* be a finite acyclic complex of vector bundles on X , there is a canonical isomorphism $\det A^* \cong \mathbf{1}_X$. More generally, if $f : A^* \rightarrow B^*$ be a quasi-isomorphism of finite complexes of vector bundles, then there is an isomorphism $\det f : \det A^* \rightarrow \det B^*$ which depends only on the homotopy class of f ;

(v) Consider

$$\begin{array}{ccccccc} \mathcal{E} : & 0 \longrightarrow & A_1^* & \longrightarrow & A^* & \longrightarrow & A_2^* \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow f & & \downarrow f_2 \\ \mathcal{E}' : & 0 \longrightarrow & B_1^* & \longrightarrow & B^* & \longrightarrow & B_2^* \longrightarrow 0 \end{array}$$

then $\det f \circ \phi_{\mathcal{E}} = \phi_{\mathcal{E}'} \circ (\det f_1 \otimes \det f_2)$;

(vi) Consider the exact sequences $\mathcal{E} : 0 \rightarrow A^* \xrightarrow{\alpha} B^* \rightarrow 0 \rightarrow 0$ and $\mathcal{E}' : 0 \rightarrow 0 \rightarrow B^* \xrightarrow{\beta} C^* \rightarrow 0$, then

$$\det \alpha = \phi_{\mathcal{E}} \circ (a \mapsto a \otimes 1), \det \beta = \phi_{\mathcal{E}'} \circ (b \mapsto 1 \otimes b).$$

Proof. These are more complicated linear algebra, we omit these here. We refer [8] XIII.4. \square

16.3.2 Constructions and properties

Proposition 16.3.5 (Determinant of the cohomology of coherent sheaves). *Aiming to construct the relative determinant of the cohomology here.*

•**Claim.** *Let $f : X \rightarrow S$ be a flat morphism and let F be a coherent sheaf on X which is flat over S . Let Z be the subset of X where F is not locally free, then Z does not contain any component of any fiber of f .*

As F locally is $\mathcal{O}_U^m \xrightarrow{\alpha} \mathcal{O}_U^n \rightarrow F|_U = \text{coker } \alpha \rightarrow 0$, If an entire component of the fiber of f over a point s were contained in Z , the rank of α would drop along the whole component. This could already be detected on a sufficiently thick infinitesimal neighborhood of s . Hence we may let $S = \text{Spec } A$ where A be an artinian local ring. Let $U = \text{Spec } B$ and $F|_U = \widetilde{M}$ and we need to show $U \not\subset Z$. (**Need to re-read. To add.**)

•**Construction.** Consider a family of nodal curves $\pi : X \rightarrow S$. Let F is flat over S . Let S covered by U such that there is an effective Cartier divisor D in $\pi^{-1}(U)$ which meets all the irreducible components of every fiber and does not contain any of them; in particular, D is relatively ample. We may replacing D with a multiple, then let $R^1\pi_*F(D) = 0$. By the Claim, we may also suppose that F is locally free at every point of D . We say such divisor admissible.

Hence $F \subset F(D)$ and let $F(D)|_D := F(D)/F$. By some cohomology and base change, we get $\pi_*F(D), \pi_*F(D)|_D$ are all locally free. We have

$$0 \rightarrow \pi_*F \rightarrow \pi_*F(D) \rightarrow \pi_*F(D)|_D \rightarrow R^1\pi_*F \rightarrow 0,$$

hence the complex $E_D^* := (\pi_*F(D) \rightarrow \pi_*F(D)|_D)$ computes the higher direct image of F . Hence we let locally $d_\pi F = \det E_D^*$.

The independence on D and the gluing map are not hard to construct and we omitted them, see [8] page 356. Hence we get the determinant $d_\pi F$ of the cohomology of F (relative to π).

Remark 16.3.6. *The flatness of F over S is unnecessary but simplifies the construction.*

Proposition 16.3.7 (Determinant of the (hyper)cohomology of complexes). *Consider a family of nodal curves $\pi : X \rightarrow S$. Similarly, Let F^* be a finite complex of coherent sheaves, flat over S . Let U be a sufficiently small open subset of S , and let D be a divisor in $\pi^{-1}(U)$ which is admissible for each one of the F^i and that $F^i \rightarrow F^{i+1}$ is a morphism of vector bundles at each point of D for each i (this is called admissible for F^*). Hence we let $E_D^{i,0} = \pi_*(F^i(D))$ and $E_D^{i,1} = \pi_*(F^i(D)|_D)$, then we get a double complex $E_D^{*,*}$. Regard it as a single complex graded by total degree, and we locally define $d_\pi F^*$ to be its determinant.*

Proposition 16.3.8. *Consider a family of nodal curves $\pi : X \rightarrow S$.*

(i) *For a coherent F with $\pi_*F, R^1\pi_*F$ are locally free, then we have*

$$d_\pi F \cong \det(R^1\pi_*F)^{-1} \otimes \det(\pi_*F);$$

(ii) *For a finite complex F^* with $\mathbb{R}^i\pi_*F^* := H^i R\pi_*F^*$ are locally free, then we have*

$$d_\pi F^* = \bigotimes_{i \in \mathbb{Z}} \det(\mathbb{R}^i\pi_*F^*)^{(-1)^i}.$$

Proof. I just prove (i) since (ii) is similar.

We split

$$0 \rightarrow \pi_*F \rightarrow \pi_*F(D) \rightarrow \pi_*F(D)|_D \rightarrow R^1\pi_*F \rightarrow 0$$

into two sequences

$$0 \rightarrow \pi_*F \rightarrow E_D^0 \rightarrow Q \rightarrow 0, 0 \rightarrow Q \rightarrow E_D^1 \rightarrow R^1\pi_*F \rightarrow 0.$$

Then we have $\det(\pi_*F) \otimes \det Q \cong \det E_D^0$ and $\det(R^1\pi_*F) \otimes \det Q \cong \det E_D^1$. Hence we have locally

$$d_\pi(F) = \det(E_D^1)^{-1} \otimes \det(E_D^0) = \det(R^1\pi_*F)^{-1} \otimes \det(\pi_*F)$$

and well done. \square

Remark 16.3.9. *These constructions are compatible with base change, hence for $s \in S$ we have*

$$d_\pi F^* \otimes \kappa(s) \cong \bigotimes_{q \in \mathbb{Z}} (\det \mathbb{H}^q(X_s, F_s^*))^{(-1)^q}.$$

Theorem 16.3.10. *Let $0 \rightarrow E^* \rightarrow F^* \rightarrow G^* \rightarrow 0$ be an exact sequence of finite complexes of coherent sheaves on X , all flat over S , then we have*

$$\phi : d_\pi(E^*) \otimes d_\pi(G^*) \cong d_\pi(F^*).$$

Proof. Not hard but it's hard to type and I omit it here. We refer [8] XIII.(4.17). \square

16.3.3 Determinant, relative duality and applications

Theorem 16.3.11. *Consider a family of nodal curves $\pi : X \rightarrow S$ and a coherent sheaf F , we have*

$$d_\pi(\omega_\pi \otimes F^\vee) \cong d_\pi(F).$$

In particular, the Hodge bundle is $d_\pi(\omega_\pi) = d_\pi(\mathcal{O}_X)$.

Proof. This is also not hard to prove by checking the construction of the determinant of cohomology. Using some canonical exact sequences and diagrams this is almost trivial. I omit these here and we refer [8] page 360. \square

Proposition 16.3.12 (Determinant and boundary of moduli). *We will describe $\mathcal{O}(\mathcal{D})$ by using the determinant of cohomology.*

Proof. Let $\pi : X \rightarrow S$ be a family of connected nodal curves of genus g .

•**Claim 1.** Ω_π^1 is S -flat.

WLOG we let S is smooth as these are pullbacked from a Kuranishi family. Shrinking S , we may assume that there exists an effective divisor D in X which cuts an ample divisor D_s on each fiber X_s and does not contain nodes of the fibers. We just need to show $\chi(X_s, \Omega_\pi^1(nD) \otimes \kappa(s))$ is independent of $s \in S$ for $n \gg 0$. By Corollary 7.4.5, we have

$$0 \rightarrow K \rightarrow \Omega_{X_s}^1 \xrightarrow{\rho_s} \omega_{X_s} \rightarrow Q \rightarrow 0$$

where $\text{supp}(K), \text{supp}(Q) \subset \{\text{nodes}\}$. As they both have one-dimensional stalks by Claim 1,2 in Corollary 7.4.5, hence we have $\chi(X_s, \Omega_\pi^1(nD) \otimes \kappa(s)) = \chi(X_s, \omega_{X_s}(nD_s)) = 2g - 2 + n \deg(D)$.

•**Claim 2.** Let $L_\pi := d_\pi(\Omega_\pi^1 \xrightarrow{\rho_\pi} \omega_\pi)$, then $L_\pi = d_\pi(\omega_\pi)d_\pi(\Omega_\pi^1)^{-1}$ and induce $\det(\rho_\pi) : d_\pi(\Omega_\pi^1) \rightarrow d_\pi(\omega_\pi)$ which is a canonical section of L_π .

As we have an exact sequence of complexes $0 \rightarrow \omega_\pi[0] \rightarrow (\Omega_\pi^1 \xrightarrow{\rho_\pi} \omega_\pi) \rightarrow \Omega_\pi^1[1] \rightarrow 0$, we get $L_\pi = d_\pi(\omega_\pi)d_\pi(\Omega_\pi^1)^{-1}$. The map $\det(\rho_\pi) : d_\pi(\Omega_\pi^1) \rightarrow d_\pi(\omega_\pi)$ can be easily constructed step by step as the construction of d_π .

•**Claim 3.** Various L_π and $\det(\rho_\pi)$ defines a line bundle L on $\overline{\mathcal{M}}_{g,n}$ with a canonical section $\det(\rho)$. As ρ_π is an isomorphism on smooth fibers, we get $L \cong \mathcal{O}(\sum_i n_i \mathcal{D}_i)$ where \mathcal{D}_i are components of \mathcal{D} with nonnegative integers n_i . We claim that all $n_i = 1$ and hence $L \cong \mathcal{O}(\mathcal{D})$.

We consider the case when S is a disk (étale locally) centered at s and all the fibers of π are smooth except for X_s , which has a single node p . All we need is to calculate n_i , the order of vanishing of $\det \rho_\pi$ at s . I omit it here and refer [8] page 363. \square

Proposition 16.3.13. *Let Γ be a connected P -pointed genus g graph. Let $H(\Gamma)$ for the set of the half-edges of Γ which are not legs. Suppose that for each v , we are given a family $\pi_v : X_v \rightarrow S$ of connected nodal L_v -pointed genus g_v curves. Let σ_l the corresponding section of π_v and \mathcal{L}_l are point bundles on S where $l \in L_v$. Let $D_l = \sigma_l(S)$ and let $\pi : X \rightarrow S$ be the family gluing via Γ by X_v . Then*

$$\mathcal{O}(\mathcal{D})_\pi \cong \left(\bigotimes_{v \in V(\Gamma)} \mathcal{O}(\mathcal{D})_{\pi_v} \right) \otimes \left(\bigotimes_{h \in H(\Gamma)} \mathcal{L}_h^{-1} \right).$$

In particular, taking Chern classes we get

$$\xi_\Gamma^* \delta = \sum_{v \in V(\Gamma)} \eta_v^* \delta - \sum_{h \in H(\Gamma)} \eta_{v(h)}^* \psi_h$$

where $\eta_\Gamma : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g,P}$ and $\eta_v : \overline{\mathcal{M}}_\Gamma \rightarrow \overline{\mathcal{M}}_{g_v, L_v}$.

Proof. Let $N = \coprod_v X_v \xrightarrow{\pi'} S$ with normalization $\nu : N \rightarrow X$. For $e = \{h, h'\} \in E(\Gamma)$ and let $\Sigma_e = \nu(D_h) = \nu(D_{h'})$, we have

$$\begin{aligned} 0 \rightarrow \omega_{\pi'} &\longrightarrow \omega_{\pi'} \left(\sum_{h \in H(\Gamma)} D_h \right) \xrightarrow{Res} \bigoplus_{\{h, h'\} \in E(\Gamma)} (\mathcal{O}_{D_h} \oplus \mathcal{O}_{D_{h'}}) \rightarrow 0 \\ 0 \rightarrow \omega_\pi &\longrightarrow \nu_* \left(\omega_{\pi'} \left(\sum_{h \in H(\Gamma)} D_h \right) \right) \longrightarrow \bigoplus_{e \in E(\Gamma)} \mathcal{O}_{\Sigma_e} \longrightarrow 0 \end{aligned}$$

Taking cohomology we get

$$\begin{aligned} d_{\pi'}(\omega_{\pi'}) &\cong d_\pi(\nu_* \omega_{\pi'}) \cong d_\pi \left(\nu_* \omega_{\pi'} \left(\sum_{h \in H(\Gamma)} D_h \right) \right) \\ d_\pi(\omega_\pi) &\cong d_\pi \left(\nu_* \omega_{\pi'} \left(\sum_{h \in H(\Gamma)} D_h \right) \right). \end{aligned}$$

Hence

$$d_\pi(\omega_\pi) \cong d_{\pi'}(\omega_{\pi'}) \cong \bigotimes_{v \in V(\Gamma)} d_{\pi_v}(\omega_{\pi_v}).$$

On the other hand, we have

$$0 \rightarrow K \rightarrow \Omega_\pi^1 \rightarrow \nu_* \Omega_{\pi'}^1 \rightarrow 0$$

which deduce

$$d_\pi(\Omega_\pi^1) \cong \left(\bigotimes_{e \in E(\Gamma)} \pi_* K_e \right) \otimes d_{\pi'} \Omega_{\pi'}^1.$$

•**Claim.** We have $\pi_* K_{\{h, h'\}} \cong \mathcal{L}_h \otimes \mathcal{L}_{h'}$.

Here we give a sketch of the claim. Consider $e = \{h, h'\}$ with local coordinates x, y , then K_e locally generated by $ydx (= -xdy)$, then we define it mapping to section $\sigma_h^*(dx) \otimes \sigma_{h'}^*(dy)$ of $\mathcal{L}_h \otimes \mathcal{L}_{h'}$. We omitted the verifying.

Finally, by the Claim 2,3 in Proposition 16.3.12 we get

$$\begin{aligned} \mathcal{O}(\mathcal{D})_\pi &= L_\pi = d_\pi(\omega_\pi) \otimes d_\pi(\Omega_\pi^1)^{-1} \\ &= \left(\bigotimes_{v \in V(\Gamma)} d_{\pi_v}(\omega_{\pi_v}) \right) \otimes \left(\left(\bigotimes_{e \in E(\Gamma)} \pi_* K_e \right) \otimes d_{\pi'} \Omega_{\pi'}^1 \right)^{-1} \\ &= \left(\bigotimes_{v \in V(\Gamma)} \mathcal{O}(\mathcal{D})_{\pi_v} \right) \otimes \left(\bigotimes_{h \in H(\Gamma)} \mathcal{L}_h^{-1} \right) \otimes (d_{\pi'} \Omega_{\pi'}^1)^{-1}. \end{aligned}$$

(How to destroy $(d_{\pi'} \Omega_{\pi'}^1)^{-1}$? Need to think this more.) □

Remark 16.3.14. We also get $\xi_\Gamma^* \lambda = \sum_{v \in V(\Gamma)} \eta_v^* \lambda$.

16.4 Deligne pairing, a quick tour

Definition 16.4.1. (a) Let C be a complete curve (need not be connected) and $D = \sum_p n_p p$ a divisor on C . If f is a rational function on C whose divisor (f) is disjoint from D , we set $f(D) := \prod_p f(p)^{n_p}$;

(b) Let $\pi : X \rightarrow S$ be a family of nodal curves and D is an effective relative Cartier divisor not containing nodes of fibers, $\pi_* \mathcal{O}(D)$ is locally free, and there is a norm map $\text{Norm}_{D/S} : \pi_* \mathcal{O}(D) \rightarrow \mathcal{O}_S$ (as $D \rightarrow S$ is proper and quasi-finite, hence finite). We also induce $\text{Norm}_{D/S} : \pi_* \mathcal{O}(D)^\times \rightarrow \mathcal{O}_S^\times$.

Hence for an divisor $D = D_1 - D_2$ where D_i are effective, then we define

$$f(D) = \text{Norm}_{D_1/S}(f) \text{Norm}_{D_2/S}(f)^{-1}$$

which is well defined as if E_1, E_2 are all effective, then $f(E_1 + E_2) = f(E_1)f(E_2)$.

Proposition 16.4.2 (Weil reciprocity). (i)[Smooth case] Let C be a smooth proper curve (need not be connected) and f, g are rational functions which are nonzero on every component of C and with disjoint divisors. Then $f((g)) = g((f))$;

(ii)[Nodal case] Let C be a possibly disconnected nodal curve, and let f and g be rational functions on C which do not vanish identically on any irreducible component of C and are regular and nonzero at all the nodes. Then, if the divisors of f and g are disjoint, we have $f((g)) = g((f))$;

(iii)[Relative case] Let $\pi : X \rightarrow S$ be a family of nodal curves and f and g are two meromorphic functions on X not vanish identically on any component of any fiber and be regular and nonzero at all the nodes, and their divisors be disjoint, then $f((g)) = g((f))$.

Proof. For (i) we refer [9] VI.B.2. For (ii), notice that what must be proved can reduce to the Weil reciprocity formula for the pullbacks of f and g to the normalization of C . For (iii), when S is reduced, we can do the same thing as single one. Otherwise, one can use the Kuranishi family and pullback. \square

Definition 16.4.3 (Deligne pairing for single case). Let L, M are two line bundles over a nodal curve C . Let V be a vector space generated by pairs (l, m) where l, m are rational sections of L, M , respectively, such that

- (a) l, m are nonzero on any component of C , and regular and nonzero at the nodes of C ;
- (b) the divisors (l) and (m) are disjoint.

Let $\langle L, M \rangle$ be the quotient of V modulo the equivalence relation generated by

$$(fl, m) \sim f((m))(l, m), \quad (l, gm) \sim g((l))(l, m)$$

where f and g are rational functions on C . This space is the Deligne pairing of L and M . The class of (l, m) denoted by $\langle l, m \rangle$.

Remark 16.4.4. (i) Actually the meromorphic section l of L defined by a data (l_i, U_i) where $l_i \in \mathcal{H}_C(U_i)$ of covering $X = \bigcup_i U_i$ such that $l_i = \psi_{ij} \cdot l_j$ where $\psi_{ij} = \psi_i \circ \psi_j^{-1}$ are cocycles of trivializations $\phi_i : L|_{U_i} \cong \mathcal{O}_{U_i}$. In other words, l is a section of $L \otimes_{\mathcal{O}_X} \mathcal{H}_X$. Hence we have canonically divisor (l) associated to l and we have trivially $\mathcal{O}_X((l)) \cong L$ (see [62] and [48]);

(ii) Two equivalence relations are called L -move and M -move, respectively.

Proposition 16.4.5. For any L, M on C , then $\dim \langle L, M \rangle = 1$ be a line.

sketch. •**Claim 1.** We have $\dim \langle L, M \rangle \leq 1$.

For any $(l, m), (l', m')$, let μ be a meromorphic divisor of M disjoint of l, l' . Hence let $\mu = gm, m' = g'\mu, l' = fl$ where f, g, g' are rational functions, then

$$(l', m') \sim g'((l'))f((\mu))g((l))(l, m),$$

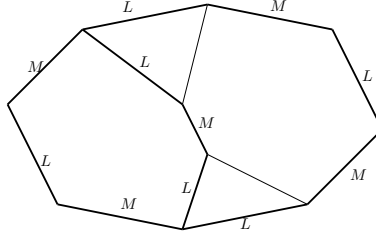
hence $\dim \langle L, M \rangle \leq 1$.

•**Claim 2.** A pair (l, m) cannot be equivalent to a strict multiple of itself (a cycle).

This is a very interesting proof by induction on the length of the cycle. After prove the case of 4 and 6 directly, we can let $n \geq 8$ and using Weil reciprocity.

This method break a n -move cycle into two cycles of length $n - 2$, then one can use the induction. This proof is not so hard and much interesting, but I omit this and the detailed proof see [8] page 368. The main idea is the following diagram:

which tell us the cycle of 8 moves broken up in two cycles of 6. \square



Definition 16.4.6 (Deligne pairing for the families). A family $\pi : X \rightarrow S$ of nodal curves and L and M are line bundles on X . For any $s \in S$ we have a rank 1 free $\mathcal{O}_{S,s}$ -module $\langle L, M \rangle_s$ by Proposition 16.4.5. For any open $U \subset S$, we define a sheaf

$$\Gamma(U, \langle L, M \rangle_\pi) := \left\{ \left\{ u_s \in \langle L, M \rangle_s : s \in U \right\} \left| \begin{array}{l} \text{for every } s \in U, \text{ there are a neighborhood } U' \\ \text{and meromorphic sections } l, m \text{ of } L, M \text{ over} \\ \pi^{-1}(U') \text{ such that } u_t = \langle l, m \rangle \text{ for every } t \in U'. \end{array} \right. \right\}.$$

This is a line bundle on S , called the Deligne pairing of L and M , denoted by $\langle L, M \rangle_\pi$.

Proposition 16.4.7. Consider a family $\pi : X \rightarrow S$ of nodal curves and $L, L_1, L_2, L_3, M, M_1, M_2$ are line bundles on X .

(i) We have canonical isomorphisms

$$\begin{aligned} \langle L_1, M \rangle_\pi \otimes \langle L_2, M \rangle_\pi &\cong \langle L_1 \otimes L_2, M \rangle_\pi \\ \langle L, M_1 \rangle_\pi \otimes \langle L, M_2 \rangle_\pi &\cong \langle L, M_1 \otimes M_2 \rangle_\pi; \end{aligned}$$

(ii) We have canonical isomorphisms $\langle L, \mathcal{O}_X \rangle_\pi \cong \mathcal{O}_S$ and $\langle \mathcal{O}_X, M \rangle_\pi \cong \mathcal{O}_S$;

(iii) Of course, we have the canonical isomorphism $\tau : \langle L, M \rangle_\pi \cong \langle M, L \rangle_\pi$ given by $\langle l, m \rangle \mapsto \langle m, l \rangle$.

In particular when $L = M$, we have $\tau(-) = (-1)^{\deg L} \cdot (-)$.

Proof. See [8] XIII (5.4),(5.5) and Proposition 5.7. \square

Theorem 16.4.8. Consider a family $\pi : X \rightarrow S$ of nodal curves and L, M are line bundles on X . Then we have a canonical isomorphism

$$\langle L, M \rangle_\pi \cong d_\pi(L \otimes M) \otimes d_\pi(L)^{-1} \otimes d_\pi(M)^{-1} \otimes d_\pi(\mathcal{O}_X)$$

compatible with base change.

Proof. See [8] XIII Theorem 5.8. \square

Corollary 16.4.9. (i) Let D be any relative divisor not passing through nodes of fibers of $\pi : X \rightarrow S$. The sheaves $\pi_*(\mathcal{O}_D)$ and $\pi_*M|_D$ are both locally free of rank equal to the degree of D over S . We may then define a line bundle on S as by setting

$$\text{Norm}_{D/S}(M|_D) := \mathcal{H}om(\det(\pi_*\mathcal{O}_D), \det(\pi_*M|_D)),$$

then we have

$$\langle \mathcal{O}_X(D), M \rangle_\pi \cong \text{Norm}_{D/S}(M|_D).$$

(ii) In particular, if we have a section σ with $D = \sigma(S)$, then for any $M \in \text{Pic}(X)$, we have

$$\langle \mathcal{O}_X(D), M \rangle_\pi \cong \sigma^*M.$$

Taking $M = \omega_\pi(D)$, we get

$$\langle \mathcal{O}_X(D), \omega_\pi \rangle_\pi \cong \langle \mathcal{O}_X(D), \mathcal{O}_X(D) \rangle_\pi^{-1}.$$

(iii) We have

$$c_1(\langle L, M \rangle_\pi) = \pi_*(c_1(L) \cdot c_1(M)).$$

Proof. (i) This is easy if we define a norm map $\text{Norm}_{D/S} : \pi_*(M|_D) \rightarrow \text{Norm}_{D/S}(M|_D)$ as $h \mapsto \det(\times h : \pi_*\mathcal{O}_D \rightarrow \pi_*(M|_D))$, then we get

$$\langle \mathcal{O}_X(D), M \rangle_\pi \cong \text{Norm}_{D/S}(M|_D), \quad \langle 1, m \rangle_\pi \mapsto \text{Norm}_{D/S}(m|_D).$$

(ii) Special case of (i).

(iii) This is a hard but difficult result, we refer [8] page 376, XIII.(5.20). \square

Corollary 16.4.10 (Some kind of Riemann-Roch). *Let $\pi : X \rightarrow S$ be a family of nodal curves, and let L be a line bundle on X . There is a canonical isomorphism of line bundles, compatible with base change:*

$$d_\pi(L)^2 \cong \langle L, L \otimes \omega_\pi^{-1} \rangle_\pi \otimes d_\pi(\mathcal{O}_X)^2.$$

Proof. As $\langle L, L \otimes \omega_\pi^{-1} \rangle_\pi \cong \langle L, L^{-1} \otimes \omega_\pi \rangle_\pi^{-1}$ by Proposition 16.4.7 (i)(ii), we then use Theorem 16.4.8 to $\langle L, L^{-1} \otimes \omega_\pi \rangle_\pi$ and we win. \square

Example 16.4.11. *Consider a family of curves $\pi : X \rightarrow S$ plus sections σ_i , corresponding to divisors $D_i = \sigma_i(S)$. We denote $\widehat{\omega}_\pi := \omega_\pi(\sum_i D_i)$ and we get $\langle \widehat{\omega}_\pi, \widehat{\omega}_\pi \rangle_\pi \in \text{Pic}(S)$. As the Deligne pairing is well behaved under base change, this defines $\langle \widehat{\omega}, \widehat{\omega} \rangle$ on $\overline{\mathcal{M}}_{g,n}$ and we denote*

$$\kappa_1 = [\langle \widehat{\omega}, \widehat{\omega} \rangle] \in \text{Pic}(\overline{\mathcal{M}}_{g,n}).$$

(For κ_a , the codimension a , can also be constructed)

Moreover, by Corollary 16.4.9, we get $[\langle \widehat{\omega}_\pi, \mathcal{O}_X(D_i) \rangle] = \psi_i$. More generally, we get

$$\left[\left\langle \widehat{\omega}_\pi^h \left(\sum_i a_i D_i \right), \widehat{\omega}_\pi^l \left(\sum_i b_i D_i \right) \right\rangle_\pi \right] = hl\kappa_1 - \sum_i a_i b_i \psi_i.$$

After this, if we let $\widetilde{\kappa}_1 = [(\omega, \omega)] \in \text{Pic}(\overline{\mathcal{M}}_{g,n})$, we have

$$\widetilde{\kappa}_1 = \kappa_1 - \psi.$$

Finally, like Remark 16.3.14 we have $\xi_\Gamma^* \kappa_1 = \sum_{v \in V(\Gamma)} \eta_v^* \kappa_1$. The proof we refer [8] page 378.

16.5 The Picard group of moduli space of curves I

Theorem 16.5.1. *Consider $H_{\nu,g,n} \subset \text{Hilb}_{\mathbb{P}^N}^{P_\nu}$ be the Hilbert scheme of ν -log-canonically embedded n -pointed stable curves of genus g where $N = (2\nu - 1)(g - 1) + \nu n$ and $P_\nu(t) = (2\nu t - 1)(g - 1) + \nu n t$ for $\nu \geq 3$. Let $H'_{\nu,g,n} \subset H_{\nu,g,n}$ be the smooth locus. Hence we have $\overline{\mathcal{M}}_{g,n} \cong [H_{\nu,g,n}/\text{PGL}(N)]$ and $\mathcal{M}_{g,n} \cong [H'_{\nu,g,n}/\text{PGL}(N)]$. Then we have group isomorphisms:*

$$\begin{aligned} \text{Pic}(\overline{\mathcal{M}}_{g,n}) &\cong \text{Pic}(H_{\nu,g,n}, \text{PGL}(N)) \cong \text{Pic}(H_{\nu,g,n})^{\text{PGL}(N)}, \\ \text{Pic}(\mathcal{M}_{g,n}) &\cong \text{Pic}(H'_{\nu,g,n}, \text{PGL}(N)) \cong \text{Pic}(H'_{\nu,g,n})^{\text{PGL}(N)}. \end{aligned}$$

Proof. The first isomorphisms of these two statements are trivial. The second isomorphism need some GIT. We refer [74] for surjectivity and [8] Proposition XIII.6.1 for injectivity. \square

Proposition 16.5.2. *For $\pi : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{M}_{g,n}$ and $\vartheta : \mathcal{M}_{g,n} \rightarrow M_{g,n}$ we have exact sequences:*

$$\begin{aligned} 0 \rightarrow \text{Pic}(\overline{M}_{g,n}) \xrightarrow{\pi^*} \text{Pic}(\overline{\mathcal{M}}_{g,n}) \rightarrow Q \rightarrow 0, \\ 0 \rightarrow \text{Pic}(M_{g,n}) \xrightarrow{\vartheta^*} \text{Pic}(\mathcal{M}_{g,n}) \rightarrow R \rightarrow 0 \end{aligned}$$

where Q, R are torsion groups. More precisely, there is a positive integer k such that

$$k \cdot \text{Pic}(\overline{\mathcal{M}}_{g,n}) \subset \text{Pic}(\overline{M}_{g,n}) \text{ and } k \cdot \text{Pic}(\mathcal{M}_{g,n}) \subset \text{Pic}(M_{g,n}).$$

In particular, one has

$$\text{Pic}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q} \cong \text{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}, \quad \text{Pic}(\mathcal{M}_{g,n}) \otimes \mathbb{Q} \cong \text{Pic}(M_{g,n}) \otimes \mathbb{Q}.$$

Proof. As the proof is the same at both cases, we just consider the case of $\text{Pic}(\overline{\mathcal{M}}_{g,n})$ and $\text{Pic}(\overline{M}_{g,n})$. As $\overline{M}_{g,n}$ covered by $U_i = B_i/G_i$ where $X_i \rightarrow B_i$ are (standard algebraic) Kuranishi families with the automorphism groups of central fiber G_i . Let $L \in \text{Pic}(\overline{M}_{g,n})$ pullback to $\overline{\mathcal{M}}_{g,n}$ is trivial hence has a nowhere vanishing global section. Hence gives a nowhere vanishing G_i -invariant section of the pullback of L to B_i by étale descent. Hence a nowhere vanishing section of L pullback to $\overline{\mathcal{M}}_{g,n}$, hence $\text{Pic}(\overline{M}_{g,n}) \rightarrow \text{Pic}(\overline{\mathcal{M}}_{g,n})$ is injective.

Next we need to find a integer k such that for any $\mathcal{L} \in \text{Pic}(\overline{\mathcal{M}}_{g,n})$ we have \mathcal{L}^k descends to a line bundle M on $\overline{M}_{g,n}$. Let $X = \coprod X_i, B = \coprod B_i$ where $X_i \rightarrow B_i$ are (standard algebraic) Kuranishi families with the automorphism groups of central fiber G_i , then $B \rightarrow \overline{\mathcal{M}}_{g,n}$ and $\coprod B_i/G_i \rightarrow \overline{M}_{g,n}$ are étale covers. Hence by étale descent we may let \mathcal{L} as line bundle L over B with descent data to $B \rightarrow \overline{\mathcal{M}}_{g,n}$. Now take $b \in B$ and consider L_b , then $\text{Aut}(X_b)$ act on L_b linearly. As L_b is just a one-dimensional vector space, hence this action is just multiplication by k_b -th roots of unity where $k_b := |\text{Aut}(X_b)|$. Hence now we let $k = \prod_i |G_i|$ and then for any b , we have $k_b|k$ by the property of the standard Kuranishi family. Hence these groups act trivially over L^k and hence \mathcal{L}^k descend to $\overline{M}_{g,n}$ by basic étale descent. \square

16.6 The Picard group of moduli space of curves II

In this section we will mainly refer Enrico Arbarello and Maurizio Cornalba's classical paper [5] in the base field \mathbb{C} . But in the positive characteristic algebraically closed field k , we have the similar result, see [73]. Actually he prove more, that is, $\text{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}_\ell \cong H_{\text{ét}}^2(\overline{M}_{g,n}, \mathbb{Q}_\ell)$ when ℓ is prime and invertible in k . But we do not care about these here.

16.6.1 Some preliminaries

Here we follows [82].

Definition 16.6.1 (Pencil). *A pencil of hypersurfaces on a variety X is a projective line $\mathbb{P}^1 \subset |L|$, where L is a line bundle on X .*

Hence a pencil of hypersurfaces on a variety X gives us $\sigma_t \in H^0(X, L)$ for all $t \in \mathbb{P}^1$, up to a coefficient in \mathbb{C}^\times . These (well-)defines the hypersurfaces $X_t \subset X$ correspond to σ_t . So we denote $(X_t)_{t \in \mathbb{P}^1}$ as this pencil. Actually we can denote $\sigma_t = \sigma_0 + t\sigma_\infty$ for $t \in \mathbb{A}^1 \subset \mathbb{P}^1$. Hence the base locus of the pencil is $B = \bigcap_{t \in \mathbb{P}^1} X_t \subset X$ defined by σ_0, σ_∞ . Let $X' = \text{Bl}_B(X) \cong \{(x, t) \in X \times \mathbb{P}^1 : x \in X_t\}$, hence if we let $f : X' \rightarrow \mathbb{P}^1$, then $f^{-1}(t) \cong X_t$.

Definition 16.6.2 (Lefschetz pencil). *A Lefschetz pencil $(X_t)_{t \in \mathbb{P}^1}$ is a pencil of hypersurfaces satisfies:*

- (i) B is smooth with $\text{codim}_X(B) = 2$;
- (ii) X_t has at most one ordinary double point as singularity.

Remark 16.6.3 (Ordinary double point). *Let X be an algebraic scheme over k with a closed $x \in X$.*

(i) *If $k = \bar{k}$, then x is called an ordinary double point if*

$$\widehat{\mathcal{O}}_{X,x} \cong k[[x_1, \dots, x_n]]/(f)$$

where $f \in \mathfrak{m}^2$ such that $f = Q + R$ where Q be a nondegenerate quadratic form and $R \in \mathfrak{m}^3$ where \mathfrak{m} be the maximal ideal of $k[[x_1, \dots, x_n]]$;

(ii) *For general k , $x \in X$ is called an ordinary double point if all points in $X \otimes_k \bar{k}$ lying over x are ordinary double points.*

Next we will introduce something about K3 surfaces. We refer [12] chapter VIII or more general book [63] for more detailed arguments.

Definition 16.6.4. *A K3 surface over k is a proper nonsingular variety X of dimension two such that*

$$\bigwedge^2 \Omega_{X/k} \cong \mathcal{O}_X, H^1(X, \mathcal{O}_X) = 0.$$

Proposition 16.6.5 (see [12] Proposition VIII.13 or [63] Lemma II.2.1). *Let X be a K3 surface and $C \subset S$ be a smooth curve of genus g , then $C^2 = 2g - 2$ and $h^0(X, \mathcal{O}_X(C)) = g + 1$.*

Proof. The statement $C^2 = 2g - 2$ follows from adjunction formula. Again by adjunction formula we get

$$\omega_C = \omega_X \otimes \mathcal{O}_X(C) \otimes \mathcal{O}_C = \mathcal{O}_X(C) \otimes \mathcal{O}_C = \mathcal{O}_X(C)|_C.$$

Hence $H^0(C, \mathcal{O}_X(C)|_C) = H^0(C, \omega_C)$. As $H^1(X, \mathcal{O}_X) = 0$ and the exact sequence $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{O}_X(C)|_C \rightarrow 0$ we get $h^0(X, \mathcal{O}_X(C)) = 1 + h^0(C, \omega_C) = g + 1$. By Riemann-Roch formula, we get $\chi(X, \mathcal{O}_X(C)) = g + 1$. As $h^2(X, \mathcal{O}_X(C)) = h^0(X, \mathcal{O}_X(-C)) = 0$, we get $h^0(X, \mathcal{O}_X(C)) \geq g + 1$. \square

Theorem 16.6.6 (Existence of K3 surfaces). *For any $g \geq 3$, there exists K3 surfaces S of degree $2g - 2$ embedded in \mathbb{P}^g .*

Proof. See [12] Theorem VIII.15. They construct K3 surfaces containing a very ample divisor D with $D^2 = 2g - 2$. \square

16.6.2 J. Harer's theorem and its corollaries

Here we follow the paper [77] and the Appendix of the Enrico Arbarello and Maurizio Cornalba's paper [5]. We just summarize the several results here and we refer the original papers [53] and [54] due to J. Harer by using the Teichmüller space (the construction one can see [49] and [8] chapter XV).

Theorem 16.6.7 (Harer's theorem). *(i) The group $\text{Pic}(M_g) \otimes \mathbb{Q}$ is freely generated by the λ ;*

(ii) The group $\text{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ is freely generated by the $\lambda, \Delta_{irr}, \Delta_i$;

(iii) The group $\text{Pic}(M_{g,n}) \otimes \mathbb{Q}$ is freely generated by the λ, ψ_i ;

(iv) The group $\text{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$ is freely generated by the $\lambda, \psi_i, \Delta_{irr}, \Delta_i$.

Remark 16.6.8. *Note that we have showed in Proposition 16.1.3 that the Hodge class λ defined over $\overline{\mathcal{M}}_{g,n}$ can not descend to $\overline{M}_{g,n}$, so the Hodge class here we defined at the meaning of Proposition 16.5.2.*

Proposition 16.6.9 (See appendix in [5]). *The group $\text{Pic}(\overline{\mathcal{M}}_{g,n})$ and $\text{Pic}(\mathcal{M}_{g,n})$ has no torsion.*

Corollary 16.6.10. *We have $\text{Pic}(\overline{\mathcal{M}}_{g,n})$ generated by rational coefficients classes $\lambda, \psi_i, \delta_{irr}, \delta_i$ and $\text{Pic}(\mathcal{M}_{g,n})$ generated by rational coefficients classes λ, ψ_i .*

Proof. Follows from the Harer's theorem 16.6.7 and Proposition 16.6.9. \square

16.6.3 The groups $\text{Pic}(\overline{\mathcal{M}}_{g,n})$ and $\text{Pic}(\mathcal{M}_{g,n})$ for $g \geq 3$

First we deal with the case of $n = 0$, as follows.

Theorem 16.6.11. *For $g \geq 3$ we have $\text{Pic}(\overline{\mathcal{M}}_g)$ is freely generated by $\lambda, \delta_{irr}, \delta_i$; the group $\text{Pic}(\mathcal{M}_g)$ is freely generated by λ .*

The most important thing is that we need to construct some special families of curves.

♣ Construct four kinds of families.

► Families of type I. Λ_n for $2 \leq n \leq g$.

Pick a smooth K3 surface Y' of degree $2n - 2$ in \mathbb{P}^n by Theorem 16.6.6 and consider a Lefschetz pencil of hyperplane sections. As Y' is smooth, one might choose generic pencil of hyperplane sections by Bertini's theorem (see [82] corollary 2.10).

Let B_s be the base locus of the pencil and let $Y = \text{Bl}_{B_s}(Y')$. Let $\phi : Y \rightarrow B := \mathbb{P}^1$. The curves of the pencil appear in Y as fibers of ϕ and the exceptional curves appear as sections E_i of ϕ .

Fix a smooth curve Γ of genus $g - n$ and a point γ on it. Construct a new surface $X = (Y \sqcup \Gamma \times \mathbb{P}^1) / (E_1 \sim \{\gamma\} \times \mathbb{P}^1)$. Hence we get a family $\Lambda_n = (f : X \rightarrow B = \mathbb{P}^1)$. As we consider the Lefschetz pencil, we find that the fibers of $\phi : Y \rightarrow B$, hence the fibers of $f : X \rightarrow B$, are all nodal curves.

• Describe λ_{Λ_n} .

First we claim that

$$f_*\omega_f \cong \phi_*\omega_\phi \oplus (\mathcal{O}_B)^{g-n}.$$

(why?) Second we claim that $\text{rank}(\phi_*\omega_\phi) = n$. As Y' be a K3 surface and the fiber of ϕ are the smooth curves $C \subset Y'$ correspond to the sections of Lefschetz pencil, hence $g(C) = p_a(C) = \frac{C^2}{2} + 1 = n$ by adjunction formula as the existence of K3 surface by Proposition 16.6.5 and Theorem 16.6.6 (hence flat by checking Hilbert polynomial. actually by our choice of Lefschetz pencil, all fibers of ϕ are smooth, hence so is ϕ). Hence $\text{rank}(\phi_*\omega_\phi) = n$. Hence we get

$$\lambda_{\Lambda_n} = \bigwedge^g f_*\omega_f = \bigwedge^n \phi_*\omega_\phi.$$

• **Compute** $\deg \lambda_{\Lambda_n}$.

First, by the Riemann-Roch of vector bundles over curves (see St 0BS6) we get

$$\chi(B, \phi_*\omega_\phi) = \deg \lambda_{\Lambda_n} + n(1 - g(B)) = \deg \lambda_{\Lambda_n} + n.$$

Second, since $R^1\phi_*\omega_\phi = \mathcal{O}_B$ we get

$$\chi(\phi_!\omega_\phi) = \chi(\phi_*\omega_\phi) - \chi(\mathcal{O}_B).$$

Finally, by Leray spectral sequence $E_2^{p,q} = H^p(B, R^q\phi_*\omega_\phi) \Rightarrow H^{p+q}(Y, \omega_\phi)$ we get the $E_2 = E_\infty$ page:

$$\begin{array}{ccc} H^0(B, R^1\phi_*\omega_\phi) & 0 & 0 \\ & \searrow & \\ H^0(B, \phi_*\omega_\phi) & H^1(B, \phi_*\omega_\phi) & \rightarrow 0 \end{array}$$

hence by the definition of $\phi_!$ we get $\chi(\phi_!\omega_\phi) = \chi(\omega_\phi)$. By Riemann-Roch of surfaces, we get

$$\chi(\omega_\phi) = \chi(\mathcal{O}_Y) + \frac{K_\phi^2 - K_\phi \cdot K_Y}{2}.$$

As ϕ is smooth, we get $\omega_Y \cong \phi^*\omega_B \otimes \omega_\phi$, hence $K_\phi \equiv_{\text{lin}} K_Y - \phi^*K_B$. Hence

$$\chi(\omega_\phi) = \chi(\mathcal{O}_Y) - \frac{\phi^*K_B^2 - K_Y \cdot \phi^*K_B}{2}.$$

By the construction of $\phi : Y \rightarrow B$, we get $\phi^*\omega_B \cong \mathcal{O}((2g(B) - 2)F)$ for a fiber F by the construction. Use the adjunction formula to F , we get $2g(F) - 2 = F^2 - F \cdot K_Y = -F \cdot K_Y$. Hence we get

$$\begin{aligned} \chi(\omega_\phi) &= \chi(\mathcal{O}_Y) - \frac{\phi^*K_B^2 - K_Y \cdot \phi^*K_B}{2} = \chi(\mathcal{O}_Y) - (g(B) - 1)(2g(F) - 2) \\ &= \chi(\mathcal{O}_Y) + 2n - 2 = 2n \end{aligned}$$

since Y is the blowing up of a K3 surface (hence birational to that K3 surface) which deduce $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_{Y'}) = 2$ as in this case it is \mathcal{O} -connected with vanishing higher direct image (this is a conclusion due to Hironaka in characteristic zero, more general, see [21]). Combining these, we get

$$\begin{aligned} \deg \lambda_{\Lambda_n} &= \chi(B, \phi_*\omega_\phi) - n = \chi(\phi_!\omega_\phi) + \chi(\mathcal{O}_B) - n \\ &= \chi(\omega_\phi) + \chi(\mathcal{O}_B) - n = n + 1. \end{aligned}$$

We win!

► **Families of type II.** F_n for $g \geq 3, 2 \leq 2n \leq g - 1$.

Let smooth curves C_1, C_2, Γ of genus $n, g - n, 1$ and fix points $x_1 \in C_1, x_2 \in C_2, \gamma \in \Gamma$. Let $Y_1 = C_1 \times \Gamma, Y_2 = \text{Bl}_{\{(\gamma, \gamma)\}}(\Gamma \times \Gamma)$ with exceptional divisor E and $Y_3 = C_2 \times \Gamma$. Let $A = \{x_1\} \times \Gamma, B = \{x_2\} \times \Gamma$ and Δ be the strict transform of the diagonal in Y_2 and S be the strict transform of $\{\gamma\} \times \Gamma$ in Y_2 . Let

$$X = \frac{Y_1 \sqcup Y_2 \sqcup Y_3}{S \sim A, \Delta \sim B}$$

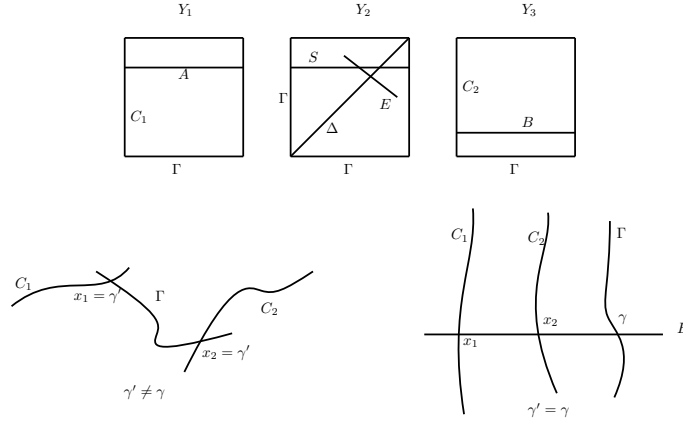
with $f : X \rightarrow \Gamma$ be the family, called F_n . The graphs of F_n and its fibers at $\gamma' \in \Gamma$ are as follows:

• **Compute** $\deg \lambda_{F_n}$.

First we have $f_*\omega_f \cong (H^0(\omega_{C_1}) \oplus H^0(\omega_{C_2}) \oplus H^0(\omega_\Gamma)) \otimes \mathcal{O}_\Gamma$. (why?) Hence $\deg \lambda_{F_n} = 0$.

• **Compute** $\deg(\delta_i)_{F_n}$.

By the arguments in [56] page 81, we have the following general principle:



Lemma 16.6.12. *Let $\pi : \mathcal{C} \rightarrow B$ be a family of stable curves over a smooth curve B which is obtained from a family $\phi : \mathcal{D} \rightarrow B$ of (not necessarily connected) nodal curves by identifying sections S_i, T_i pairwise. For each j , let Σ_j denote the image of S_j in \mathcal{C} . Suppose the locus of singular points of type i in the fibers of π is*

$$[p_1, \dots, p_m] \cup \bigcup_j \Sigma_j$$

where the p_i are distinct points not belonging to $\bigcup_j \Sigma_j$. Then

$$(\delta_i)_\pi = \bigotimes_j (\phi_*(N_{S_j}) \otimes \phi_*(N_{T_j})) \left(\sum_l n_l \pi(p_l) \right)$$

where N_S be the normal bundle and \mathcal{C} is of form $xy = t^{n_i}$ near p_l .

Now we will use this to compute $\deg(\delta_i)_{F_n}$. Actually by adjunction formula we get

$$\begin{aligned} A^2 &= 2g(A) - 2 - A \cdot K_{Y_1} = -A \cdot (p^* K_{C_1} + q^* K_\Gamma) = 0, \\ B^2 &= 2g(B) - 2 - B \cdot K_{Y_2} = -B \cdot (p^* K_{C_2} + q^* K_\Gamma) = 0. \end{aligned}$$

By Proposition A.3.5, we have

$$\begin{aligned} \Delta^2 &= 2g(\Delta) - 2 - \Delta \cdot K_{Y_2} = -\Delta \cdot (p^* K_\Gamma + q^* K_\Gamma + E) = -1, \\ S^2 &= 2g(S) - 2 - S \cdot K_{Y_2} = -S \cdot (p^* K_\Gamma + q^* K_\Gamma + E) = -1. \end{aligned}$$

Hence we have

$$\deg N_A = \deg N_B = 0, \deg N_S = \deg N_\Delta = -1.$$

Hence by the Lemma we get

$$\deg(\delta_{irr})_{F_n} = 0, \deg(\delta_1)_{F_n} = \begin{cases} 1, & n > 1; \\ 0, & g - n - 1 > n = 1; \\ -1, & g - n - 1 = n = 1(g = 3). \end{cases}$$

$$\deg(\delta_n)_{F_n} = \begin{cases} -1, & g - n - 1 > n > 1; \\ 0, & g - n - 1 > n = 1; \\ -2, & g - n - 1 = n > 1; \\ -1, & g - n - 1 = n = 1(g = 3), \end{cases} \quad \deg(\delta_{n+1})_{F_n} = -1(\text{if } g - n - 1 > n).$$

And other cases are all 0.

► **Families of type III. The family F .**

Consider a general pencil of conics in \mathbb{P}^2 with four base points. Blowing up these points in the plane we get $\psi : X \rightarrow \mathbb{P}^2$ with exceptional lines E_1, \dots, E_4 . Moreover we consider the resulting conic bundle $\phi : X \rightarrow \mathbb{P}^1$. Hence we have

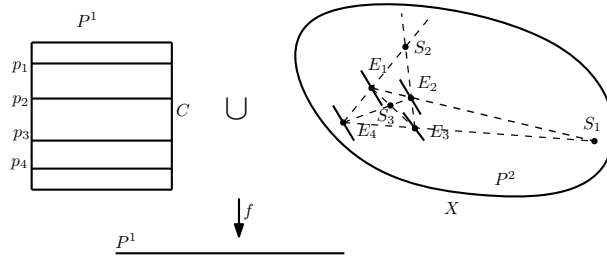
$$\begin{aligned} \omega_\phi &= \omega_X \otimes \phi^* \mathcal{O}_{\mathbb{P}^1}(-2)^{-1} = \psi^* \omega_{\mathbb{P}^2} \otimes \mathcal{O}_X \left(\sum E_i \right) \otimes \phi^* \mathcal{O}_{\mathbb{P}^1}(2) \\ &= \psi^* \omega_{\mathbb{P}^2} \otimes \mathcal{O}_X \left(\sum E_i \right) \otimes \psi^* \mathcal{O}_{\mathbb{P}^2}(4) \otimes \mathcal{O}_X \left(-2 \sum E_i \right) = \psi^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_X \left(-\sum E_i \right). \end{aligned}$$

Now we let C be a fixed smooth curve of genus $g - 3$ with four fixed points p_1, \dots, p_4 on it, let

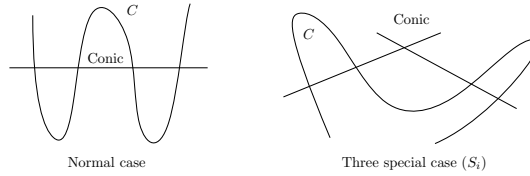
$$Y = \frac{X \sqcup (C \times \mathbb{P}^1)}{E_i \sim \{p_i\} \times \mathbb{P}^1 (i = 1, \dots, 4)}$$

and consider $f : Y \rightarrow \mathbb{P}^1$ a family of curves of genus g . We call this family F .

We consider the fibers of F . First we draw the picture of the family F , then we find that there are exactly three special points such that the conics are not smooth, hence we have two different types of fibers. The following picture is the family $f : Y \rightarrow \mathbb{P}^1$:



Hence we have two kinds of fibers as follows:

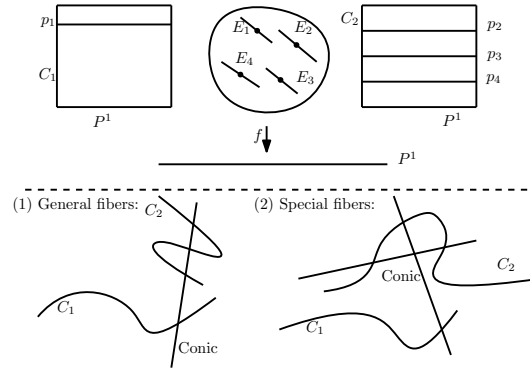


- **Compute** $\deg \lambda_F$.
In fact $f_* \omega_f \rightarrow H^0(\omega_C(\sum p_i)) \otimes \mathcal{O}_{\mathbb{P}^1}$ is injective, hence an isomorphism. Hence $\deg \lambda_F = 0$.
- **Compute** $\deg(\delta_i)_F$.
As that three special points, hence we get $\deg(\delta_{irr})_F = 3 + \sum \deg N_{E_i} = 3 + \sum E_i^2 = -1$. Moreover, it's easy to see that $\deg(\delta_i)_F = 0$ for $i > 0$.
- **Families of type IV. The family F' .**
Let C_1 be an smooth elliptic curve and C_2 be a smooth curves of genus $g - 3$. Let $p_1 \in C_1$ and $p_2, p_3, p_4 \in C_2$. We consider the similar X in the construction of F , we let

$$Y = \frac{X \sqcup (C_1 \times \mathbb{P}^1) \sqcup (C_2 \times \mathbb{P}^1)}{E_i \sim \{p_i\} \times \mathbb{P}^1, i = 1, \dots, 4},$$

to get a family of stable curves $f : Y \rightarrow \mathbb{P}^1$. We call this family F' , as follows:
There are two kinds of fibers as before.

- **Compute** $\deg \lambda_{F'}$.
Similar as F , we get $f_* \omega_f$ is trivial. Hence $\deg \lambda_{F'} = 0$.
- **Compute** $\deg(\delta_i)_{F'}$.
As that three special points, hence we get $\deg(\delta_{irr})_{F'} = 3 + \sum_{i \geq 2} E_i^2 = 0$. Moreover, we have $\deg(\delta_1)_{F'} = \deg N_{E_1} = E_1^2 = -1$. Finally we get $\deg(\delta_i)_{F'} = 0$ for all $i > 1$.
- ♣ **Back to the proof of the theorem.**



Let $k = \lfloor g/2 \rfloor$ and let $G_i = (C_i \rightarrow S_i)$ are $k + 2$ families of stable curves. We denote the matrix

$$\eta(G_1, \dots, G_{k+2}) = \begin{pmatrix} \deg \lambda_{G_1} & \deg(\delta_{irr})_{G_1} & \cdots & \deg(\delta_k)_{G_1} \\ \deg \lambda_{G_2} & \deg(\delta_{irr})_{G_2} & \cdots & \deg(\delta_k)_{G_2} \\ \vdots & \vdots & \ddots & \vdots \\ \deg \lambda_{G_{k+2}} & \deg(\delta_{irr})_{G_{k+2}} & \cdots & \deg(\delta_k)_{G_{k+2}} \end{pmatrix}.$$

Proof of the theorem. For our families of curves we find that λ, δ_i are linearly independent. By Harer's result (Corollary 16.6.10) we have that $\text{Pic}(\overline{\mathcal{M}}_g)$ is generated by the rational coefficients of the linear combinations of λ, δ_i . So we let $\xi \in \text{Pic}(\overline{\mathcal{M}}_g)$ with $\xi = a\lambda + b_0\delta_{irr} + \sum b_i\delta_i$ where $a, b_i \in \mathbb{Q}$. Now we first let that we have constructed two different sets of $k + 2$ families of stable curves G_i such that two $\det \eta$ s are relative prime. Let $d_i = \deg \xi_{G_i}$, then

$$\begin{pmatrix} d_1 \\ \vdots \\ d_{k+2} \end{pmatrix} = \eta \begin{pmatrix} a \\ b_0 \\ \vdots \\ d_k \end{pmatrix}.$$

As $d_i \in \mathbb{Z}$, then so are $a \det \eta, b_0 \det \eta, \dots, b_k \det \eta$. As two $\det \eta$ s are relative prime, then $a, b_i \in \mathbb{Z}$ and we win! Now we just need to construct these two different sets of $k + 2$ families.

- **When g is odd and $g = 2m + 1$.**

We consider $\eta_n := \eta(\Lambda_n, F, F_1, \dots, F_m)$ where n is an integer between 2 and $k = \lfloor g/2 \rfloor$. When have

$$\det \eta_n = \det \begin{pmatrix} n+1 & \cdots & & & & & & & \\ 0 & -1 & \cdots & & & & & & \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdots & & \\ 0 & 0 & 1 & -1 & -1 & 0 & \cdots & & \\ & & & 1 & 0 & -1 & -1 & \cdots & \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & 1 & 0 & \cdots & 0 & -1 & -1 & \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & -2 & \end{pmatrix} = (-1)^{m+1}(n+1).$$

Taking $n = 2, 3$ and well done.

- **When g is even and $g = 2m + 2$.**

We consider $\eta_n := \eta(\Lambda_n, F, F', F_1, \dots, F_m)$ where n is an integer between 2 and $k = \lfloor g/2 \rfloor$. When

have

$$\det \eta_n = \det \begin{pmatrix} n+1 & \cdots & & & & & & & \\ 0 & -1 & \cdots & & & & & & \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & & \\ 0 & 0 & 0 & -1 & 0 & 0 & \cdots & & \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdots & & \\ & & & 1 & 0 & -1 & -1 & 0 & \cdots \\ \vdots & & & & & & & & \vdots \\ 0 & 0 & 1 & 0 & \cdots & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & -2 \end{pmatrix} = (-1)^m(n+1).$$

Taking $n = 2, 3$ and well done. □

Now we come to the general case. Here we just give a sketch and the detailed proof we refer the section 3 in the original paper [5].

Theorem 16.6.13. *For every $g \geq 3$, the group $\text{Pic}(\overline{\mathcal{M}}_{g,n})$ is freely generated by $\lambda, \psi_i, \delta_j$ and $\text{Pic}(\mathcal{M}_{g,n})$ is freely generated by λ, ψ_i .*

Remark 16.6.14. *As the marked points here are in order (instead of the case $\overline{\mathcal{M}}_{g,P}$), we now need to make the boundary divisor more explicitly. The class δ_{irr} is the locus that the partial normalization is connected. The class $\delta_{\alpha; i_1, \dots, i_a}$ are the locus that the partial normalization have two connected components, one of them is of genus α with marked points p_{i_1}, \dots, p_{i_a} and another one is of genus $g - \alpha$ with remaining marked points. Of course we will let $0 \leq \alpha \leq \lfloor g/2 \rfloor$, $0 \leq a \leq n$, $i_1 < \dots < i_a$ and $a \geq 2$ when $\alpha = 0$.*

♣ **Step 1. Define the forgetting map $\vartheta : \text{Pic}(\overline{\mathcal{M}}_{g,n}) \rightarrow \text{Pic}(\overline{\mathcal{M}}_{g,n+1})$.**

Actually this of course induced by the forgetful map (is some kind of blow down). Moreover along this map, we have the following fundamental relations.

$$\begin{cases} \vartheta(\lambda) = \lambda, \\ \vartheta(\psi_i) = \psi_i - \delta_{0; i, n+1}, & i = 1, \dots, n, \\ \vartheta(\delta_{irr}) = \delta_{irr}, \\ \vartheta(\delta_\alpha) = \delta_\alpha, & \text{if } \alpha = g/2, n = 0, \\ \vartheta(\delta_{\alpha; i_1, \dots, i_a}) = \delta_{\alpha; i_1, \dots, i_a} + \delta_{\alpha; i_1, \dots, i_a, n+1}, & \text{otherwise.} \end{cases}$$

♣ **Step 2. Preparation I.**

Pick a smooth family $F = (f : \mathcal{C} \rightarrow S, \sigma_i) \in \mathcal{M}_{g,n}(S)$ and consider the pullback of σ_i in $\mathcal{C} \times_S \mathcal{C} \rightarrow \mathcal{C}$ as σ'_i . Let

$$X = \text{Bl}_{\cup_i (\Delta \cap \sigma'_i(\mathcal{C}))}(\mathcal{C} \times_S \mathcal{C})$$

and consider the diagram

$$\begin{array}{ccccc} X = \text{Bl}_{\cup_i (\Delta \cap \sigma'_i(\mathcal{C}))}(\mathcal{C} \times_S \mathcal{C}) & \longrightarrow & \mathcal{C} \times_S \mathcal{C} & \longrightarrow & \mathcal{C} \\ & \searrow \phi & \downarrow \sigma'_i & & \downarrow \sigma_i \\ & & \mathcal{C} & \xrightarrow{f} & S \end{array}$$

where $\widehat{\Delta}, \tau_i$ are strict transform of Δ, σ'_i . We let $F' = (\phi : X \rightarrow \mathcal{C}, \tau_i, \widehat{\Delta})$.

Definition 16.6.15. *Let $L \in \text{Pic}(\mathcal{M}_{g,n+1})$. We shall say that L is trivial on smooth curves if $L|_{F'}$ is trivial whenever S consists of a single point.*

Lemma 16.6.16. *Let L be a line bundle on $\overline{\mathcal{M}}_{g,n+1}$. If L is trivial on smooth curves there exists a line bundle \mathcal{L} on $\overline{\mathcal{M}}_{g,n}$ such that $L \equiv \vartheta(\mathcal{L}) \text{ mod boundary classes}$. Conversely, if there is \mathcal{L} on $\overline{\mathcal{M}}_{g,n}$ such that $L - \vartheta(\mathcal{L})$ is an integral linear combination of boundary classes other than the $\delta_{0; i, n+1}$, then L is trivial on smooth curves.*

Proof. See [5] Lemma 2. □

♣ **Step 3. Preparation II.**

Let X be a smooth K3 surface of degree $d = 2g - 2$ in \mathbb{P}^g such that $\text{Pic}(X) \cong \mathbb{Z} \cdot L$ where L be a hyperplane section, by [63] Example II.3.9. Pick a Lefschetz pencil of hyperplane sections on X . Blowing up the base locus to get Y' with exceptional curves E_1, \dots, E_d as sections of $Y' \rightarrow \mathbb{P}^1$. Hence $\text{Pic}(Y')$ freely generated by a fiber and the E_i by Proposition A.3.6.

Notice that as one varies the Lefschetz pencil the monodromy action on the base points of the pencil, and hence on the E_i , is given by the full symmetric group.

Let $Y = Y' - \bigcup\{\text{Singular fibers}\}$ and \mathbb{P} be the projection of Y over \mathbb{P}^1 . Let $\psi : Y \rightarrow \mathbb{P}$. We write E_i instead of $E_i \cap Y$. Hence we get the E_i freely generate $\text{Pic}(Y)$ as we have the exact sequence

$$\mathbb{Z}^k \rightarrow \text{Pic}(Y') \rightarrow \text{Pic}(Y) \rightarrow 0$$

where k be the numbers components of singular fibers.

♣ **Step 4. Proof for $n \leq 2g - 2$ by induction on n .**

As $n = 0$ is proved we let so is n when $n \leq 2g - 3$. We just need to show that $\text{Pic}(\overline{\mathcal{M}}_{g,n+1})$ is generated, over \mathbb{Z} , by $\vartheta(\text{Pic}(\overline{\mathcal{M}}_{g,n}))$, ψ_{n+1} and the boundary classes. Let $\mu \in \text{Pic}(\overline{\mathcal{M}}_{g,n+1})$. As $n \leq 2g - 2 = d$, we let

$$f : \mathcal{Y} := \text{Bl}_{\bigcup_{i=1}^n (\Delta \cap E_i)}(Y \times_{\mathbb{P}} Y) \rightarrow Y, \widehat{\Delta}, \widehat{E}_1, \dots, \widehat{E}_n$$

as the construction in **Preparation I**.

As μ_f is an integral linear combination of E_1, \dots, E_d , by monodromy, the coefficients of E_{n+1}, \dots, E_d are all equal, that is,

$$\mu_f = \sum_{i \leq n} a_i E_i + a_{n+1} \sum_{i > n} E_i.$$

On the other hand we can express $(\psi_i)_f, (\psi_{n+1})_f, (\delta_{0;j,n+1})_f$ as the combinations of E_i . So if we let

$$\mu = \sum \alpha_j \psi_j + \beta \lambda + \sum \gamma_j \delta_{0;j,n+1} + \dots$$

where $\alpha_j, \beta, \gamma_j \in \mathbb{Q}$ by Harer's theorem, then we can get some relations (we will omit it here). In particular, we get $\alpha_{n+1}, \alpha_j + \gamma_j \in \mathbb{Z}$ for $j \leq n$.

Set

$$\mu' = \mu - \alpha_{n+1} \psi_{n+1} - \sum (\alpha_j + \gamma_j) \delta_{0;j,n+1},$$

then by these relations, we get $(\mu')_f = 0$, and similarly, on any fibers of f . On the other hand, since we have

$$\mu = \alpha_{n+1} \psi_{n+1} + \sum \alpha_j \vartheta(\psi_j) + \beta \vartheta(\lambda) + \sum (\alpha_j + \gamma_j) \delta_{0;j,n+1} + \dots$$

by several relations in **Step 1**. Hence μ' a \mathbb{Q} -coefficients linear combination of classes in $\vartheta(\text{Pic}(\overline{\mathcal{M}}_{g,n}))$ and boundary classes not of the form $\delta_{0;j,n+1}$. By Lemma 16.6.16 there exists $\xi \in \text{Pic}(\overline{\mathcal{M}}_{g,n})$ such that $\mu' \equiv \vartheta(\xi) \pmod{\text{boundary classes}}$, hence

$$\mu \equiv \alpha_{n+1} \psi_{n+1} + \vartheta(\xi) \pmod{\text{boundary classes}}$$

and we win!

♣ **Step 5. Proof for $n > 2g - 2$ by induction on n .**

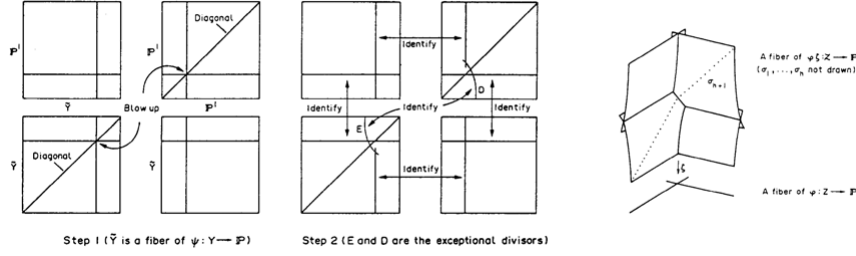
We assume that this is proved for some $n \geq 2g - 2$ and we consider $n + 1$. The main idea is similar as the previous case and we just give a construction of the family of curves we considered here and omit all details.

Consider the same $\psi : Y \rightarrow \mathbb{P}, E_1, \dots, E_d$ and let $Q = \mathbb{P}^1 \times \mathbb{P} \rightarrow \mathbb{P}$ with sections D_{2g-4}, \dots, D_n . Let

$$\phi : Z := \frac{Y \sqcup Q}{E_{2g-3} \sim D_{2g-4}}, E_1, \dots, E_{2g-4} \rightarrow \mathbb{P}, D_{2g-3}, \dots, D_n.$$

Consider $\zeta : \mathcal{Z} \rightarrow Z, \sigma_1, \dots, \sigma_{n+1}$ constructed via $Z \times_{\mathbb{P}} Z$ as follows

Like the previous case, we can have some relations and then, we will use the Lemma 16.6.16.



16.7 The tautological & canonical class

Here we will follow the section XIII.7 in [8].

• **Situation A.** Let $j : Y \rightarrow Z$ be a codimension r closed immersion of smooth schemes with \mathcal{G} be a coherent sheaf on Y . By Grothendieck-Riemann-Roch theorem we get

$$\text{ch}(j_*\mathcal{G}) = \text{ch}(j!\mathcal{G}) = j_*(\text{ch}(\mathcal{G})\text{td}(Y))\text{td}(Z)^{-1}.$$

Hence we get $c_i(j_*\mathcal{G}) = 0$ when $i < r$ and $c_r(j_*\mathcal{G}) = (-1)^{r-1}(r-1)!\text{rank}(\mathcal{G})[Y]$ by the codimension reasons.

• **Situation B.** For any $(f : X \rightarrow H, \tau_1, \dots, \tau_n) \in \overline{\mathcal{M}}_{g,n}(H)$ where X, H are smooth and any coherent sheaf \mathcal{F} on X , by Grothendieck-Riemann-Roch theorem we get

$$\text{ch}(f_!\mathcal{F}) = f_*(\text{ch}(\mathcal{F}) \cdot \text{td}(\Omega_{X/H}^\vee)).$$

Consider the degree 1 terms, we get

$$c_1(f_!\mathcal{F}) = f_* \left(\frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F}) - \frac{c_1(\mathcal{F})c_1(\Omega_f^1)}{2} + \frac{c_1(\Omega_f^1)^2 + c_2(\Omega_f^1)}{12} \right).$$

Next let Σ be the locus of nodes with ideal sheaf \mathcal{I} , then by Corollary 7.4.5 we have

$$0 \rightarrow \Omega_f^1 \rightarrow \omega_f \rightarrow \omega_f \otimes \mathcal{O}_\Sigma \rightarrow 0$$

as X is smooth. Now consider $j : \Sigma \rightarrow X$ and $\mathcal{G} = \omega_f \otimes \mathcal{O}_\Sigma$ in **Situation A** and Whitney formula we get $c_1(\Omega_f^1) = c_1(\omega_f)$ and $c_2(\Omega_f^1) = [\Sigma]$. (Why the locus Σ is smooth?)

Theorem 16.7.1 (Mumford). For $2g - 2 + n > 0$ we have $\kappa_1 = 12\lambda + \psi - \delta$ in $\text{Pic}(\overline{\mathcal{M}}_{g,n})$.

Proof. By the previous situations, let $\mathcal{F} = \omega_f$ we get

$$c_1(f_!\omega_f) = f_* \left(\frac{c_1(\omega_f)^2}{2} - \frac{c_1(\omega_f)c_1(\omega_f^1)}{2} + \frac{c_1(\omega_f)^2 + [\Sigma]}{12} \right) = f_* \left(\frac{c_1(\omega_f)^2 + [\Sigma]}{12} \right).$$

By Corollary 16.4.9(iii) and the definition of κ_1 (Example 16.4.11) we get

$$\lambda = \frac{\kappa_1 - \psi + \delta}{12} \Rightarrow \kappa_1 = 12\lambda + \psi - \delta$$

in $\text{Pic}(\overline{\mathcal{M}}_{g,n})$. □

Theorem 16.7.2 (Mumford). For $2g - 2 + n > 0$ we have $K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda + \psi - 2\delta$ in $\text{Pic}(\overline{\mathcal{M}}_{g,n})$.

Proof. In the case of **Situation B** we let $f : X \rightarrow H$ with divisor of sections D and $\mathcal{F} = \Omega_f^1 \otimes \omega_f(D)$. By Proposition 9.2.9 and Serre duality we get $f_! \mathcal{F} = f_* \mathcal{F}$. Hence again by Example 16.2.2 we get

$$K_{\overline{\mathcal{M}}_{g,n}} = c_1(f_*(\Omega_f^1 \otimes \omega_f(D))) = c_1(f_* \mathcal{F}) = c_1(f_! \mathcal{F}).$$

By the same work in **Situation B** we have $c_1(\Omega_f^1 \otimes \omega_f(D)) = c_1(\omega_f^2(D))$ and $c_2(\Omega_f^1 \otimes \omega_f(D)) = [\Sigma]$. Hence again we have

$$K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda + \psi - 2\delta$$

in $\text{Pic}(\overline{\mathcal{M}}_{g,n})$. □

Corollary 16.7.3. *For $g \geq 1$ and $g+n \geq 4$, we have $K_{\overline{M}_{g,n}} = 13\lambda + \psi - 2\delta - \delta_{1,\emptyset}$ in $\text{Pic}(\overline{M}_{g,n})$.*

Proof. We need to following results due to [8] Proposition XII.2.5:

• **Fact.** If $g \geq 1$ and $g+n \geq 4$, consider the locus parameterizing curves with nontrivial automorphism group Ξ , then the only divisor components of Ξ is $\Delta_{1,\emptyset}$.

Back to the problem, then this follows $K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda + \psi - 2\delta$ in $\text{Pic}(\overline{\mathcal{M}}_{g,n})$, this fact and Riemann-Hurwitz Theorem! □

Remark 16.7.4. *Note that when we talking about δ instead of Δ over coarse moduli space $\overline{M}_{g,n}$, we means that $\delta_1 = \frac{1}{2}\Delta_1$ and $\delta = \Delta_{irr} + \frac{1}{2}\Delta_1 + \Delta_2 + \dots$. Hence here $K_{\overline{M}_g} = 13\lambda - 2\Delta + \frac{1}{2}\Delta_1$.*

16.8 A glimpse of ample & nef divisors and F -conjecture

Here we will summary (without proofs) some results about ample divisors over the coarse moduli space $\overline{M}_{g,n}$. We will follows the idea in [46] here.

Theorem 16.8.1 (Cornalba-Harris [24], 1988). *The class $a\lambda - b\delta \in \text{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ has non-negative degree on every curve in \overline{M}_g not contained in the boundary $\Delta = \overline{M}_g \setminus M_g$ if and only if $a \geq (8 + \frac{4}{g})b$ and is ample if and only if $a > 11b > 0$.*

Remark 16.8.2. *By Lemma 6.1 in [25], the Hodge class λ is big and nef. Note that by this result, λ itself is not ample, but since it is big it is a sum of an ample and an effective divisor.*

Definition 16.8.3. (i) *The strata consisting of curves with $3g - 4 + n$ nodes form curves in $\overline{M}_{g,n}$ called F -curves (in honor of Faber and Fulton);*

(ii) *The locus of flag curves is the image $\overline{F}_{g,n}$ of the morphism*

$$\overline{M}_{0,g+n}/\mathfrak{S}_g \rightarrow \overline{M}_{g,n}$$

obtained by attaching g copies of the pointed rational elliptic curve at the g -unordered points.

Remark 16.8.4. *For (i), since the locus of curves with k nodes has codimension k in $\overline{M}_{g,n}$ and $\dim(\overline{M}_{g,n}) = 3g - 3 + n$, so we just consider the 1-dimensional locus.*

Actually by classical Nakai-Moishezon criterion we know that a divisor D on $\overline{M}_{g,n}$ is ample if and only if $D^{\dim K} \cdot K > 0$ for all integral subscheme $K \subset \overline{M}_{g,n}$. But Fulton's Conjecture asserts more remarkable thing:

Conjecture 1 (F -Conjecture). *A divisor D on $\overline{M}_{g,n}$ is ample if and only if $D \cdot C > 0$ for every F -curve on $\overline{M}_{g,n}$.*

In the paper [46] they showed that we just need to consider the case $n = 0$:

Theorem 16.8.5 (Gibney-Keel-Morrison, 2001). *A divisor D on $\overline{M}_{g,n}$ is nef if and only if D has non-negative intersection with all the F -curves and the restriction $D|_{\overline{F}_{g,n}}$ is nef. In particular, the F -conjecture for $g = 0$ implies the F -conjecture for all g .*

(But although $n = 0$, this problem is also open and difficult) In particular, this result can deduce many ad hoc examples. We may call the divisor which has non-negative intersection number with F -curves are called F -nef. Hence the F -conjecture asserts that the F -nef cone of divisors is the same as the nef cone of divisors of $\overline{M}_{g,n}$.

Corollary 16.8.6. *Let D be an F -nef divisor $a\lambda - \sum b_i \delta_i$ on \overline{M}_g . Assume further for each coefficient b_i , $1 \leq i \leq \lfloor g/2 \rfloor$, that either $b_i = 0$ or $b_i \geq b_{irr}$. Then D is nef.*

Proof. See [46] Proposition 6.1. □

Corollary 16.8.7. (i) *The ray $10\lambda - 2\delta + \delta_{irr}$ is nef on \overline{M}_g for all $g \geq 2$;*
(ii) *(Cornalba-Harris). The class $11\lambda - \delta$ is nef on \overline{M}_g for all $g \geq 2$.*

Proof. See [46] Corollary 6.2, 6.3. □

We should also remark that the F -conjecture is known for small genus and small numbers of points thanks to the work of Keel, McKernan, Farkas and Gibney:

Theorem 16.8.8 (Keel-McKernan with Gibney-Keel-Morrison and Farkas). *The F -conjecture holds for $\overline{M}_{g,n}$ when the pair (g, n) is of form:*

- (i) (g, n) for $g + n \leq 7$;
- (ii) $(g, 0)$ for $g \leq 24$;
- (iii) $(g, 1)$ for $g \leq 8$ or $(6, 2)$.

Proof. See Corollary (0.4) in [46], Theorem 1 in [38] and the results in [45]. In fact, Gibney has reduced the conjecture on a given \overline{M}_g to an entirely combinatorial question which can be checked by computer. □

Chapter 17

The Kodaira dimension of moduli space of curves

17.1 Summary of the results of kodaira dimension

Here as an introduction we will summary some results of the types of $\overline{M}_{g,n}$ where we will prove and we may not prove.

First, $\overline{M}_{g,n}$ is uniruled or even unirational for some small values of g and n :

Theorem 17.1.1 (Summaried in [14]). *Here we have a table about these. Let $\overline{M}_{g,n}$ is rational if $0 \leq n \leq a(g)$; is unirational if $0 \leq n \leq b(g)$ and uniruled if $0 \leq n \leq c(g)$:*

g	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$a(g)$	12	14	15	12	8										
$b(g)$	12	14	15	12	15	11	8	9	3	10	1	0	2		
$c(g)$	12	14	15	14	15	13	12	10	9	10	5	3	2	2	0

Remark 17.1.2. *Hence the Kodaira dimension of \overline{M}_g is negative for $g \leq 15$.*

Let's back to consider the Kodaira dimension and general typeness of $\overline{M}_{g,n}$.

Theorem 17.1.3 (Belorousski [13], 1998; Bini-Fontanari [16], 2004). *We have*

$$\kappa(\overline{M}_{1,n}) = \begin{cases} -\infty, & 1 \leq n \leq 10; \\ 0, & n = 11; \\ 1, & n \geq 12. \end{cases}$$

Corollary 17.1.4 (Bini-Fontanari [16], 2004). *For $n \geq 1$, $\overline{M}_{1,n}$ is never of general type.*

Theorem 17.1.5 (Summaried in [77]). *Let $\overline{M}_{g,n}$ is of general type for all $n \geq m(g)$ given in the following table:*

g	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$m(g)$	16	15	16	15	14	13	11	12	11	11	10	10	9	9	9	7	6	4	4	1

Now we consider the case when $n = 0$, that is, the space \overline{M}_g .

Theorem 17.1.6 (Mumford-Eisenbud-Harris [56][34], 1987; Farkas-Jensen-Payne [39], 2020). *The space \overline{M}_g is of general type when $g \geq 22$.*

Remark 17.1.7. (i) The Mumford-Eisenbud-Harris theorem proved that \overline{M}_g is of general type when $g \geq 24$ and has positive Kodaira dimension when $g = 23$. Further more, Farkas-Jensen-Payne proved that \overline{M}_g is of general type when $g = 22, 23$;

(ii) The remaining cases are $16 \leq g \leq 21$. Actually the Kodaira dimension of \overline{M}_g are still open for $16 \leq g \leq 21$ (Chang and Ran also argued that \overline{M}_{16} is uniruled, but Tseng recently found a fatal computational error in this argument arXiv:1905.00449, and this case is again open).

The first main aim of the chapter is to show the Mumford-Eisenbud-Harris theorem, that is, \overline{M}_g is of general type when $g \geq 24$. We will refer [33] and [34] by using limit linear series instead of admissible covers in [56].

17.2 The theorem of Harris-Mumford-Eisenbud

We omitted the theory of limite linear series and focus on the whole structure of the proof.

Theorem 17.2.1 (Mumford-Eisenbud-Harris [56][34], 1987). *The space \overline{M}_g is of general type when $g \geq 24$.*

Remark 17.2.2. *In papers [56][34] they show that M_g is of general type when $g \geq 24$. But their criterion also can derived that \overline{M}_g is of general type.*

To prove this, just need to show that $K_{\overline{M}_g}$ is big (see Definition A.3.7). Recall that by Corollary 16.7.3, we get

$$K_{\overline{M}_g} = 13\lambda - 2\delta - \delta_1 = 13\lambda - 2\delta_{irr} - 3\delta_1 - 2 \sum_{i=2}^{\lfloor g/2 \rfloor} \delta_i.$$

We also know that big if and only if it is numerically equivalent to the sum of an ample and an effective divisor.

Theorem 17.2.3 (Criterion in [34]). *The space \overline{M}_g is of general type if there exists an effective divisor D over it with class*

$$D = a\lambda - b_0\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i\delta_i$$

such that

$$\frac{a}{b_1} < \frac{13}{3}, \quad \frac{a}{b_i} < \frac{13}{2} \text{ for all } i.$$

Proof. By Theorem 16.8.1 we know that $(11 + \varepsilon)\lambda - \delta$ is ample, then hence λ is big. Now we let we have such an effective divisor D on \overline{M}_g . Let $c = \frac{p}{q}$ be a rational number such that

$$\max \left\{ \frac{2}{b_0}, \frac{3}{b_1}, \frac{2}{b_i} \right\} < c < \frac{13}{a},$$

and we find that

$$qK_{\overline{M}_g} - pD = (13q - ap)\lambda + (pb_0 - 2q)\delta_{irr} + (pb_1 - 3q)\delta_1 + \sum_{i=2}^{\lfloor g/2 \rfloor} (pb_i - 2q)\delta_i.$$

As λ is big and D is effective, then $K_{\overline{M}_g}$ is big. Hence \overline{M}_g is of general type. \square

So we need to find such effective divisor D for any $g \geq 24$.

♣ Construction A. Brill-Noether divisors D_s^r .

If $g + 1$ is composite, we can write $g = (r + 1)(s - 1) - 1$ for $s \geq 3, r > 0$ and let $d = rs - 1$. Let $S = \{[C] \in M_g : [C] \text{ admits } \mathfrak{g}_d^r\}$. Consider D_s^r be the union of the codimension 1 components

of $\bar{S} \subset \bar{M}_g$. This divisor called Brill-Noether divisor since in this case the Brill-Noether number $\rho = g - (r + 1)(g - d + r) = -1$.

Theorem A.(Brill-Noetherian Ray Theorem) In this situation, there exists some rational number $c > 0$ such that

$$D_s^r = c \left((g + 3)\lambda - \frac{g + 1}{6}\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g - i)\delta_i \right).$$

♣ **Construction B. Petri divisors E_s^r .**

If $g + 1 > 2$ is not composite (in particular g is even), then we let $g = (r + 1)(s - 1)$ and $d = rs$. Let

$$T = \left\{ [C] \in M_g \left| \begin{array}{l} [C] \text{ admits } L = (\mathcal{L}, V) \text{ be a } \mathfrak{g}_d^r \text{ such that the map} \\ V \otimes H^0(C, K_C \otimes L^{-1}) \rightarrow H^0(C, K_C) \text{ is not injective} \end{array} \right. \right\}.$$

Then consider E_s^r be the union of the codimension 1 components of $\bar{T} \subset \bar{M}_g$. In this case the Brill-Noether number $\rho = 0$.

Theorem B. If $g = 2(d - 1)$ then we have

$$E_d^1 = c \left(e\lambda - f_0\delta_{irr} - \sum_{i=1}^{g/2} f_i\delta_i \right)$$

where $c = 2 \frac{(2d-4)!}{d!(d-2)!}$, $e = 6d^2 + d - 6$ and $f_0 = d(d - 1)$, $f_1 = (2d - 3)(3d - 2)$, $f_2 = 3(d - 2)(4d - 3)$ and for $e \leq i \leq g/2$ we have $f_i > f_{i-1}$.

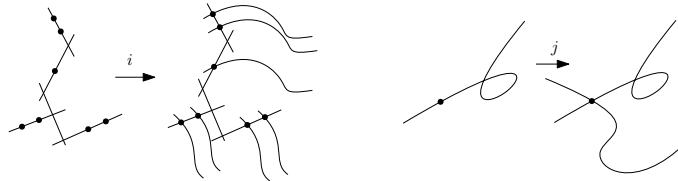
♣ **The proof of the main theorem.**

Assume that we have proved the **Theorem A** and **B**, then we give an easy proof of the main theorem 17.2.1 by using the Theorem 17.2.3.

Proof of Theorem 17.2.1. First, for $g \geq 28$ and even, consider $E_{(g/2)+1}^1$ and one can easy to calculate that it satisfied the Theorem 17.2.3; Second, for $g \geq 24$ and odd, then $D_{(g+1)/2}^1$ satisfied the Theorem 17.2.3; For $g = 24$, then D_6^4 satisfied the Theorem 17.2.3 and for $g = 26$, then D_{10}^2 satisfied the Theorem 17.2.3. Hence for any $g \geq 24$, there exists such divisor. Hence for any $g \geq 24$ the space \bar{M}_g is of general type. \square

♣ **Preparation for the proof of Theorem A and B.**

We just prove **Theorem A** and give a sketch of **Theorem B** since it is so complicated. Consider $i : \bar{M}_{0,g} \rightarrow \bar{M}_g$ be the map by attaching g copies of a fixed pointed elliptic curve at each of the marked points; And $j : \bar{M}_{2,1} \rightarrow \bar{M}_g$ be the map by attaching a fixed general smooth pointed curve of genus $g - 2$ at the marked point:



Let $W \subset \bar{M}_{2,1}$ be the closure of the locus for which the marked point is Weierstrass point of the underlying curves in $M_{2,1}$.

Theorem 17.2.4. Let $D \subset \bar{M}_g$ be an effective divisor as

$$D = a\lambda - b_0\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i\delta_i.$$

(i) If $\text{supp}(j^*D) \subset W$, then $a = 5b_1 - 2b_2$ and $b_0 = \frac{b_1}{2} - \frac{b_2}{6}$. Further, if we write $j^*D = qW$ for some rational number q , then $b_2 = 3q$.

(ii) If $i^*D = 0$, then

$$b_i = \frac{i(g-i)}{g-1}b_1 \text{ for } i = 2, \dots, \left\lfloor \frac{g}{2} \right\rfloor.$$

Proof. See [34] Theorem 2.1 and Theorem 3.1 or the final several parts of the section 6.F in [55]. \square

♣ **Proof of Theorem A.**

Corollary 17.2.5. Let $D \subset \overline{M}_g$ be an effective divisor as $D = a\lambda - b_0\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i\delta_i$ satisfies both (i)(ii) in the Theorem 17.2.4, then there exists some rational c such that

$$D = c \left((g+3)\lambda - \frac{g+1}{6}\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i \right).$$

Proof. Almost trivial. Use the second relation to write b_2 in terms of b_1 . Then use the first to show that $a/b_0 = 6 + 12/(g+1)$. Then show that if $a = g+3$, then $b_0 = (g+1)/6$ and $b_1 = 1$. The remaining coefficients are then immediate from the second set of relations. \square

Proposition 17.2.6. Let $g = (r+1)(s-1) - 1$ for $s \geq 3, r > 0$, then D_s^r doesn't meet either $i(\overline{M}_{0,g})$ or $j(\overline{M}_{2,1} \setminus W)$. Moreover, $j^*(D_s^r)$ is a positive multiple of the class of W .

Proof. See [34] Proposition 4.1. and [55] Proposition 6.68. \square

Proof of Theorem A. By the first statement in Proposition 17.2.6 and Corollary 17.2.5, we get

$$D_s^r = c \left((g+3)\lambda - \frac{g+1}{6}\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i \right)$$

for some rational c . To show $c > 0$, we use the second statement in Proposition 17.2.6. This statement shows that the coefficient cb_2 of δ_2 in the Theorem is positive. As D_s^r is effective, hence $c > 0$. Well done. \square

♣ **Sketch of Theorem B.**

Let $g = 2d - 2 = 2k$ and let $E_d^1 = a\lambda - b_0\delta_{irr} - \sum_{i=1}^{d-1} b_i\delta_i$. Since when $g \leq 2$ is trivial, we let $g \geq 4$.

• **Step A.** First show that $j^*E_d^1 \subset W$. By Theorem 17.2.4 we get

$$R_A : \quad a = 5b_1 - 2b_2 \text{ and } b_0 = \frac{b_1}{2} - \frac{b_2}{6}.$$

• **Step B.** Choose some kind of family of curves $C_j \subset \overline{M}_{0,g}$ and restricting to them. Then we get some relations:

$$R_{B0} : \quad b_j - 2b_{j+1} + b_{j+2} = -2b_1 + b_2 + 2 \frac{(2k-2)!}{k!(k-1)!} - \begin{cases} 2 \frac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k+l-1)!}, & j = 2l \text{ even;} \\ 0, & j \text{ odd.} \end{cases}$$

(Where b_{k+1} is interpreted as $= b_{k-1}$) These give the following formulas for b_2, \dots, b_k in terms of b_1 :

$$R_{B1} : \quad b_2 = \frac{4k-4}{2k-1}b_1 - 2 \frac{(2k-1)!}{(k-2)!(k+1)!};$$

for $3 \leq i \leq k$ we have

$$R_{B2} : \quad b_i = -i(i-2)b_1 + \frac{i(i-1)}{2}b_2 + (i-2)(i-1) \frac{(2k-2)!}{k!(k-1)!} \\ - \sum_{l=1}^{\lfloor (i-2)/2 \rfloor} 2(i-1-2l) \frac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k+l-1)!}.$$

Hence the relations R_A, R_{B_1}, R_{B_2} express all the coefficients in terms of b_1 . But since R_{B_1}, R_{B_2} are inhomogeneous, this is not enough to check the criterion given in the introduction.

• **Step C.** We consider another 1-dimensional family of curves and restrict E_d^1 into that family, then we can get

$$R_C : (4k-2)b_0 - b_1 = (4k+2) \frac{(2k-1)!}{(k-2)!(k+1)!}.$$

• **The final proof of the Theorem B.**

Proof of Theorem B. Combining R_A, R_{B_1}, R_C we can get

$$a = c(6k^2 + 13k + 1), b_0 = ck(k+1), b_1 = c(2k-1)(3k+1), b_2 = 3c(k-1)(4k+1)$$

where $c = 2 \frac{(2k-2)!}{(k+1)!(k-1)!}$. Moreover, by R_{B_0} we get

$$\begin{aligned} b_j - 2b_{j+1} + b_{j+2} &\leq -2b_1 + b_2 + 2 \frac{(2k-2)!}{k!(k-1)!} \\ &\leq -12k \frac{(2k-2)!}{(k+1)!(k-1)!} < 0, \end{aligned}$$

so the sequence of b_i is convex. But since $b_{k+1} = b_{k-1}$ and taking $j = k-1$ gives $b_k > b_{k-1}$, hence we get Theorem B. \square

Chapter 18

About Hassett-Keel program

We will work over algebraically closed field of characteristic zero.

18.1 A glimpse of Hassett-Keel program of \overline{M}_g

Here we follow the survey [37] and the introduction of the paper [60][61] and [4][2][3]. These are called Hassett-Keel program aiming to give modular interpretations of the log canonical models of \overline{M}_g , with the ultimate goal of giving a modular interpretation of the canonical model for the case $g \gg 0$.

By the Theorem of Mumford-Eisenbud-Harris, we get \overline{M}_g is of general type when $g \gg 0$. Fixed this kind of g . By the paper BCHM[18] we get the canonical ring

$$R(\overline{M}_g) = \bigoplus_{n \geq 0} H^0(\overline{M}_g, nK_{\overline{M}_g})$$

is finitely generated. Hence we can consider its canonical model, more generally, its log canonical models

$$\overline{M}_g(\alpha) = \text{Proj} \left(\bigoplus_{n \geq 0} H^0 \left(\overline{\mathcal{M}}_g, \left[n(K_{\overline{\mathcal{M}}_g} + \alpha\delta) \right] \right) \right)$$

for $\alpha \in \mathbb{Q} \cap [0, 1]$.

Before consider these, we first introduce some more stability of curves other than Deligne-Mumford's.

Definition 18.1.1 (Some types of elliptic chains). (i) *Elliptic tail* be a connected subcurve of genus 1 which meets the rest curves at only one node;

(ii) *Elliptic bridge* be a connected subcurve of genus 1 which meets the rest curves at only two nodes;

(iii) *Open elliptic chain of length l* is a 2-pointed subcurve (C', p, q) such that $C = E_1 \cup_{a_1} \cup \dots \cup_{a_{l-1}} E_l$ where E_i are connected genus one curves such that $E_i \cap E_{i+1}$ is a node and $E_i \cap E_j = \emptyset$ for $|i - j| > 1$ and $p \in E_1, q \in E_l$ are smooth points;

(iv) *Open tacnodal elliptic chain of length l* is a 2-pointed subcurve (C', p, q) such that $C = E_1 \cup_{a_1} \cup \dots \cup_{a_{l-1}} E_l$ where E_i are connected genus one curves with nodes, cusps, or tacnodes as singularities such that $E_i \cap E_{i+1}$ is a tacnode and $E_i \cap E_j = \emptyset$ for $|i - j| > 1$ and $p \in E_1, q \in E_l$ are smooth points with $\omega_{C'}(p + q)$ is ample.

Definition 18.1.2. Let C be a projective connected curve of arithmetic genus $g \geq 3$, with nodes, cusps, and tacnodes as singularities. We say C admits an open (tacnodal) elliptic chain if there is an open (tacnodal) elliptic chain (C', p, q) and a morphism $i : C' \rightarrow C$ such that

(i) i is an isomorphism over $C' \setminus \{p, q\}$ onto its image;

(ii) $i(p), i(q)$ are nodes of C ; we allow the case $i(p) = i(q)$, in which case C is said to be a closed (tacnodal) elliptic chain.

C admits a weak tacnodal elliptic chain if there exists $i : C' \rightarrow C$ as above with the second condition replaced by

(ii') $i(p)$ is a tacnode and $i(q)$ is a node of C ;
(ii'') $i(p) = i(q)$ is a tacnode of C , in which case C is said to be a closed weak tacnodal elliptic chain.

Definition 18.1.3 (pseudo-stable). A complete curve is pseudo-stable if

- (1) it is connected, reduced, and has only nodes and ordinary cusps as singularities;
- (2) admits no elliptic tails and its the canonical sheaf of the curve is ample.

Definition 18.1.4 (c-stable). A complete curve is c-semistable if

- (1) C has nodes, cusps, and tacnodes as singularities and ω_C is ample;
 - (2) a connected genus one subcurve meets the rest of the curve in at least two points (not counting multiplicity).
- It is said to be c-stable if it is c-semistable and has no tacnodes or elliptic bridges.

Remark 18.1.5. A curve is c-stable if and only if it is pseudo-stable and has no elliptic bridges.

Definition 18.1.6 (h-stable). A complete curve is h-semistable if it is c-semistable and admits no tacnodal elliptic chains. It is said to be h-stable if it is h-semistable and admits no weak tacnodal elliptic chains.

Actually in [60][61], Hassett and Hyeon showing that:

Theorem 18.1.7 (Hassett-Hyeon).

$$\overline{M}_g(\alpha) \cong \begin{cases} \overline{M}_g, & \text{if } \alpha \in (9/11, 1] \\ \overline{M}_g^{ps}, & \text{if } \alpha \in (7/10, 9/11] \\ \overline{M}_g^c, & \text{if } \alpha = 7/10 \\ \overline{M}_g^h, & \text{if } \alpha \in (2/3 - \varepsilon, 7/10) \end{cases}$$

where $\overline{M}_g^{ps}, \overline{M}_g^c, \overline{M}_g^h$ are the moduli spaces of pseudostable, c-semistable, and h-semistable curves, respectively.

In these works, new projective moduli spaces of curves are constructed using GIT. Actually one of the most appealing features of the Hassett-Keel program is the way that it ties together different compactifications of M_g obtained by varying the parameters in the GIT constructions of Mumford's.

In the series of papers [4][2][3], recall that in Part II, the theory of Deligne-Mumford stabilization has three steps:

- (a) Prove that the functor of stable curves is a proper Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}$;
- (b) Use the Keel-Mori theorem to show that $\overline{\mathcal{M}}_{g,n}$ has a coarse moduli space $\overline{M}_{g,n}$;
- (c) Find some line bundle on $\overline{\mathcal{M}}_{g,n}$ descends to an ample line bundle on $\overline{M}_{g,n}$.

In these papers, they proved a general existence theorem for good moduli spaces of non-separated algebraic stacks that can be viewed as a generalization of the Keel-Mori theorem. Hence we can run the modified version of the previous standard three-step procedure to construct moduli interpretations for the log canonical models:

$$\overline{M}_{g,n}(\alpha) = \text{Proj} \left(\bigoplus_{n \geq 0} H^0 \left(\overline{\mathcal{M}}_g, \left[n(K_{\overline{\mathcal{M}}_g} + \alpha\delta + (1-\alpha)\psi) \right] \right) \right)$$

Actually, for all $\alpha > 2/3 + \varepsilon$, where $0 < \varepsilon \ll 1$, we have

- (a) Construct an algebraic stack $\overline{\mathcal{M}}_{g,n}(\alpha)$ of α -stable curves (see [4]);
- (b) Construct a good moduli space $\overline{\mathcal{M}}_{g,n}(\alpha) \rightarrow \overline{M}_{g,n}(\alpha)$;
- (c) Show that $K_{\overline{\mathcal{M}}_g} + \alpha\delta + (1-\alpha)\psi$ on $\overline{\mathcal{M}}_{g,n}(\alpha)$ descends to an ample line bundle on $\overline{M}_{g,n}(\alpha)$,

and conclude that $\overline{M}_{g,n}(\alpha) \cong \overline{M}_{g,n}(\alpha)$.

Here in the intervals $(9/11, 1)$, $(7/10, 9/11)$, $(2/3, 7/10)$ and $(2/3 - \varepsilon, 2/3)$, the definition of α -stability does not change, hence so are $\overline{\mathcal{M}}_{g,n}(\alpha)$ and $\overline{M}_{g,n}(\alpha)$.

18.2 Log canonical models of Deligne-Mumford stacks

This section taken from the appendix of paper [60]. In this section, a scheme means a separated scheme of finite type over k and a stack means a separated Deligne-Mumford stack of finite type over k .

Definition 18.2.1. A birational morphism of stacks is a morphism $f : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ such that there exist dense open substacks $\mathcal{U}_i \subset \mathcal{X}_i$ with $\mathcal{U}_2 = f^{-1}(\mathcal{U}_1)$ and $f : \mathcal{U}_1 \rightarrow \mathcal{U}_2$ an isomorphism.

There is a largest open substack $\mathcal{U} \subset \mathcal{X}_1$ such that f an isomorphism, and we let the complement is exceptional locus $\text{Exc}(f)$. For any closed substack $\mathcal{D} \subset \mathcal{X}_2$ such that $f(\mathcal{U}) \cap \mathcal{D}$ dense in \mathcal{D} , the birational transform $f_*^{-1}\mathcal{D}$ is the closure of $f^{-1}(f(\mathcal{U}) \cap \mathcal{D})$ in \mathcal{X}_1 .

We say that \mathcal{X}_1 and \mathcal{X}_2 are properly birational if there exists a stack \mathcal{Y} and birational proper morphisms $g_i : \mathcal{Y} \rightarrow \mathcal{X}_i$.

Theorem 18.2.2 (Resolution of singularities). For a reduced stack \mathcal{X} , there exists a smooth stack \mathcal{X}' and a birational proper map $f : \mathcal{X}' \rightarrow \mathcal{X}$ such that

- (i) locus $\text{Exc}(f)$ can be taken to be a normal crossings divisor;
- (ii) If $\mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed substack with ideal sheaf $\mathcal{I}_{\mathcal{Z}} \subset \mathcal{O}_{\mathcal{X}}$, then there exists an divisor $\mathcal{D}' \subset \mathcal{X}'$ such that $f^*\mathcal{I}_{\mathcal{Z}} \cong \mathcal{O}_{\mathcal{X}'}(-\mathcal{D}')$ and $\text{Exc}(f) \cup \mathcal{D}'$ is simple normal crossings.

Proof. This need the functorial resolving singularities and such procedures commute with étale maps (see Page 329 in [59]), here we give an idea. One have an étale presentation $R \rightrightarrows U$. After resolving R, U to be R', U' , we can get $R' \rightrightarrows U'$ for a smooth stack \mathcal{X}' . \square

Proposition 18.2.3. Let X be a connected normal separated scheme of finite type over k . Let $D = \sum_j a_j D_j$ be a \mathbb{Q} -divisor on X , with the D_j distinct and reduced and $a_j \in [0, 1]$. Then the following properties of pair (X, D) are local on the étale topology:

- (i) X is normal;
- (ii) D_j are codimension one reduced closed subschemes;
- (iii) for some $m > 0$ the divisor $m(K_X + D)$ is Cartier.

We will call (X, D) admissible if it satisfying (i)(ii)(iii).

Pick a coherent sheaf \mathcal{F} over X , then following properties are local on the étale topology:

- (a) \mathcal{F} is locally free;
- (b) \mathcal{F} with S_k condition for $k > 0$.

If X integral with a integral Weil divisor, then $\mathcal{O}_X(D)$ be a reflexive S_2 of rank 1 and the formation of $\mathcal{O}_X(D)$ commutes with étale maps.

Let (X, D) admissible, then being terminal, canonical, klt, plt, lc are local on the étale topology.

Proof. These are the results of descent theory, we refer St 0238 and [66] 5.20. \square

Definition 18.2.4. Let $(\mathcal{X}, \mathcal{D})$ is proper and admissible, then define its log canonical ring is

$$R(\mathcal{X}, \mathcal{D}) = \bigoplus_{m \geq 0} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(mK_{\mathcal{X}} + [m\mathcal{D}])).$$

We define $(\mathcal{X}, \mathcal{D})$ is terminal, canonical, klt, plt, lc if it admits an étale presentation with this property. We say it is strictly lc if it is lc and $\mathcal{X} \setminus \bigcup_j \mathcal{D}_j$ is canonical.

Similar as the normal birational geometry as in [66], we have:

Proposition 18.2.5. The admissible pair $(\mathcal{X}, \mathcal{D})$ is terminal, canonical, klt, and lc if and only if there exists a log resolution $f : \mathcal{X}' \rightarrow \mathcal{X}$ such that

- (i) $\text{Exc}(f) = \bigcup \mathcal{E}_j$ is a divisor, $\text{Exc}(f) \cup f^{-1}\mathcal{D}$ is simple normal crossings and $\sum_j f_*^{-1}\mathcal{D}_j$ is smooth;
- (ii) we have

$$K_{\mathcal{X}'} + \sum_j a_j f_*^{-1}\mathcal{D}_j \sim_{\mathbb{Q}} f^*(K_{\mathcal{X}} + \mathcal{D}) + \sum_i d_i \mathcal{E}_i$$

such that $(a_j < 1$ and $d_i > 0)$, $(a_j \leq 1$ and $d_i \geq 0)$, $(a_j < 1$ and $d_i > -1)$ and $(a_j \leq 1$ and $d_i \geq -1)$.

Proof. See [66] Corollary 2.32. \square

Remark 18.2.6. These $d_i =: d(\mathcal{E}_i; \mathcal{X}, \mathcal{D})$ is called discrepancy if replaced by any proper birational $\mathcal{Y} \rightarrow \mathcal{X}$. We will define $d(\mathcal{D}_j; \mathcal{X}, \mathcal{D}) = -a_j$ and $d(\mathcal{D}_0; \mathcal{X}, \mathcal{D}) = 0$ for divisors in \mathcal{X} and not in \mathcal{D} .

Definition 18.2.7. Two admissible pairs $(\mathcal{X}, \mathcal{D})$ and $(\mathcal{X}', \mathcal{D}')$ are properly birational if there exists an admissible pair $(\mathcal{Y}, \mathcal{B})$ and proper birational morphisms $f : \mathcal{Y} \rightarrow \mathcal{X}$ and $f' : \mathcal{Y} \rightarrow \mathcal{X}'$ such that $K_{\mathcal{Y}} + \mathcal{B} - f^*(K_{\mathcal{X}} + \mathcal{D})$ is f -exceptional and effective, similar as $K_{\mathcal{X}'} + \mathcal{D}'$.

Proposition 18.2.8. If two admissible pairs $(\mathcal{X}, \mathcal{D})$ and $(\mathcal{X}', \mathcal{D}')$ are properly birational, then the natural morphism is an isomorphism of graded rings:

$$R(\mathcal{X}, \mathcal{D}) \cong R(\mathcal{X}', \mathcal{D}').$$

Proof. This is directly, see [60] Proposition A.12. \square

Proposition 18.2.9. Let $(\mathcal{X}, \mathcal{D})$ be a proper lc (or klt) pair with coarse moduli space $\pi : \mathcal{X} \rightarrow X$. Then there exists an effective \mathbb{Q} -divisor $D' = \sum_{\ell} c_{\ell} D'_{\ell}$, $0 \leq c_{\ell} \leq 1$ of X such that

(i) pair (X, D') is lc (or klt);

(ii) for each $m > 0$ such that $m(K_X + D')$ is integral and Cartier, we have

$$m(K_{\mathcal{X}} + \mathcal{D}) = \pi^*m(K_X + D');$$

(iii) for any $m \geq 0$ we have

$$\Gamma(X, mK_X + \lfloor mD' \rfloor) \cong \Gamma(\mathcal{X}, mK_{\mathcal{X}} + \lfloor m\mathcal{D} \rfloor).$$

Together these we have

$$\pi^* : R(X, D') \cong R(\mathcal{X}, \mathcal{D}).$$

Proof. Omitted, see [60] Proposition A.13. \square

Corollary 18.2.10 (Basepoint-freeness for stacks). Let $(\mathcal{X}, \mathcal{D})$ be a proper klt pair and $K_{\mathcal{X}} + \mathcal{D}$ is nef and big (i.e., $K_X + D'$ is nef and big). Consider the stack and coarse moduli space

$$\mathcal{Y} := [(\mathrm{Spec}R(\mathcal{X}, \mathcal{D}) \setminus \{0\})/\mathbb{G}_m], Y := \mathrm{Proj}R(\mathcal{X}, \mathcal{D}) = \mathrm{Proj}R(X, D'),$$

where the action of \mathbb{G}_m arises from the grading, then there is a morphism of stacks $\psi : \mathcal{Y} \rightarrow \mathcal{X}$ inducing on coarse moduli spaces the contraction from X to the log canonical model of (X, D') .

Proof. Use [66] Theorem 3.3. \square

18.3 The first result for $9/11 < \alpha \leq 1$

Here we just consider $g \geq 4$. First we need to point out the following well-known result:

Lemma 18.3.1. For $g \geq 4$ we consider the canonical map $f : \overline{\mathcal{M}}_g \rightarrow \overline{M}_g$, then we have

$$f^* \left(K_{\overline{M}_g} + \alpha(\Delta_{irr} + \Delta_2 + \cdots + \Delta_{\lfloor g/2 \rfloor}) + \frac{1+\alpha}{2}\Delta_1 \right) = K_{\overline{\mathcal{M}}_g} + \alpha\delta.$$

Proof. Similar as Corollary 16.7.3, the only divisor components of the locus parameterizing curves with nontrivial automorphism group is Δ_1 . Hence by Riemann-Hurwitz theorem we have

$$\begin{aligned} & f^* \left(K_{\overline{M}_g} + \alpha(\Delta_{irr} + \Delta_2 + \cdots + \Delta_{\lfloor g/2 \rfloor}) + \frac{1+\alpha}{2}\Delta_1 \right) \\ &= K_{\overline{\mathcal{M}}_g} - \delta_1 + \alpha(\delta_{irr} + \delta_2 + \cdots + \delta_{\lfloor g/2 \rfloor}) + \frac{1+\alpha}{2}f^*\Delta_1 \\ &= K_{\overline{\mathcal{M}}_g} - \delta_1 + \alpha(\delta_{irr} + \delta_2 + \cdots + \delta_{\lfloor g/2 \rfloor}) + (1+\alpha)\delta_1 = K_{\overline{\mathcal{M}}_g} + \alpha\delta, \end{aligned}$$

so we get the conclusion. \square

Fact. By Lemma 18.3.1, the universal property of the coarse moduli space implies that sections of invertible sheaves on $\overline{\mathcal{M}}_g$ are all pullbacks of sections of the corresponding reflexive sheaves on \overline{M}_g . Hence by Proposition 18.2.9 we get the log canonical model of $\overline{\mathcal{M}}_g$ with respect to $K_{\overline{\mathcal{M}}_g} + \alpha\delta$ can thus be identified with the log canonical model of \overline{M}_g with respect to (we will use both of them)

$$K(\overline{M}_g; \Delta, \alpha) := K_{\overline{M}_g} + \alpha(\Delta_{irr} + \Delta_2 + \cdots + \Delta_{\lfloor g/2 \rfloor}) + \frac{1+\alpha}{2}\Delta_1.$$

Theorem 18.3.2 (Mumford-Cornalba-Harris). *If $9/11 < \alpha \leq 1$, then $\overline{M}_g(\alpha) \cong \overline{M}_g$.*

Proof. By Remark 16.7.4 and Theorem 16.8.1, we know that the divisor

$$\begin{aligned} a\lambda - \Delta + \frac{1}{2}\Delta_1 &= \frac{a}{13} \left(K_{\overline{M}_g} + \left(2 - \frac{13}{a}\right) (\Delta_{irr} + \Delta_2 + \cdots) + \left(\frac{3}{2} - \frac{13}{2a}\right) \Delta_1 \right) \\ &= \frac{a}{13} K \left(\overline{M}_g; \Delta, \left(2 - \frac{13}{a}\right) \right) \end{aligned}$$

is ample if and only if $a > 11$. Hence $K(\overline{M}_g; \Delta, \alpha)$ is ample if and only if $9/11 < \alpha \leq 1$. As \overline{M}_g is proper, we get $\overline{M}_g(\alpha) \cong \overline{M}_g$ by [47] Proposition 13.48. \square

Remark 18.3.3. *As $(\overline{\mathcal{M}}_g, \alpha\delta)$ is lc and proper, the log canonical model (in sense of [66] Definition 3.50) is unique and is \overline{M}_g here by [66] Theorem 3.52.*

18.4 The main results for $7/10 < \alpha \leq 9/11$

We will focus on the paper [60] and work out this paper. Here we just consider $g \geq 4$.

When $\alpha = 9/11$, the divisor $K(\overline{M}_g; \Delta, \alpha)$ is not ample and pair $(\overline{\mathcal{M}}_g, \alpha\delta)$ is a klt pair. Hence using the log MMP argument, we need to find some extremal ray to obtain a contraction!

By Theorem 16.8.1, when $\alpha = 9/11$, then $K(\overline{M}_g; \Delta, \alpha) = 11\lambda - \delta$ is nef and big (when $g \geq 22$ as then it's log general type). By the base-point free theorem we get it is semi-ample.

Fix a (C_2, p) in $\overline{M}_{g-1,1}$ and consider

$$\overline{M}_{1,1} \times \overline{M}_{g-1,1} \rightarrow \overline{M}_g.$$

Consider $C = C_1 \cap_p C_2$ be curves with elliptic tails and (C_1, p) in $\overline{M}_{1,1} \cong \mathbb{P}^1$. Hence these curves parameterizing by a rational curve $R(C_2, p) \subset \overline{M}_g$.

Lemma 18.4.1. *In \overline{M}_g , the class $R = [R(C_2, p)]$ is independent of (C_2, p) and we have $\lambda \cdot R = 1/12$, $\delta_{irr} \cdot R = 1$, $\delta_1 \cdot R = -1/12$ and $\delta_i \cdot R = 0$ for all $i \geq 2$. In particular we have $(K_{\overline{\mathcal{M}}_g} + 9/11\delta) \cdot R = 0$.*

Proof. \square

In fact, in [46] (Proposition 6.4) they find that:

Proposition 18.4.2. *When $g \geq 5$, the only divisorial contraction $\gamma : \overline{M}_g \rightarrow X$ with $\rho(\overline{M}_g/X) = 1$ with X projective is the blowdown of the elliptic tails, contraction of R .*

Now what is X ? Actually the construction of γ as follows. As $D := K(\overline{M}_g; \Delta, \alpha)$ is semi-ample, replacing a multiple we get a morphism $f : \overline{M}_g \rightarrow P := \mathbb{P}(\Gamma(\overline{M}_g, D))$. By taking Stein factorization, we get

$$\overline{M}_g \rightarrow \underline{\text{Spec}}_P f_* \mathcal{O}_{\overline{M}_g} \rightarrow P = \mathbb{P}(\Gamma(\overline{M}_g, D)).$$

The morphism $\overline{M}_g \rightarrow \underline{\text{Spec}}_P f_* \mathcal{O}_{\overline{M}_g}$ is the contraction map. We claim that

$$\underline{\text{Spec}}_P f_* \mathcal{O}_{\overline{M}_g} \cong \text{Proj} \bigoplus_n \Gamma(\overline{M}_g, nD).$$

First we need to find a canonical morphism $r : \overline{M}_g \rightarrow \text{Proj} \bigoplus_n \Gamma(\overline{M}_g, nD)$ and analyze its properties. This is a standard scheme theory. For simplicity, we denote $A_d = \Gamma(\overline{M}_g, dD)$ and $A = \bigoplus_n \Gamma(\overline{M}_g, nD)$. Then the map $\mathcal{O}_{\overline{M}_g} A \rightarrow \bigoplus_d \mathcal{O}(dD) = \text{Sym}(\mathcal{O}(D))$ induce the morphism

$$r : \overline{M}_g \cong \text{ProjSym}(\mathcal{O}(D)) \rightarrow \underline{\text{Proj}} \mathcal{O}_{\overline{M}_g} A = \text{Proj} A \times \overline{M}_g \rightarrow \text{Proj} A$$

which induce $r^{-1}(D_+(f)) = (\overline{M}_g)_f$ and the restriction map $r_f : (\overline{M}_g)_f \rightarrow D_+(f)$ yield $A_{(f)} \cong \Gamma((\overline{M}_g)_f, \mathcal{O}_{\overline{M}_g})$ where $f \in A_+$ and $(\overline{M}_g)_f$ means the non-vanishing locus of f (for more details of the general case of this, we refer the previous part of the Proposition 13.48 in[47]).

Next, we have the canonical map $\pi : \text{Proj} \bigoplus_n \Gamma(\overline{M}_g, nD) \rightarrow \mathbb{P}(\Gamma(\overline{M}_g, D))$ induced by surjection $\text{Sym} \Gamma(\overline{M}_g, D) \rightarrow \bigoplus_n \Gamma(\overline{M}_g, nD)$. Easy to see that we have the following commutative diagram:

$$\begin{array}{ccc} \text{Spec}_P f_* \mathcal{O}_{\overline{M}_g} & \xleftarrow{g} \overline{M}_g & \xrightarrow{r} \text{Proj} \bigoplus_n \Gamma(\overline{M}_g, nD) \\ & \searrow h & \swarrow \pi \\ & & P = \mathbb{P}(\Gamma(\overline{M}_g, D)) \end{array}$$

Let f induced by the sections f_i , then pick the standard coordinates x_i of P we have $h^{-1}(D'_+(x_i)) = D_+(f_i) = \text{Spec} A_{(f_i)} \cong \text{Spec} \Gamma((\overline{M}_g)_{f_i}, \mathcal{O}_{\overline{M}_g})$. One the other hand, since $f^{-1}(D'_+(x_i)) = (\overline{M}_g)_{f_i}$, we also have $h^{-1}(D'_+(x_i)) = \text{Spec} \Gamma((\overline{M}_g)_{f_i}, \mathcal{O}_{\overline{M}_g})$. As the intersection parts are automatically coincident, we get the claim. Hence we can restate the result:

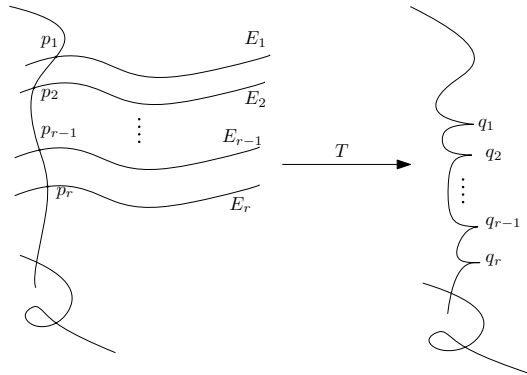
Proposition 18.4.3. *When $g \geq 5$, the only divisorial contraction of \overline{M}_g is $\gamma : \overline{M}_g \rightarrow \overline{M}_g(9/11)$ with $\overline{M}_g(9/11)$ projective is the blowdown of the elliptic tails Δ_1 .*

As γ is a divisorial contraction and \overline{M}_g is a quotient of smooth variety by a finite group (by E. Looijenda, see [71]), the space $\overline{M}_g(9/11)$ is \mathbb{Q} -factorial. Then we may ask that is $\overline{M}_g(9/11)$ the coarse moduli space of moduli stack of some kind of curves which is the compactification of moduli space of smooth curves? Actually this is one of our main theorems. Here we state the main theorems we will prove.

Theorem 18.4.4. *Let $\overline{\mathcal{M}}_g^{\text{ps}}$ be the stack of pseudostable curves. Then there is a morphism of stacks*

$$T : \overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{M}}_g^{\text{ps}}$$

which is an isomorphism in the complement of δ_1 . And for a stable curve $C \in \delta_1(\text{Speck})$, the curve $T(\text{Speck})(C)$ is obtained by replacing each elliptic tail of C with a cusp:



The coarse moduli space $\overline{M}_g^{\text{ps}} \cong \overline{M}_g(9/11)$ and the induced map

$$T : \overline{M}_g \rightarrow \overline{M}_g^{\text{ps}}$$

coincides with the extremal contraction $\gamma : \overline{M}_g \rightarrow \overline{M}_g(9/11)$.

Theorem 18.4.5. For $7/10 < \alpha \leq 9/11$, then $\overline{M}_g(\alpha) \cong \overline{M}_g^{ps}$ as projective varieties.

18.4.1 Results about the moduli stack of pseudostable curves

Define the moduli stack of pseudostable curves $\overline{\mathcal{M}}_g^{ps}$ as

$$\overline{\mathcal{M}}_g^{ps}(S) := \left\{ f : \mathcal{C} \rightarrow S \mid \begin{array}{l} f \text{ is proper and flat and the geometric fibers} \\ \text{of } f \text{ are pseudostable of genus } g. \end{array} \right\}.$$

The main reference we use about the moduli stack of pseudostable curves is paper [76].

Theorem 18.4.6 (D. Schubert, 1991). *The stack $\overline{\mathcal{M}}_g^{ps}$ is a separated Deligne-Mumford stack of finite type over k . By Keel-Mori's theorem, they have a coarse moduli space \overline{M}_g^{ps} . Actually we have $\overline{\mathcal{M}}_g^{ps} \cong [\mathbf{chow}_3^s / \mathrm{PGL}_{5g-5}]$ and $\overline{M}_g^{ps} \cong \mathbf{chow}_3 // \mathrm{SL}_{5g-5}$ where \mathbf{chow}_3 be the Chow variety and \mathbf{chow}_3^s be the GIT-stable locus.*

Remark 18.4.7. We may use the Hilbert scheme instead of Chow variety, we refer [61].

18.4.2 Constructing the morphism of moduli stacks

Lemma 18.4.8 (Single case). *Let C be a genus $g > 2$ stable curve of with elliptic tails E_1, \dots, E_r and let D be the union of the components of C p_i the node where E_i meets D , away from the elliptic tails an. Then there exists a unique curve $\mathcal{T}(C)$ characterized by the following properties:*

- (a) *there is a birational morphism $\nu : D \rightarrow \mathcal{T}(C)$, which is an isomorphism away from p_1, \dots, p_r ;*
- (b) *ν is bijective and maps each $p_i \in D$ to a cusp $q_i \in \mathcal{T}(C)$.*

There is a unique replacement morphism $\xi_C : C \rightarrow \mathcal{T}(C)$ with $\xi_C|_D = \nu$ and $\xi_C|_{E_i}$ are constant. Note that $\mathcal{T}(C)$ has arithmetic genus g .

Proof. It suffices to determine the subrings $\mathcal{O}_{\mathcal{T}(C), q_i} \subset \mathcal{O}_{D, p_i}$ (some kind of anti-normalization). Let \mathfrak{m}_{D, p_i} be the maximal ideal and let $\mathcal{O}_{\mathcal{T}(C), q_i}$ generated by the constants and \mathfrak{m}_{D, p_i}^2 as an algebra. Then the maximal ideal of it generated by two elements x, y such that $x^2 = y^3$ (let t be the uniformizer, then \mathfrak{m}_{D, p_i}^2 generated by t^2, t^3). Conversely, any germ of a cuspidal curve normalized by (D, p_i) is obtained in this way. Hence ν is the normalization of the cusps q_1, \dots, q_r . \square

(I) Reduce to the concrete case

Now we need to construct the (1-)morphism of stacks

$$\mathcal{T} : \overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{M}}_g^{ps}$$

assigns to each stable curve $f : \mathcal{C} \rightarrow S$ a pseudostable curve $\mathcal{T}(f) : \mathcal{T}(C) \rightarrow S$.

The main tool is the (family-case) replacement S -morphism $\xi_C : \mathcal{C} \rightarrow \mathcal{T}(C)$ satisfies the following properties:

- (a) Over the open subset $S_0 \subset S$ mapping to the complement of δ_1 , the $\xi_C|_{S_0}$ is an isomorphism;
- (b) for the $s \in S$ mapping to δ_1 , the morphism ξ_{C_s} is the morphism as Lemma 18.4.8;
- (c) ξ_C is compatible with base change and isomorphisms.

Pick a étale cover U of $\overline{\mathcal{M}}_g$ (as it is a Deligne-Mumford stack) with representation $R \rightrightarrows U$ and the universal family $\pi : Z \rightarrow U$ where R encodes the isomorphisms among the fibers of π .

Proposition 18.4.9. *To construct \mathcal{T} and ξ over $\overline{\mathcal{M}}_g$, it suffices to construct $\mathcal{T}(\pi) : \mathcal{T}(Z) \rightarrow U$ and ξ_Z compatibly with the isomorphism relation R .*

Proof. For any family of stable curves $f : \mathcal{C} \rightarrow S$ correspond to $\rho_f : S \rightarrow \overline{\mathcal{M}}_g$, consider the cartesian of stacks:

$$\begin{array}{ccc} S' := S \times_{\overline{\mathcal{M}}_g} U & \xrightarrow{\mathrm{pr}_U} & U \\ \mathrm{pr}_S \downarrow & \lrcorner & \downarrow \rho_\pi \\ S & \xrightarrow{\rho_f} & \overline{\mathcal{M}}_g \end{array}$$

and we get the stable curves

$$\begin{array}{ccc} \mathrm{pr}_S^* \mathcal{C} & \xrightarrow{\cong} & \mathrm{pr}_U^* Z \\ & \searrow \mathrm{pr}_S^* f & \swarrow \mathrm{pr}_U^* \pi \\ & & S' \end{array}$$

After base change, we get $\mathrm{pr}_U^* \mathcal{T}(\pi) : \mathrm{pr}_U^* \mathcal{T}(Z) \rightarrow S'$ and $\mathrm{pr}_U^* \xi_Z : \mathrm{pr}_U^* Z \rightarrow \mathrm{pr}_U^* \mathcal{T}(Z)$. Hence they correspond to $\mathrm{pr}_S^* f : \mathrm{pr}_S^* \mathcal{C} \rightarrow S'$. As pr_S is the base change of ρ_π , it is étale, hence faithfully flat. As \mathcal{T} is compatible with isomorphisms, by the descent theory we get

$$\mathcal{T}(f) : \mathcal{T}(\mathcal{C}) \rightarrow S, \quad \xi_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{T}(\mathcal{C})$$

as desired. \square

(II) Setup step. Find a appropriate line bundle

Consider

$$\begin{array}{ccc} Z & \xrightarrow[\Gamma]{\mu_\pi} & \mathcal{U}_g = \overline{\mathcal{M}}_{g,1} \supset \delta_{1,\{1\}} \\ \pi \downarrow & & \mathrm{univ} \downarrow \\ U & \xrightarrow{\rho_\pi} & \overline{\mathcal{M}}_g \end{array}$$

with elliptic tail $\delta_{1,\{1\}}$. Let $E := \mu_\pi^* \delta_{1,\{1\}}$ (is Cartier, omitted) and $W = \rho_\pi^* \delta_{1,\{1\}}$ and consider $L := \omega_\pi \otimes \mathcal{O}(E) = \omega_\pi(E)$. Like Lemma 2.1.1, we find that over $U \setminus W$ we have

$$k_n := \mathrm{rank} \pi_* L^{\otimes n} = \begin{cases} g & n = 1, \\ (2n-1)(g-1) & n \geq 2. \end{cases}, \quad R^1 \pi_* L^{\otimes n} \cong \begin{cases} \mathcal{O}_U & n = 1, \\ 0 & n \geq 2. \end{cases}$$

Hence we need to discover that happens near W as we will use $\pi_* L^{\otimes n}$ to define our maps.

(III) Locally freeness I. Some cohomology

Note that the locally freeness and cohomology can descend along the faithfully flat extension. Fix $u_0 \in W$, then $Z_{u_0} := C$ has r elliptic tails E_i and the remaining component we denote D . After some faithfully flat extension, we may assume that π has sections $s_1, \dots, s_r : U \rightarrow Z$ such that $s_i(u_0) \in E_i$ are smooth points (Why?).

Now we have

$$0 \rightarrow L^{\otimes n} \rightarrow L^{\otimes n}(s_1 + \dots + s_r) \rightarrow L^{\otimes n}(s_1 + \dots + s_r)|_{\{s_1, \dots, s_r\}} \rightarrow 0$$

Over this fiber C , we have

$$0 \rightarrow L^{\otimes n}|_C \rightarrow L^{\otimes n}(s_1 + \dots + s_r)|_C \rightarrow L^{\otimes n}(s_1 + \dots + s_r)|_{\{s_1(u_0), \dots, s_r(u_0)\}} \rightarrow 0$$

and the last term support over a finite sets $\{s_1(u_0), \dots, s_r(u_0)\}$. On the other hand, we have (Why?)

$$\begin{aligned} & H^1(C, L^{\otimes n}(s_1 + \dots + s_r)|_C) \\ &= H^1(D, L^{\otimes n}|_D) \oplus \left(\bigoplus_{i=1}^r H^1(E_i, \mathcal{O}_{E_i}(s_i(u_0))) \right) = 0. \end{aligned}$$

Then we get an exact sequence

$$0 \rightarrow \pi_* L^{\otimes n} \rightarrow F^0 \rightarrow F^1 \rightarrow R^1 \pi_* L^{\otimes n} \rightarrow 0$$

with $F^0 := \pi_*(L^{\otimes n}(s_1 + \dots + s_r))$ and $F^1 := \pi_*(L^{\otimes n}(s_1 + \dots + s_r)|_{s_1 + \dots + s_r})$ locally free of rank r_0, r_1 . Let $Q = \mathrm{coker}(\pi_* L^{\otimes n} \rightarrow F^0)$ is a subsheaf of a locally-free sheaf, hence is locally-free of rank $r_0 - k_n$ away from a subset $Y \subset U$ of codimension ≥ 2 . Note that Y is contained in the locus where $R^1 \pi_* L^{\otimes n}$ fails to be locally free, and thus is a subset of W .

(IV) Locally freeness II. Some limit linear series

Let B be a spectrum of a DVR with closed point 0 and generic point b . Consider the map $\beta : (B, 0) \rightarrow (U, u_0)$ such that $0 \mapsto u_0$ and $b \notin W$. Consider the fiber product

$$\begin{array}{ccc} Z_B & \xrightarrow{\beta'} & Z \\ \pi_B \downarrow & \ulcorner & \downarrow \pi \\ B & \xrightarrow{\beta} & U \end{array}$$

and let $L_B = (\beta')^*L$. As $\pi_{B,*}L_B^{\otimes n}$ is locally free of rank k_n as it is a subsheaf of β^*F_0 and thus torsion free and then flat (over Dedekind domain).

We need the following result:

Proposition 18.4.10 ([26] 4.3iii). *For $n \geq 1$, let $V_n := \pi_{B,*}L_B^{\otimes n}|_0$ can be naturally identified with*

$$\Gamma \left(D, \omega_D^{\otimes n} \left((2n-2) \sum_{j=1}^r p_j \right) \right) + \text{span}(\sigma_1^n, \dots, \sigma_r^n)$$

as a subspace of $H^0(C, L^{\otimes n}|_C)$ where σ_j be a section of V_1 such that $\sigma_i(p_i) \neq 0$ and σ_j vanishes to order two at p_i for $i \neq j$.

Corollary 18.4.11. *For $n \geq 2$, the linear series V_n defines $C \rightarrow \mathbb{P}^{k_n-1}$ with image $\mathcal{T}(C)$. The induced $C \rightarrow \mathcal{T}(C)$ is the morphism in Lemma 18.4.8.*

Proof. For stable curve (D, p_1, \dots, p_r) , unless $g(D) = 2, r = 0$ (which can not occur here) the line bundle $(\omega_D(p_1 + \dots + p_r))^2$ is very ample. Hence V_n can induce an embedding of $D \setminus \{p_1, \dots, p_r\}$ into the projective space. By Proposition 18.4.10 we get that the vanishing sequence of V_n near p_j are all $(0, 2, 3, \dots)$ and then induce p_j into cusps. As p_j separated by σ_j^n , these cusps are different ones. As these sections are constant ($\neq 0$) along the E_i , so each E_i is collapsed to the corresponding cusp. \square

(V) Locally freeness III. Finish the locally freeness

Proposition 18.4.12. *For integer $n \geq 1$, the sheaf $\pi_*L^{\otimes n}$ is locally free of rank k_n .*

Proof. We denote $\underline{\text{Grass}}(m, F)$ be the geometric Grassmannian represented the functor of rank m subbundles of F . Recall the exact sequence

$$0 \rightarrow \pi_*L^{\otimes n} \rightarrow F^0 \xrightarrow{\varphi} F^1 \rightarrow R^1\pi_*L^{\otimes n} \rightarrow 0$$

with F^0 and F^1 locally free of rank r_0, r_1 . Let $Q = \text{coker}(\pi_*L^{\otimes n} \rightarrow F^0)$ is locally-free of rank $r_0 - k_n$ away from a subset $Y \subset U$ of codimension ≥ 2 .

Then we induce the morphism

$$\begin{aligned} \tau : U \setminus Y &\rightarrow \underline{\text{Grass}}(r_0 - k_n, (F^0)^\vee) \times \underline{\text{Grass}}(r_0 - k_n, F^1) \\ &\hookrightarrow \mathbb{P} \left(\bigwedge^{r_0 - k_n} (F^0)^\vee \right) \times \mathbb{P} \left(\bigwedge^{r_0 - k_n} F^1 \right) \\ &\hookrightarrow \mathbb{P} \left(\mathcal{H}om \left(\bigwedge^{r_0 - k_n} F^0, \bigwedge^{r_0 - k_n} F^1 \right) \right). \end{aligned}$$

Consider as a rational map

$$U \dashrightarrow \mathbb{P} \left(\mathcal{H}om \left(\bigwedge^{r_0 - k_n} F^0, \bigwedge^{r_0 - k_n} F^1 \right) \right),$$

it is given by the section $\bigwedge^{r_0-k_n} \varphi$ and the indeterminacy locus of it is precisely the zero locus of $\bigwedge^{r_0-k_n} \varphi$ defined by $(r_0-k_n) \times (r_0-k_n)$ -minors of φ (which is one of the Fitting ideals \mathcal{I} of $R^1\pi_*L^{\otimes n}$). Blowing up the base locus as follows:

$$\begin{array}{ccc} B & \xrightarrow{\beta'_i} & U' := \text{Bl}_{\mathcal{I}}(U) \\ & \searrow \beta_i & \downarrow \sigma \\ & & U \end{array} \quad \begin{array}{c} \xrightarrow{\tau'} \\ \dashrightarrow \tau \end{array} \quad \begin{array}{c} \\ \\ \text{Grass}(r_0 - k_n, (F^0)^\vee) \times \text{Grass}(r_0 - k_n, F^1) \end{array}$$

Now we claim that σ is an isomorphism.

As U normal (it is étale over smooth stack $\overline{\mathcal{M}}_g$) and σ is proper birational, using the Zariski's main theorem, we just need to show that σ is quasi-finite. Let it is not quasi-finite and for two different closed points $X_1, x_2 \in U'$ such that τ is not defined at $u := \sigma(x_1) = \sigma(x_2)$. Pick a DVR $B = \text{Spec} \Delta$ with closed point 0 and generic point η . Then let $\beta'_i : B \rightarrow U'$ such that $0 \mapsto x_i$ and η is not in the locus of δ_1 . Let $\beta_i = \sigma \circ \beta'_i$ and using the functorial, we find that $\tau' \circ \beta'_i$ correspond to

$$0 \rightarrow K_i \rightarrow \beta_i^* F^0 \rightarrow Q_i \rightarrow 0,$$

where K_i is of rank k_n . Consider the fiber product

$$\begin{array}{ccc} Z_B & \xrightarrow{\mu_i} & Z \\ \pi_i \downarrow & \ulcorner & \downarrow \pi \\ B & \xrightarrow{\beta_i} & U \end{array}$$

then we get the exact sequence

$$0 \rightarrow \pi_{i,*} \mu_i^* L^{\otimes n} \rightarrow \beta_i^* F^0 \xrightarrow{\beta_i^* \varphi} \beta_i^* F^1 \rightarrow R^1(\pi_i)_* \mu_i^* L^{\otimes n} \rightarrow 0.$$

Note that $\beta_i^* F^0 / \pi_{i,*} \mu_i^* L^{\otimes n}$ is locally free and $\pi_{i,*} \mu_i^* L^{\otimes n}$ and K_i are agree over $B \setminus \{0\}$ as subbundles of $\beta_i^* F^0$. Hence they are isomorphic over entire B . By Proposition 18.4.10, $(\pi_{i,*} \mu_i^* L^{\otimes n})_0$ are identified in $H^0(Z|_{\beta_i(0)}, L^{\otimes n}|_{Z|_{\beta_i(0)}})$, then $\tau' \circ \beta'_1(0) = \tau' \circ \beta'_2(0)$ and hence there exists a subbundle $K \subset F^0$ over U such that $K|_{U \setminus Y} \cong \pi_* L^{\otimes n}|_{U \setminus Y}$. Let $j : U \setminus Y \hookrightarrow U$, then $j_*(\pi_* L^{\otimes n}|_{U \setminus Y}) \cong \pi_* L^{\otimes n}$ as it is reflexive and $\text{codim}_U(Y) \geq 2$. Hence we have

$$K \cong j_*(K|_{U \setminus Y}) = (\pi_* L^{\otimes n}|_{U \setminus Y}) \cong \pi_* L^{\otimes n}$$

hence a vector bundle. \square

(VI) Finish the construction

Proposition 18.4.13. *For $n \geq 2$, the sections of $L^{\otimes n}$ relative to π induced $Z \rightarrow \mathbb{P}(\pi_* L^{\otimes n})$ over U which factors as*

$$Z \xrightarrow{\xi_Z} \mathcal{T}(Z) \hookrightarrow \mathbb{P}(\pi_* L^{\otimes n}).$$

Proof. Now by Proposition 18.4.12, $\pi_* L^{\otimes n}$ is locally free. By the argument in Proposition 18.4.10 and Corollary 18.4.11, using the Stein factorization we get

$$Z \xrightarrow{\xi_Z} \mathcal{T}(Z) \hookrightarrow \mathbb{P}(\pi_* L^{\otimes n}).$$

By the functoriality of dualizing sheaf and $\delta_{1,\{1\}}$ on the moduli stack $\overline{\mathcal{M}}_{g,1}$, we find that this construction is compatible with isomorphism relation and commutes with base extension. \square

18.4.3 Moduli of pseudostable curves as log canonical models

Proof of Theorem 18.4.4. By Proposition 18.4.13, we just need to identify $\overline{\mathcal{M}}_g^{ps}$ and $\overline{\mathcal{M}}_g(9/11)$.

By Lemma 18.4.8, we know that $\mathcal{T} : \overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{M}}_g^{ps}$ implies

- (a) T is isomorphism over $\overline{\mathcal{M}}_g \setminus \Delta_1$;
- (b) T takes locus of elliptic tails into cusps as in Lemma 18.4.8;
- (c) $\mathcal{T}(C) = \mathcal{T}(C')$ if and only if the number of elliptic tails are the same and the remaining pointed stable curves $(D; p_1, \dots, p_r) \cong (D'; p'_1, \dots, p'_r)$.

Hence T is also an extremal contraction of the locus of elliptic tails. By Remark 1.26 in [66], we know that $\gamma : \overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{M}}_g(9/11)$ are the same as T . Well done. \square

Now we turn to prove Theorem 18.4.5. Let δ^{ps} be the boundary divisor of $\overline{\mathcal{M}}_g^{ps}$, it is the image of δ under \mathcal{T} . We will write $K_{\overline{\mathcal{M}}_g} + \alpha\delta$ instead of $K(\overline{\mathcal{M}}_g; \Delta, \alpha)$ over $\overline{\mathcal{M}}_g$ since they are the same things, after a pull-back.

Lemma 18.4.14 (Log Discrepancy Formula).

$$K_{\overline{\mathcal{M}}_g} + \alpha\delta = T^*(K_{\overline{\mathcal{M}}_g^{ps}} + \alpha\delta^{ps}) + (9 - 11\alpha)\delta_1.$$

Proof. First, it's easy to see that

$$K_{\overline{\mathcal{M}}_g} + \alpha\delta = T^*(K_{\overline{\mathcal{M}}_g^{ps}} + \alpha\delta^{ps}) + c\delta_1.$$

Then by Lemma 18.4.1 we find that

$$\begin{aligned} (K_{\overline{\mathcal{M}}_g} + \alpha\delta) \cdot R &= 13\lambda \cdot R + (\alpha - 2)R \cdot \delta \\ &= 13/12 + (\alpha - 2)(1 - 1/12) = \frac{11\alpha - 9}{12} \end{aligned}$$

and $\delta_1 \cdot R = -1/12$. Hence $c = 9 - 11\alpha$. \square

Hence if $(\overline{\mathcal{M}}_g, \alpha\delta - (9 - 11\alpha)\delta_1)$ is lc pair and $\alpha \leq 9/11$, then so is $(\overline{\mathcal{M}}_g^{ps}, \alpha\delta^{ps})$. This is right as reducing the coefficient of δ_1 can only increase the discrepancies of divisors lying over δ_1 ; this does not affect whether the singularities are log canonical. Now we just need to analyze whether $K_{\overline{\mathcal{M}}_g^{ps}} + \alpha\delta^{ps}$ is ample for $7/10 < \alpha < 9/11$.

We have

$$T^*\text{NS}(\overline{\mathcal{M}}_g^{ps}) = R^\perp \subset \text{NS}(\overline{\mathcal{M}}_g),$$

then $\text{Amp}(\overline{\mathcal{M}}_g^{ps}) = \text{Int}(R^\perp \cap \text{Nef}(\overline{\mathcal{M}}_g))$. Actually, if the Fulton's Conjecture 1 holds, then we finish the proof. Of course we can not use this for now and fortunately, we do not need the full strength of the conjecture and we just need to use the following result (Corollary 16.8.6):

Proposition 18.4.15 ([46] Proposition 6.1). *Let $D = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i\delta_i$ over $\overline{\mathcal{M}}_g$ such that if $1 \leq i \leq g/2$ then either $b_i = 0$ or $b_i > b_0$. If D has non-negative intersection with all F -curves, then D is nef.*

Actually the intersection numbers of $D = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i\delta_i$ with the F -curves has the following results (let $b_i = b_{g-i}$ for $i > g/2$):

(a)	$a/12 - b_0 + b_1/12$	family of elliptic tails
(b)	b_0	
(c)	b_i	for $g - 2 \geq i$
(d)	$2b_0 - b_{i+1}$	for $g - 2 \geq 2i$
(e)	$b_i + b_j - b_{i+j}$	for $i, j \geq 1, i + j \leq g - 1$
(f)	$b_i + b_j + b_k + b_l - b_{i+j} - b_{i+k} - b_{i+l}$	for $i, j, k \geq 1, i + j + k + l = g$

Proof of Theorem 18.4.5. We just need to show that for $7/10 < \alpha \leq 9/11$, the divisor

$$D := K_{\overline{\mathcal{M}}_g} + \alpha\delta - (9 - 11\alpha)\delta_1 = 13\lambda + (\alpha - 2)\delta - (9 - 11\alpha)\delta_1$$

lies in $\text{Int}(R^\perp \cap \text{Nef}(\overline{\mathcal{M}}_g))$. Then we just need to check Proposition 18.4.15. In our case, we have $a = 13$ and $b_i = \begin{cases} 2 - \alpha, & i \neq 1; \\ 11 - 12\alpha, & i = 1 \end{cases}$. Then our divisor satisfied the conditions in Proposition 18.4.15, and we just need to check the table above!

- (i) We do not consider (a) as it is elliptic tails;
- (ii) for (b)(c)(f), if $\alpha \leq 11/12$ then they are positive;
- (iii) for (d), if $\alpha > 7/10$ it is positive;
- (iv) for (e), if $\alpha > 7/10$ it is positive.

Hence well done! □

Chapter 19

More geometry of moduli space of curves

19.1 A glimpse of some results using Teichmüller theory

Theorem 19.1.1 (Boggi-Pikaart, 2000, [19]). *The stack $\overline{\mathcal{M}}_{g,n}$ has no non-trivial étale cover, hence it is simply connected.*

Corollary 19.1.2. *The coarse moduli space $\overline{M}_{g,n}$ is simply connected.*

Theorem 19.1.3 (Harer-Zagier 1986, [52]; Bini-Harer 2011, [17]). *For non-negative integers $g, n, n > 2 - 2g$, we have: (i) The orbifold Euler characteristic of $\mathcal{M}_{g,n}$ is*

$$\chi(\mathcal{M}_{g,n}) = (-1)^n \frac{(2g-1)B_{2g}}{(2g)!} (2g+n-3)!;$$

(ii) *The orbifold Euler characteristic of $\overline{\mathcal{M}}_{g,n}$ is*

$$\chi(\overline{\mathcal{M}}_{g,n}) = \sum_{G \text{ of type } (g,n)} \frac{\prod_v \chi(\mathcal{M}_{g_v, L_v})}{\#\text{Aut}(G)}.$$

19.2 Intersection theory of moduli space of curves

We will refer [44] and [8] chapter XVII. We work over \mathbb{C} .

19.2.1 Chow rings of moduli space of curves

In the classical book [43] section 8.3 (Example 8.3.12) we know that the intersection product in the Chow ring only defined on the non-singular case and the rational coefficient Chow ring can be defined over the quotient variety X/G of a non-singular variety X by a finite group G of automorphisms with $p: X \rightarrow X/G$, that is,

$$V \cdot W := \frac{p_*(p^*[V] \cdot p^*[W])}{\#G}.$$

(In fact we have $\text{CH}^*(X/G) \otimes \mathbb{Q} = (\text{CH}^*(X) \otimes \mathbb{Q})^G$.)

Now we consider smooth quotient X/G and its stack $[X/G]$ with canonical map $p: [X/G] \rightarrow X/G$. We define $\text{CH}^*([X/G])_{\mathbb{Q}} := \text{CH}^*(X/G) \otimes \mathbb{Q}$. Concerning Chern classes of vector bundles on a stack $[X/G]$, recall that such a vector bundle can be viewed as a G -equivariant bundle E on X , meaning that E is a vector bundle on X plus a lifting of the G -action on X . The Chern classes of E are naturally G -invariant elements of $\text{CH}^*(X)$ and therefore give well-defined Chern classes $c_i(E) \in \text{CH}^*([X/G])_{\mathbb{Q}}$.

Now back to the moduli of curves. Now $M_{g,n}$ and $\overline{M}_{g,n}$ are all singular! Foundations of these due to D. Mumford. But $M_{g,n}$ can be written as a non-singular variety (the ℓ -level structure, omit it here) quotient by a finite group, hence we have define as above. But $\overline{M}_{g,n}$ is not that easy. Mumford use that, $\overline{M}_{g,n}$ is étale locally as a non-singular variety (Kuranishi family) quotient by a finite group by Theorem 15.3.1 and globally it is a quotient of Cohen-Macaulay variety by a finite group, to define the rational Chow ring. However, E. Looijenda (cf. [71]) showed that $\overline{M}_{g,n}$ is also be a globally quotient of a smooth variety $\overline{\mathfrak{M}}_g/G$ by a finite group (non abelian ℓ -level structure)! Hence we can define $\text{CH}^*(M_g) \otimes \mathbb{Q}$ and $\text{CH}^*(\overline{M}_g) \otimes \mathbb{Q}$ now.

For moduli stacks, unfortunately, although this works for coarse moduli space, the moduli stack in general not quotients of smooth varieties modulo finite groups! Let then $M \cong \mathfrak{M}/G$ be a moduli space of (stable or smooth) curves, and \mathcal{M} the corresponding moduli stack. We then have natural morphisms

$$[\mathfrak{M}/G] \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\beta} M = \mathfrak{M}/G$$

by the definition of coarse moduli space. Hence we define

$$\text{CH}^*(\mathcal{M})_{\mathbb{Q}} := \text{CH}^*([\mathfrak{M}/G])_{\mathbb{Q}} = \text{CH}^*(M)_{\mathbb{Q}}.$$

Noet that here we have

$$\text{CH}^1(\mathcal{M})_{\mathbb{Q}} = \text{CH}^1(M)_{\mathbb{Q}} = \text{Pic}(\mathcal{M}) \otimes \mathbb{Q} = \text{Pic}(M) \otimes \mathbb{Q}.$$

19.2.2 Basic classes

Fixed the moduli stack $\overline{\mathcal{M}}_{g,P}$ with sections $\sigma_p : \overline{\mathcal{M}}_{g,P} \rightarrow \mathcal{U}_{g,P} \cong \overline{\mathcal{M}}_{g,P \cup \{x\}}$. Let $\Sigma_p := \text{Im}(\sigma_p)$ and $\Sigma = \sum_{p \in P} \Sigma_p$. Fix the dualizing sheaf ω_{π} with universal family $\pi : \mathcal{U}_{g,P} \rightarrow \overline{\mathcal{M}}_{g,P}$.

Example 19.2.1 (Mumford-Morita-Miller classes). (1) Let $\psi_p := \sigma_p^* c_1(\omega_{\pi}) \in \text{CH}^1(\overline{\mathcal{M}}_{g,P})_{\mathbb{Q}}$;
(2) Let $\kappa_i := \pi_*(c_1(\omega_{\pi}(\Sigma))^{i+1}) \in \text{CH}^i(\overline{\mathcal{M}}_{g,P})_{\mathbb{Q}}$.

Remark 19.2.2. (i) We have $\kappa_0 = (2g - 2 + \#P)[\overline{\mathcal{M}}_{g,P}]$
(ii) Also we let $\tilde{\kappa}_i = \kappa_i - \sum \psi_p^i$.

Example 19.2.3 (λ -classes). We let $\lambda_i := c_i(\pi_* \omega_{\pi}) = c_i(\pi_! \omega_{\pi}) \in \text{CH}^i(\overline{\mathcal{M}}_{g,P})_{\mathbb{Q}}$. More generally, we let

$$\lambda_i(\nu) := c_i(\pi_! \omega_{\pi}^{\nu}) \in \text{CH}^i(\overline{\mathcal{M}}_{g,P})_{\mathbb{Q}}.$$

Hence we may define these over the coarse moduli space.

19.2.3 Tautological relations

Relations after forgetful maps

Proposition 19.2.4. Let $\pi : \overline{\mathcal{M}}_{g,P \cup \{x\}} \rightarrow \overline{\mathcal{M}}_{g,P}$ and let $a_t \in \mathbb{Z}_{\geq 0}$ labelled by $t \in P \cup \{x\}$.

(i) (String Equation)

$$\pi_* \left(\prod_{p \in P} \psi_p^{a_p} \right) = \sum_{a_x > 0} \psi_p^{a_p - 1} \prod_{q \neq p} \psi_q^{a_q};$$

(ii) (Generalized Dilaton Equation)

$$\pi_* \left(\psi_x^{a_x + 1} \prod_{p \in P} \psi_p^{a_p} \right) = \kappa_{a_x} \prod_{p \in P} \psi_p^{a_p}.$$

Proof. Omitted, see Proposition XVII.4.9 in [8]. □

Remark 19.2.5. *The standard dilaton equation is the special case of the generalized one in which $a_x = 0$, that is,*

$$\pi_* \left(\psi_x \prod_{p \in P} \psi_p^{a_p} \right) = (2g - 2 + \sharp P) \prod_{p \in P} \psi_p^{a_p}.$$

Corollary 19.2.6. *Let $\pi : \overline{\mathcal{M}}_{g, P \cup \{x\}} \rightarrow \overline{\mathcal{M}}_{g, P}$, then*

- (i) $\pi^*(\kappa_a) = \kappa_a - \psi_x^a$;
- (ii) $\pi_{x,*}(\kappa_a) = \kappa_{a-1}$;
- (iii) $\pi^*(\psi_p a) = \psi_p - \delta_{0, \{p, x\}}$;
- (iv) *If we let $\pi_n : \overline{\mathcal{M}}_{g, n} \rightarrow \overline{\mathcal{M}}_{g, n-1}$, then*

$$(\pi_{k+1} \cdots \pi_n)_*(\psi_1^{a_1} \cdots \psi_k^{a_k} \psi_{k+1}^{a_{k+1}+1} \cdots \psi_n^{a_n+1}) = \psi_1^{a_1} \cdots \psi_k^{a_k} \sum_{\sigma \in \mathfrak{S}_k} \kappa_\sigma$$

where $\kappa_\sigma = \kappa_{\sharp \gamma_1} \cdots \kappa_{\sharp \gamma_{\nu(\sigma)}}$ with σ as a product of $\nu(\sigma)$ disjoint cycles γ_i .

Proof. Omitted, (i)(ii) follows from string equation and see XVII.4.17,18 in [8]. (iii) follows from XVII.4.6 in [8]. (iv) follows directly from dilaton equation. \square

Remark 19.2.7. *Now combine (iv) and string equation, we find that the intersection numbers of the classes ψ_i and κ_i on a fixed $\overline{\mathcal{M}}_{g, n}$ is completely determined by the intersection theory of the ψ_i alone on all the $\overline{\mathcal{M}}_{g, \nu}$ with $\nu \geq n$, and conversely. This is a generalization of E. Witten's remark.*

Proposition 19.2.8 (Some relations of boundaries). *Let $\pi : \overline{\mathcal{M}}_{g, P \cup \{x\}} \rightarrow \overline{\mathcal{M}}_{g, P}$ and Γ be a dual graph with Γ_ν is the $P \cup \{x\}$ -marked graph obtained from Γ by letting $P_\nu \cup \{x\}$ be the index set for the vertex ν . Then*

$$\pi^* \delta_\Gamma = \sum_{\nu \in V} \delta_{\Gamma_\nu}.$$

In particular, we have $\pi^ \delta_{irr} = \delta_{irr}$ and $\pi^* \delta_{a, A} = \delta_{a, A} + \delta_{a, A \cup \{x\}}$.*

Proof. Trivial. \square

Relations after general gluing maps

Proposition 19.2.9. *Consider the morphisms*

$$\begin{array}{c} \overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g_v, L_v} \xrightarrow{\eta_v} \overline{\mathcal{M}}_{g_v, L_v} \\ \downarrow \xi_\Gamma \\ \overline{\mathcal{M}}_{g, P} \end{array}$$

Then:

- (i) $\xi_\Gamma^* \psi_p = \psi_p$;
- (ii) $\xi_\Gamma^* \kappa_a = \sum_{v \in V(\Gamma)} \eta_v^* \kappa_a$;
- (iii) *by the notations as before, we have*

$$\begin{array}{c} \overline{\mathcal{M}}_{\Gamma\Gamma'} := \overline{\mathcal{M}}_\Gamma \times_{\overline{\mathcal{M}}_{g, P}} \overline{\mathcal{M}}_{\Gamma'} \\ \parallel \\ \prod_{\Lambda \in G_{\Gamma\Gamma'}} \overline{\mathcal{M}}_\Lambda \xrightarrow{\prod \xi_{\Lambda\Gamma}} \overline{\mathcal{M}}_\Gamma \end{array}$$

then

$$\xi_\Gamma^* \delta_{\Gamma'} = \sum_{\Lambda \in G_{\Gamma\Gamma'}} \xi_{\Lambda\Gamma,*} \left(\prod_{\{l, l'\} \in E(\Lambda)} (-\eta_{v(l)}^* \psi_l - \eta_{v(l')}^* \psi_{l'}) \right).$$

Remark 19.2.10. *The special case of the relations of gluing maps we refer Lemma XVII.4.35,36 in [8].*

19.2.4 The tautological ring and relative theorems and conjectures

Here we refer [79] and chapter XVII and XX in [8].

Definition 19.2.11. *The system of tautological rings $(R^*(\overline{\mathcal{M}}_{g,n})) \subset (\text{CH}^*(\overline{\mathcal{M}}_{g,\nu})_{\mathbb{Q}})_{g,n}$ is the smallest system of \mathbb{Q} -algebras satisfying*

- (i) $\psi_1, \dots, \psi_n \in R^*(\overline{\mathcal{M}}_{g,n})$;
- (ii) *the system is closed under pushforwards by morphisms $\pi, \xi_{irr}, \xi_{a,A}$.*

Define

$$F_g := \sum_{n \geq 0} \frac{1}{n!} \sum_{\#\mathbf{k}=3g-3+n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \right) t_{k_1} \dots t_{k_n}$$

and $F := \sum_g F_g \lambda^{2g-2}$. This is Witten's free energy. For convenience, we will set $\lambda = 1$.

Theorem 19.2.12 (Witten's Conjecture (Kontsevich's Theorem)). *We have the following system of PDEs:*

$$(2n+1) \frac{\partial^3 F}{\partial t_n \partial t_0^2} = \left(\frac{\partial^2 F}{\partial t_{n-1} \partial t_0} \right) \left(\frac{\partial^3 F}{\partial t_0^3} \right) + 2 \left(\frac{\partial^3 F}{\partial t_{n-1} \partial t_0^2} \right) \left(\frac{\partial^2 F}{\partial t_0^2} \right) + \frac{1}{4} \left(\frac{\partial^5 F}{\partial t_{n-1} \partial t_0^4} \right).$$

Sketch of Kontsevich's Proof. Kontsevich's Proof consisted of two parts. The first part was to prove a combinatorial formula for the gravitational descendants. Let $G_{g,n}$ be the set of isomorphism classes of trivalent ribbon graphs of genus g with n faces and together with a numbering $\text{Faces}(G) \cong [n]$. Denote by $V(G)$ the set of vertices of a graph $G \in G_{g,n}$. Let us introduce formal variables $\lambda_i, i \in [n]$. For an edge $e \in \text{Edges}(G)$, let $\lambda(e) = \frac{1}{\lambda_i + \lambda_j}$ where i and j are the numbers of faces adjacent to e . Then we have (Kontsevich's combinatorial formula):

$$\sum_{\#\mathbf{k}=3g-3+n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \prod_{\ell=1}^n \frac{(2k_\ell - 1)!!}{\lambda_\ell^{2k_\ell+1}} = \sum_{G \in G_{g,n}} \frac{2^{\#\text{Edges}(G) - \#V(G)}}{\#\text{Aut}(G)} \prod_{e \in \text{Edges}(G)} \lambda(e).$$

The second step of Kontsevich's proof was to translate the combinatorial formula into a matrix integral. Then, by using non-trivial analytical tools and the theory of τ -functions of the KdV hierarchy, he was able to prove that $\exp(F)$ is a τ -function of the KdV hierarchy and, hence, the free energy F satisfies our equation. \square

Remark 19.2.13. (i) *Using these PDEs along with a straightforward geometric fact known as the string equation and the initial condition $\int_{\overline{\mathcal{M}}_{0,3}} 1 = 1$, as $\overline{\mathcal{M}}_{0,3}$ is a point, all top intersections are quickly recursively determined;*

(ii) *There was a later reformulation of Witten's conjecture using Virasoro equations, see [8];*

(iii) *Okounkov and Pandharipande in 2001 showed the Witten's conjecture using the ELSV formula which relates intersection numbers with Hurwitz numbers (as a corollary, they get $R^{3g-3+n}(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$);*

(iv) *Mirzakhani in 2004 showed the Witten's conjecture with a formula which relates intersection numbers with volumes of moduli spaces.*

There are several conjectures about Poincaré duality of moduli of curves:

Conjecture 2 (Poincaré duality conjecture for stable curves (Hain-Looijenga)). *Let $d = \dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$, then*

- (I) $R^i(\overline{\mathcal{M}}_{g,n}) = 0$ for $i > d$ (which is obvious);
- (II) $R^d(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$ (true by ELSV formula);
- (III) *the natural pairing $R^i(\overline{\mathcal{M}}_{g,n}) \times R^{d-i}(\overline{\mathcal{M}}_{g,n}) \rightarrow R^d(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$ is perfect.*

This conjecture was motivated by an earlier conjecture by Faber:

Conjecture 3 (Faber's conjecture for smooth curves). (I) $R^*(\mathcal{M}_g)$ is a dimension $g - 2$ Poincaré ring;

(II) The $\lfloor g/3 \rfloor$ classes $\kappa_1, \dots, \kappa_{\lfloor g/3 \rfloor}$ generated the ring, with no relations in degree $\lfloor g/3 \rfloor$ (is true in cohomology by Morita, Ionel; Ionel's proof should extend to the Chow ring without difficulty);

(III) We have

$$\psi_1^{d_1+1} \dots \psi_k^{d_k+1} = \frac{(2g-3+k)!(2g-1)!!}{(2g-1)! \prod_{j=1}^k (2d_j+1)!!} = \sum_{\sigma \in \mathfrak{S}_k} \kappa_\sigma$$

which defined as before (proved by Getzler, Pandharipande, Givental).

Remark 19.2.14. E. Looijenga showed that $R^i(\mathcal{M}_g) = 0$ for $i > g - 2$ and $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$.

19.3 Cohomology of moduli space of curves

19.3.1 Collections of results about cohomology groups

We will consider about the rational (singular)-cohomology (or homology) groups of moduli space of smooth or stable curves over \mathbb{C} . We note that as in [31], we have $H^i(\mathcal{M}) \otimes \mathbb{Q} \cong H^i(M, \mathbb{Q}) := H^i(M(\mathbb{C}), \mathbb{Q})$, so we may just consider the coarse moduli space.

Theorem 19.3.1 (Harer 1986). For $i > 4g - 5$, we have $H_i(M_{g,n}, \mathbb{Q}) = 0$ for $i > c(g, n)$ where

$$c(g, n) = \begin{cases} n - 3, & g = 0; \\ 4g - 5, & g > 0, n = 0; \\ 4g - 4 - n, & g > 0, n > 0. \end{cases}$$

Proof. Using the construction of Teichmüller method, see chapter XIX Theorem 2.2 in [8]. \square

Theorem 19.3.2. The following hold true:

(i) (Mumford, Harer) $H^1(M_{g,n}; \mathbb{Q}) = 0$ for any $g \geq 1$ and any n such that $2g - 2 + n > 0$;

(ii) (Harer) $H^2(M_{g,n}; \mathbb{Q})$ is freely generated by $\kappa_1, \psi_1, \dots, \psi_n$ for any $g \geq 3$ and any n . $H^2(M_{2,n}; \mathbb{Q})$ is freely generated by ψ_1, \dots, ψ_n for any n , while $H^2(M_{1,n}; \mathbb{Q})$ vanishes for all n .

Proof. For the modern proof using Deligne's spectral sequence, we refer Theorem 10 in [7]. \square

Theorem 19.3.3 (Arbarello-Cornalba [6], 1998; Bergström-Faber-Payne [15], 2022; Canning-Larson-Payne [20], 2022). (i) The cohomology groups $H^k(\overline{M}_{g,n}) = 0$ for all odd $k \leq 9$;

(ii) The cohomology groups $H^2(\overline{M}_{g,n})$ generated by κ_1, ψ_i and δ_{irr} and $\delta_{a,A}$ for $0 \leq a \leq g$ and $2a - 2 + \sharp A \geq 0$ and $2(g - a) - 2 + \sharp A^c \geq 0$, with relations

(ii-a) If $g > 2$ all relations are generated by $\delta_{a,A} = \delta_{g-a, A^c}$;

(ii-b) If $g = 2$ all relations are generated by $\delta_{a,A} = \delta_{g-a, A^c}$ and

$$5\kappa_1 = 5\psi + \delta_{irr} - 5 \sum_A \delta_{0,A} + 7 \sum_A \delta_{1,A};$$

(ii-c) If $g = 1$ all relations are generated by $\delta_{a,A} = \delta_{g-a, A^c}$ and

$$\kappa_1 = \psi - 5 \sum_A \delta_{0,A}, 12\psi_p = \delta_{irr} + 12 \sum_{p \in S, \sharp S \geq 2} \delta_{0,S}, p \in \{1, \dots, n\};$$

(ii-d) If $g = 0$ all relations are generated by $\delta_{a,A} = \delta_{g-a, A^c}$ and

$$c(g, n) = \begin{cases} \kappa_1 = \sum_{x,y \notin A} (\sharp A - 1) \delta_{0,A}, & x, y \in \{1, \dots, n\}, x \neq y; \\ \psi_z = \sum_{z \in A; x,y \notin A} \delta_{0,A}, & x, y, z \in \{1, \dots, n\}, x, y, z \text{ distinct}; \\ \delta_{irr} = 0. & \end{cases}$$

(iii) The group $H^{11}(\overline{M}_{g,n})$ nonzero if and only if $g = 1$ and $n \geq 11$, and in this case we have $H^{11}(\overline{M}_{1,n}) \cong (H^{11}(\overline{M}_{1,11}))^{\oplus \binom{n-1}{10}}$;

(iv) Assume $k \leq 11$. Let g_1, \dots, g_k be distinct positive integers, and set $g = 1 + g_1 + \dots + g_k$. Then $H^{11+2k}(\overline{M}_{g,n}) \neq 0$ for $n \geq 11 - k$.

Proof. (i)(ii) In paper [6], E. Arbarello and M. Cornalba showed when $k = 1, 2, 3, 5$ and in paper [15], Jonas Bergström, Carel Faber and Sam Payne showed when $k = 7, 9$.

(iii)(iv) In paper [20], Samir Canning, Hannah Larson and Sam Payne showed these. \square

Remark 19.3.4. *The funny point is that, in papers [15] and [20], they using some number-theoretic method, such as Hasse-Weil zeta functions.*

Chapter 20

Alterations and the moduli space of stable curves

20.1 Extensions of stable curves

In this chapter we will consider the stack we have used over \mathbb{Z} -schemes. And for simplicity, we consider the case without marked points.

Recall that we have proved the stable reduction of stable curves over a spectrum of DVR:

Theorem 20.1.1 (Stable reduction). *Let R be a DVR with fraction field K and $\Delta = \text{Spec}(R), \Delta^* = \text{Spec}(K)$. If $(\mathcal{C}^* \rightarrow \Delta^*)$ is a family of stable curves of genus g , then there exists a finite cover $\Delta' \rightarrow \Delta$ of spectrums of DVRs and a family $(\mathcal{C}' \rightarrow \Delta')$ of stable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \rightarrow \Delta'^*$. As*

$$\begin{array}{ccc}
 \Delta'^* & \longrightarrow & \Delta^* & \xrightarrow{(\mathcal{C}^* \rightarrow \Delta^*)} & \overline{\mathcal{M}}_{g,n} \\
 \downarrow & & \downarrow & \nearrow & \\
 \Delta' & \longrightarrow & \Delta & &
 \end{array}$$

Now how can we let the DVR to be an integral scheme S ? We will use a technical lemma (le lemme de Gabber) to show the case when S is a normal integral scheme directly. Then we will give a canonical way to construct like in le lemme de Gabber. These are all have some same methods with de Jong's alteration.

Theorem 20.1.2. *Let S be a normal integral scheme with a generic point $\eta = \text{Spec}K$ and C_η be a stable curve over η . Then there exists a generically étale, generically finite, proper and dominant morphism of integral schemes $S' \rightarrow S$ with generic point $\eta' = \text{Spec}K'$ such that the stable curve $C_{\eta'} \times_{\eta'} \eta'$ extends to a stable curve over S' .*

First we give some comments. Actually we find that C is already extended to the family of stable curves in $\overline{\mathcal{M}}_g$! So if we consider some compact neighborhood S' of C in $\overline{\mathcal{M}}_g$, then we have such S' extended C .

$$\begin{array}{ccc}
 \eta & \longrightarrow & S' & \hookrightarrow & \overline{\mathcal{M}}_g \\
 & \searrow & \downarrow & & \\
 & & S & &
 \end{array}$$

But now $S' \rightarrow S$ need not be proper surjective and even S' may not be a scheme! Hence we need to find some finite cover of scheme over this stack to deal with this problem.

Proof. Consider $\eta \rightarrow \overline{\mathcal{M}}_g$ induced by C_η . By Gabber's Lemma C.2.1, we find a finite surjective generically étale morphism from a scheme $M \rightarrow \overline{\mathcal{M}}_g$. Now M is proper as the stack $\overline{\mathcal{M}}_g$ is proper. Hence we obtain a finite separable K -scheme $\eta \times_{\overline{\mathcal{M}}_g} M$ and there exists a finite separable extension

K'/K such that $\eta' := \text{Spec}K' \rightarrow \eta$ factor through $\eta \times_{\overline{\mathcal{M}}_g} M$. Hence the curve $C'_\eta := C_\eta \times_\eta \eta'$ induced from a family of curve $C'_M \rightarrow M$ by $M \rightarrow \overline{\mathcal{M}}_g$.

Hence we claim that there exists a proper, dominant and generically finite morphism of integral schemes $S' \rightarrow S$ with generic point η' such that the map $\eta' \rightarrow M$ factor through $S' \rightarrow M$.

Now consider $\eta \rightarrow S_1 = \text{Norm}_{\eta'}(S) \rightarrow S$ be the normalization. Consider S' be the closure of the image of diagonal $\Delta : \eta' \rightarrow S_1 \times_{\text{Spec}\mathbb{Z}} M$. Hence we have:

$$\begin{array}{ccccc}
 \eta' & & & & \\
 \downarrow & \searrow \Delta & & \searrow & \\
 & & S' := \overline{\text{Im}(\Delta)} & \longrightarrow & M \\
 & & \downarrow & & \downarrow \\
 S & \longleftarrow & S_1 = \text{Norm}_{\eta'}(S) & & \overline{\mathcal{M}}_g
 \end{array}$$

Now we find that $S' \rightarrow S$ is generically étale, generically finite and dominant. Hence we just need to show $S' \rightarrow S$ is proper. This comes from the fact that $S_1 \rightarrow S$ is finite (this follows from St 032L) and $S' \rightarrow S_1$ is proper as the composition of the base change of $M \rightarrow \text{Spec}\mathbb{Z}$ and a closed immersion. This gives us the claim. Now let C' induced by C'_M along $S' \rightarrow M$, then well done. \square

Remark 20.1.3. (i) In the original case the author said that this construction holds for any integral scheme S , but here *why $S_1 \rightarrow S$ is of finite type?*

(ii) Actually, de Jong's alteration is defined as generically finite, proper and dominant. Hence we find a generically étale alteration $S' \rightarrow S$ now.

Now we construct that canonical construction of M . See He Tongmu's Answer.

Here we will follows [29]. We all knew that the obstruction of $\overline{\mathcal{M}}_g$ to be a scheme (or algebraic space) is that it has non-trivial automorphisms. So we need to get a ℓ -level structure to kill these automorphisms.

Let $\ell \geq 3$ is invertible on T , then we will call a family of genus g smooth curves $X \rightarrow T$ with ℓ -level structure if $\text{Pic}_S^0(X)[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{\oplus 2g}$ (when over some algebraically closed field, one can let $H_{\text{ét}}^1(C, \mathbb{Z}/\ell\mathbb{Z}) \cong (\mathbb{Z}/\ell\mathbb{Z})^{\oplus 2g}$ or $R_{\text{ét}}^1 f_* (\mathbb{Z}/\ell\mathbb{Z}) \cong (\mathbb{Z}/\ell\mathbb{Z})^{\oplus 2g}$ for families). Then we consider the stack ${}_\ell\mathcal{M}_g$ over $\text{Spec}\mathbb{Z}[1/\ell]$ parameterizing smooth curves of genus g with ℓ -level structure. Hence we have:

$$\begin{array}{ccc}
 {}_\ell\mathcal{M}_g & & \\
 \downarrow & & \\
 \mathcal{M}_g & \hookrightarrow & \overline{\mathcal{M}}_g
 \end{array}$$

By Lemma 3.5.1 in [29], we get ${}_\ell\mathcal{M}_g$ is an algebraic space. since Deligne proved that all automorphisms are killed after adding the ℓ -level structure.

Let ${}_\ell\overline{\mathcal{M}}_g = \text{Norm}_{\overline{\mathcal{M}}_g}({}_\ell\mathcal{M}_g)$, Deligne also proved that ${}_\ell\overline{\mathcal{M}}_g$ is a proper algebraic space in Corollary 3.6 in [29]. By the algebraic-space version of Chow's lemma, one get a projective scheme M over \mathbb{Z} as:

$$\begin{array}{ccccc}
 {}_\ell\mathcal{M}_g & \longrightarrow & {}_\ell\overline{\mathcal{M}}_g & \longleftarrow & M \\
 \downarrow & & \downarrow & & \\
 \mathcal{M}_g & \longrightarrow & \overline{\mathcal{M}}_g & &
 \end{array}$$

Hence we give a canonical construction of M now!

Remark 20.1.4. As the paper [29] is of French, so *we will omit the proof for now and will add them later!*

20.2 Alteration, the basic notions

We will mainly refer B. Conrad's notes [23] about A. Johan de Jong's alteration theory.

Definition 20.2.1. Let S be a regular scheme and D be a effective Cartier divisor (as a closed subscheme) with (reduced) irreducible components $\{D_i\}_{i \in I}$.

(i) We called D is strict normal crossings divisor if D is reduced and for any finite subset $J \subset I$, the subscheme $D_J := \bigcap_{i \in J} D_i$ is regular and of codimension $\sharp(J)$;

(ii) We called D is normal crossings divisor if for all $p \in D$, there exists an étale neighborhood $S' \rightarrow S$ of p such that $(S', D' := D \times_S S')$ is a strict normal crossings divisor.

Remark 20.2.2. (i) When we consider the (locally) noetherian case, we can also let D_J is regular and for all $p \in D_J$ we have $\dim \mathcal{O}_{D_J, p} = \dim \mathcal{O}_{S, p} - \sharp(J)$. This because $\mathcal{O}_{S, p}$ is regular and hence Cohen-Macaulay, so it is catenary;

(ii) By St 0BIA, we can replace this definition as for all $p \in D$ the local ring $\mathcal{O}_{S, p}$ is regular and there exists a regular system of parameters $x_1, \dots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ such that D is cut out by $x_1 \dots x_r$ in $\mathcal{O}_{S, p}$.

Definition 20.2.3 (Modification and Alteration). Let X be an integral scheme.

(i) An modification of X is a map $f : X' \rightarrow X$ such that X' is an integral scheme and f is proper birational;

(ii) An alteration of X is a map $g : X'' \rightarrow X$ such that X'' is an integral scheme and g is proper dominant and generically finite.

Remark 20.2.4. Hence for any alteration $g : X'' \rightarrow X$ of X , after the normalization, one get g can factored as $X'' \xrightarrow{f} X' \xrightarrow{h} X$ where f is a modification and g is finite.

Now we introduce the main theorem of de Jong's first paper about alterations. ACTually this is some kind of weakened form of Hironaka's resolution of singularities in any characteristic.

Theorem 20.2.5 (A. Johan de Jong - 1996). Let X be a variety over a field k . For $Z \subset X$ be any proper closed subset, there exists an alteration $f : X' \rightarrow X$ and an open immersion $j : X' \hookrightarrow \overline{X}'$ such that

(i) \overline{X}' is regular and projective;

(ii) The subscheme

$$\left(j(f^{-1}(Z)) \cup (\overline{X}' \setminus j(X')) \right)_{\text{red}} \subset \overline{X}'$$

is a strict normal crossings divisor.

20.3 de Jong's main theorem and moduli of curves

Actually to prove the de Jong's main theorem, we need to use the general theory of moduli stack of curves! There are many useful conclusion at the way proving this theorem and in this section we will give a sketch of the proof, follows [23] and the original paper is [27]. First we recall the theorem.

Theorem 20.3.1 (A. Johan de Jong - 1996). Let X be a variety over a field k . For $Z \subset X$ be any proper closed subset, there exists an alteration $f : X' \rightarrow X$ and an open immersion $j : X' \hookrightarrow \overline{X}'$ such that

(i) \overline{X}' is regular and projective;

(ii) The subscheme

$$\left(j(f^{-1}(Z)) \cup (\overline{X}' \setminus j(X')) \right)_{\text{red}} \subset \overline{X}'$$

is a strict normal crossings divisor.

Remark 20.3.2 (Outline of the proof). First we use Chow's lemma and some reduction one can let X is quasi-projective and $k = \overline{k}$; then the main method is induction of $d = \dim X$.

When $d \leq 1$, then using the normalization and we win. When $d \geq 2$, we will blowing up X appropriately to get a surjection $f : X \rightarrow \mathbb{P}_k^{d-1}$ with all geometric fibers are connected of dimension 1 (but may have bad singularities and may non-reduced) and X_η is smooth for generic point η .

Hence using some properties of moduli stack $\overline{\mathcal{M}}_{g,n}$ and semistable reduction of curves, we can find $f' : X_1 \rightarrow Y$ with $X_1 \twoheadrightarrow X$ and $Y \rightarrow \mathbb{P}_k^{d-1}$ are alterations with commutative diagram

$$\begin{array}{ccccc} X' = X_1 \times_Y Y' & \longrightarrow & X_1 & \xrightarrow{\text{alteration}} & X \\ \downarrow & \lrcorner & \downarrow f' & & \downarrow f \\ Y' & \xrightarrow{\text{alteration}} & Y & \xrightarrow{\text{alteration}} & \mathbb{P}_k^{d-1} \end{array}$$

where f' is flat with semistable geometrically connected fibers of dimension 1 (and smooth generic fiber). By induction, we can find a regular alteration of $Y' \rightarrow Y$, so $X' := X \times_Y Y'$ (which remains integral) is an alteration of X that is semistable over a regular Y' (with smooth generic fiber). Then, in this nice situation, we resolve X' by hand using some carefully-chosen blow-ups.

♣ **Step 1. Some reductions.**

We omit the reduction of the base field into algebraically closed (See [23] 3.2). Now we first go through the reduction to the projective case.

By Chow’s lemma one get a modification $\pi : X' \rightarrow X$ with X' integral and quasi-projective. So after replacing Z by $\pi^{-1}(Z)$ we may let X is quasi-projective. After taking Nagata compactification, we get a dense open immersion $X \hookrightarrow \overline{X}$ where \overline{X} is projective and let $\overline{Z} := \overline{X} \setminus (X \setminus Z)$, we will show that we can replace (X, Z) into $(\overline{X}, \overline{Z})$.

If we have an alteration $\phi_1 : \overline{X}_1 \rightarrow \overline{X}$ with \overline{X}_1 is smooth and $\phi_1^{-1}(\overline{Z}) \subset \overline{X}_1$ is a strict normal crossings divisor. Consider $X_1 := \phi_1^{-1}(X) \subset \overline{X}_1$, we know that

$$\begin{array}{c} X_1 \hookrightarrow \overline{X}_1 \\ \downarrow \phi_1|_{X_1} \\ X \end{array}$$

is an alteration and X_1 is smooth. We just need to check the property of Z . As the construction of \overline{Z} , we get

$$\partial_{\overline{X}_1}(X_1 \setminus \phi_1^{-1}(Z)) = \partial(\overline{X}_1 \setminus \phi_1^{-1}(\overline{Z}))$$

and it is a strict normal crossings divisor.

Next after blowing up $X' = \text{Bl}_Z(X)$ and let $Z' = E$, we can reduce to the case that Z is a Cartier divisor. Again after taking normalization $\tilde{X} \rightarrow X$ (which is a finite birational map as X is a variety), we can let X is normal (how about the sncd here?).

Lemma 20.3.3. *If $\phi : X' \rightarrow X$ be an alteration and $Z \subset Z' \subsetneq X$ such that $\phi^{-1}(Z') \subset X'$ is a sncd, then so is $\phi^{-1}(Z) \subset X'$.*

Proof. Trivial. □

Hence by this lemma, under our running assumptions, we can increase Z without loss of generality. Now after some reductions we now let X be a projective normal variety over some algebraically closed field k with Z be a Cartier divisor which can be increased.

♣ **Step 2. Construct good curve fibrations I.**

The main result is:

Lemma 20.3.4 (de Jong [27], 4.11). *Consider the pairs (X, Z) as in Step 1, except X normal. There exists a finite subset $S \subset X^{\text{sm}}(k) \setminus Z(k)$ such that for the diagram*

$$\begin{array}{ccc} & X' = \text{Bl}_S(X) & \\ f \swarrow & & \searrow \phi \\ \mathbb{P}^{d-1} & & X \end{array}$$

the fibers of f are pure dimension 1 and generically smooth, and $F : \phi^{-1}(Z) \rightarrow \mathbb{P}^{d-1}$ is generically étale and finite.

Furthermore, if X is normal then we can arrange that there exists a dense open $U \subset \mathbb{P}^{d-1}$ such that $f^{-1}(U) \rightarrow U$ is smooth with geometrically connected fibers.

(Reader can first consider $\text{Bl}_p \mathbb{P}^d = \{(q, l) \in \mathbb{P}^d \times \mathbb{P}^{d-1} : q \in l\}$ and see what happens) This is a technical and difficult result. Here we just give a basic idea and the conclusions we used. One can see paper [27] 4.11 or notes [23] Lemma 4.1.

• **Projections from points**

First we have the projection from point

$$X \hookrightarrow \mathbb{P}^N \setminus \{p\} \xrightarrow{\pi_p} \mathbb{P}^{N-1} = \{l : p \in l\}$$

as in the blowing up $\text{Bl}_p \mathbb{P}^N \rightarrow \mathbb{P}^{N-1}$ with $(\pi_p|_X)^{-1}(\{l\}) = l \cap X$. Then we have:

Proposition 20.3.5 (de Jong [27], Prop.2.11). *Let k be a field and a closed subscheme $X \subset \mathbb{P}_k^N$ such that is generically smooth of pure dimension $d \leq N - 1$. Then there exists a dense open $U \subset \mathbb{P}^N \setminus X$ so that for all finite separable extensions k'/k and points $p \in U(k')$, the projection*

$$\pi'_p : X_{k'} \rightarrow \mathbb{P}_{k'}^{N-1}$$

- (i) is birational onto its image if $d < N - 1$;
- (ii) is generically étale onto $\mathbb{P}_{k'}^{N-1}$ if $d = N - 1$.

Sketch of proof. We just give a sketch of (ii). First reduce to the case k is separably closed.

(ii) As the non-smooth locus $X^{\text{sing}} = X \setminus X^{\text{sm}}$ has codimension ≥ 2 in \mathbb{P}^N , most lines $l \subset \mathbb{P}^N$ miss X^{sing} . Let $V = X^{\text{sm}} = \coprod V_j$ are open subsets where the smooth loci of irreducible components of X avoid the other irreducible components of X . Hence for most lines $l \subset \mathbb{P}_k^N$ we have

$$l \cap X = \coprod_j (l \cap V_j).$$

By some kind of Bertini theorem (see [23] Theorem 5.5) to the Grassmannian $\text{Gr}(N - 1, N)$, one get a line $l_0 \subset \mathbb{P}_k^N$ such that $l \cap X = \coprod_j (l \cap V_j)$ with all $l \cap V_j \neq \emptyset$ and étale over k (hence are finite sets). Hence after choosing $p \in l_0 \setminus X$, we get $\pi_p : X \rightarrow \mathbb{P}^{N-1} = \{l : p \in l\}$ such that $\pi_p^{-1}(\{l_0\})$ is étale and meets X inside X^{sm} touching every irreducible component of X . By openness on the source for the quasi-finite maps, we conclude that π_p is generically quasi-finite and hence dominant (and even surjective, due to properness).

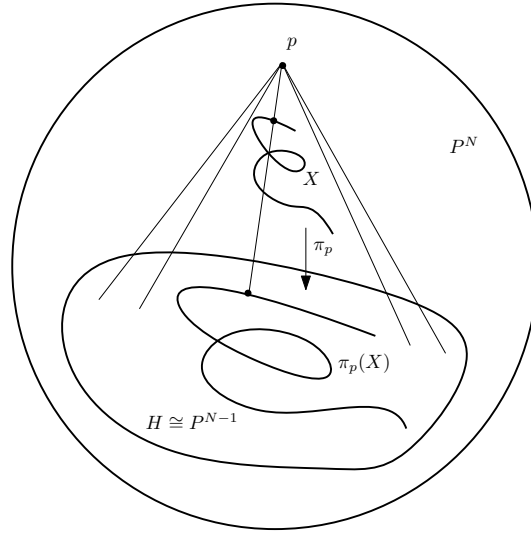
Since π_p is a dominant map, it is flat on some dense open W . Next after analyze $\Omega_{\pi_p}^1$ using Bertini theorem, one get the open locus where W meet the open complement of the support of the coherent sheaf $\Omega_{\pi_p}^1$ to touch each irreducible component of X . Hence π_p is étale at these points.

Finally, consider

$$\begin{array}{ccc} & \{(x, l) \in \mathbb{P}^N \times \text{Gr}(N - 1, N) : x \in l\} & \\ & \swarrow p_1 \qquad \searrow p_2 & \\ \mathbb{P}^N & & \text{Gr}(N - 1, N) \end{array}$$

and using some kind of Bertini theorem, one get some open $\Omega \subset \text{Gr}(N - 1, N)$ such that $p_1(p_2^{-1}(\Omega)) \cap (\mathbb{P}^N \setminus X)$ is the locus of points p we have been using sweeps out. Hence it is at least a dense open locus in \mathbb{P}^N . \square

• **Sketch of the Lemma de Jong 4.11**



Use $N - d$ times Proposition 20.3.5, we get

$$\begin{array}{ccc} Z & \longrightarrow & \pi(Z) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & \mathbb{P}^d \end{array}$$

where the bottom map is finite and generically étale and the top map is birational.

Pick $\Omega \subset \mathbb{P}^d \setminus \pi(Z)$ be a dense open such that $\pi^{-1}(\Omega) \rightarrow \Omega$ is étale. Choose $\xi \in \Omega(k)$ such that $\pi_\xi : \pi(Z) \rightarrow \mathbb{P}^{d-1} = \{l : \xi \in l\}$ finite and generically étale. Hence the natural map $\text{Bl}_\xi \mathbb{P}^d \rightarrow \mathbb{P}^{d-1}$ is generically étale when restrict it to $\pi(Z)$.

Let $S = \pi^{-1}(\xi)$, we find that $S \subset X^{\text{sm}}$. Easy to see that $\text{Bl}_S(X)$ is normal as it is a gluing $X \setminus S$ and $\text{Bl}_S(X^{\text{sm}})$ (which is smooth when we see this étale locally) via $X^{\text{sm}} \setminus S$.

As blowing up commute with flat base-change, we get the cartesian

$$\begin{array}{ccc} X' := \text{Bl}_S(X) & \longrightarrow & \text{Bl}_\xi \mathbb{P}^d \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & \mathbb{P}^d \end{array}$$

and hence

$$X' = \text{Bl}_S(X) = \{(x, l) \in X \times \mathbb{P}^{d-1} : x \in \pi^{-1}(l)\}$$

which is a similar description as $\text{Bl}_\xi \mathbb{P}^d$.

Next consider

$$\begin{array}{ccc} & X' & \\ \swarrow & & \searrow \\ X & & \mathbb{P}^{d-1} \end{array} \begin{array}{l} \\ \text{pr}_1 \\ \\ f \end{array}$$

and hence $f|_Z$ is generically étale and finite.

Easy to see that $\pi^{-1}(l)$ is pure dimension 1. Second, we claim $\pi^{-1}(l)$ is generically smooth. This because $\pi^{-1}(l) \rightarrow l$ is finite, hence irreducible component of $\pi^{-1}(l)$ surjectively onto l for dimension reasons. As it is also the base change of $\pi : X \rightarrow \mathbb{P}^d$, which is generically étale, hence the claim is right. Finally, as $f^{-1}(\{l\}) = \pi^{-1}(l)$, we conclude the first part of the Lemma.

For the second part, we need to show that $f : X' \rightarrow \mathbb{P}^{d-1}$ is smooth over some dense open $U \subset \mathbb{P}^{d-1}$ and f is its own Stein factorization (we will omit this).

As the relative smooth locus of f is open and f is proper, we just need to show that this open set contain some siber X'_y . By some commutative algebra (see [27] Lemma 2.8) we just need to show that X'_y is smooth. Hence we need to find $l \subset \mathbb{P}^d$ through ξ such that $\pi^{-1}(l)$ is smooth.

As X is normal, we get $\dim(X \setminus X^{\text{sm}}) \geq d - 2$. As π is constructed by projecting from $N - d + 1$ independent points in \mathbb{P}^N , and ξ corresponds to another point. Overall, this corresponds to $(N - d + 1)$ planes in \mathbb{P}^N . Use Bertini's theorem many times and well done.

♣ **Step 3. Construct good curve fibrations II.**

The main result are follows:

Lemma 20.3.6 (The three-point lemma, de Jong [27] Lemma 4.13). *Suppose we have $f : X \rightarrow Y$ is a map of projective varieties over $k = \bar{k}$ with $d = \dim X \geq 2$. Let all fibers of f are geometrically connected of pure dimension 1 and the smooth locus $\text{sm}(f)$ is fiberwise dense over Y . Then there exists a Cartier divisor $D \subset X$ such that*

- (i) $D \rightarrow Y$ is finite and generically étale;
- (ii) for all geometrical points $y \in Y$, $\text{sm}(f) \cap D$ meets each irreducible component of X_y in at least 3 points.

Proof. This is very interesting but we will omit it, the details see [23] Lemma 6.2 or [27] Lemma 4.13. □

Remark 20.3.7. *Hence in our case we may replace Z by $(Z \cup D)_{\text{red}}$.*

Lemma 20.3.8. *By passing to further alteration Y of \mathbb{P}^{d-1} , one can let $Z = \bigcup_{i=1}^r \sigma_i(Y)$ for sections $\sigma_i : Y \rightarrow X$ such that $\{\sigma_i(\eta_Y)\}$ are pairwise distinct (which is preserved under further alterations of Y).*

Proof. Omitted, see [23] Lemma 7.2. □

Hence we have the folloing now:

► **Situation.** Now we get the following situation of the alteration theorem of de Jong:

- (1) X is a projective normal variety over $k = \bar{k}$ of dimension $d \geq 2$ with $Z \subset X$ a Cartier divisor;
- (2) A surjective $f : X \rightarrow Y$ where Y is a projective variety of dimension $d - 1$ such that

- * All fibers of f are geometrically connected of pure dimension 1;
- * The smooth locus $\text{sm}(f)$ is fiberwise dense over Y ;
- * f is smooth over a dense open $U \subset Y$;
- * $Z \rightarrow Y$ is finite surjective and generically étale;
- * $\text{sm}(f) \cap Z$ meets each X_y for geometrical points $y \in Y$ in at least 3 points per irreducible component of X_y ;
- * $Z = \bigcup_{i=1}^r \sigma_i(Y)$ for sections $\sigma_i : Y \rightarrow X$ such that $\{\sigma_i(\eta_Y)\}$ are pairwise distinct (which is preserved under further alterations of Y).

♣ **Step 4. Construct good curve fibrations III.**

Chapter 21

A glimpse of Kontsevich's moduli space of stable maps

Part IV
Appendix

Appendix A

Some basic result in scheme theory

A.1 Some corollaries of semi-continuity theorem

Review A.1.1 (Cohomology and Base Change, see [58] III.12.11). *Let $f : X \rightarrow Y$ be a proper and finitely presented morphism of schemes with a finitely presented sheaf on X which is flat over Y . Let a point $y \in Y$ and $i \in \mathbb{Z}$, the comparison map $\phi_y^i : R^i f_* F \otimes \kappa(y) \rightarrow H^i(X_y, F_y)$ is surjective. Then*

(i) *There is an open neighborhood $V \subset Y$ of y such that for any morphism $g : Y' \rightarrow V$ of schemes, the comparison map $\phi_{Y'}^i : g^* R^i f_* F \rightarrow R^i f'_*(g')^* F$ is an isomorphism. In particular ϕ_y^i is an isomorphism;*

(ii) *ϕ_y^{i-1} is surjective if and only if $R^i f_* F$ is locally free in some neighborhood of y .*

Review A.1.2 (Grauert's Corollary). (See [1] A.7.16) *Let $f : X \rightarrow Y$ be a flat proper morphism of noetherian schemes such that $h^0(X_y, \mathcal{O}_y) = 1$ for all $y \in Y$ ($\Leftrightarrow \mathcal{O}_Y = f_* \mathcal{O}_X$ and stable under base-change) (resp. the geometric fibers are integral).*

For a line bundle L on X , consider the functor $(\text{Sch}/Y) \rightarrow (\text{Sets})$ by sending $T \rightarrow Y$ to $\{\}$ if L_T is the pullback of a line bundle on T and to \emptyset otherwise. Then this functor is representable by a locally closed (resp. closed) subscheme of Y .*

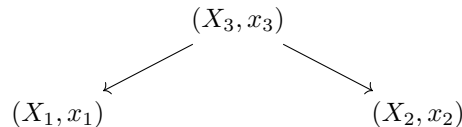
A.2 Artin approximation and its corollaries

Definition A.2.1. *Let $A \rightarrow B$ of noetherian rings is called geometrically regular if it is flat and for every prime ideal $\mathfrak{p} \subset A$ and any finite field extension $K/\kappa(\mathfrak{p})$, the fiber $B \otimes_A K$ is regular.*

A noetherian local ring R is called a G -ring if $A \rightarrow \hat{A}$ is geometrically regular.

Theorem A.2.2 (Artin approximation, see [1] A.10.9). *Let S be a scheme and $s \in S$ be a point such that $\mathcal{O}_{S,s}$ is G -ring. Let $F : (\text{Sch}/S) \rightarrow (\text{Sets})$ be a colimit preserving contravariant functor (commutes with systems of \mathcal{O}_S -algebras) and $\hat{\xi} \in F(\text{Spec} \hat{\mathcal{O}}_{S,s})$. For any integer $N \geq 0$, there exists an étale morphism $(S', s') \rightarrow (S, s)$ and $\xi' \in F(S')$ with $\kappa(s) = \kappa(s')$ such that the restrictions of $\hat{\xi}$ and ξ' to $\text{Spec}(\hat{\mathcal{O}}_{S,s}/\mathfrak{m}_s^{N+1})$ are equal.*

Corollary A.2.3 (See [1] A.10.13). *Let X_1, X_2 be schemes of finite type over S and let $s \in S$ be a point such that $\mathcal{O}_{S,s}$ is a G -ring. If $x_1 \in X_1, x_2 \in X_2$ are points over s such that $\hat{\mathcal{O}}_{X_1, x_1}$ and $\hat{\mathcal{O}}_{X_2, x_2}$ are isomorphic as $\mathcal{O}_{S,s}$ -algebras, then there exists a common residually-trivial étale neighborhood as*



A.3 Miscellany

Review A.3.1. *Let k be a field and X be a proper geometrically connected and geometrically reduced k -scheme, then $\Gamma(X, \mathcal{O}_X) = k$.*

Proof. This is almost trivial. See [47] Proposition 12.66 or St 0BUG in [11]. \square

Review A.3.2 (Openness of ampleness). *Let $X \rightarrow S$ be a proper morphism of schemes and L be a line bundle over X . Let S is noetherian. If for some $s \in S$, the fiber L_s over X_s is ample (resp. very ample), then exists an open neighborhood U of s such that L_U is ample (resp. very ample) over X_U .*

Proposition A.3.3 (St 0C45). *Let X be a locally Noetherian scheme of dimension 1 with normalization $f : \tilde{X} \rightarrow X$. Then*

- (1) *f is integral (finite if X is reduced locally finite type over a field), surjective, and induced a bijection on irreducible components;*
- (2) *there is a factorization $\tilde{X} \rightarrow X_{\text{red}} \rightarrow X$ and the morphism $\tilde{X} \rightarrow X_{\text{red}}$ is the normalization of X_{red} and birational;*
- (3) *for every closed point $x \in X$, stalk $(f_* \mathcal{O}_{\tilde{X}})_x$ is the integral closure of $\mathcal{O}_{X,x}$ in total ring of fractions of $(\mathcal{O}_{X,x})_{\text{red}} = \mathcal{O}_{X_{\text{red}},x}$;*
- (4) *\tilde{X} is a disjoint union of integral normal Noetherian schemes.*

Proposition A.3.4 (0B5V). *Let R be a Noetherian ring. Let $f : X \rightarrow Y$ be a morphism of schemes proper over R . Let L be an invertible \mathcal{O}_Y -module. Assume f is finite and surjective. Then L is ample if and only if f^*L is ample.*

Proposition A.3.5 (Canonical bundle and blowing ups). *Let X be a regular variety and let Y be a regular subvariety of codimension $r \geq 2$. Let $\pi : X' = \text{Bl}_Y X \rightarrow X$ be the blowing up with exceptional divisor E , then*

$$\omega_{X'} \cong \pi^* \omega_X \otimes \mathcal{O}_{X'}((r-1)E).$$

Proof. This is Exercise II.8.5 in [58]. \square

Proposition A.3.6 (Picard groups and blowing ups). *Let X be a regular variety and let Y be a regular subvariety of $\text{codim}_X(Y) = r \geq 2$. Let $\pi : X' = \text{Bl}_Y X \rightarrow X$ be the blowing up of X along Y and let E be the exceptional divisor. Then the map $\pi^* : \text{Pic}(X) \rightarrow \text{Pic}(X')$ given by functoriality of the Picard group and the map $\mathbb{Z} \rightarrow \text{Pic}(X')$ defined by $n \mapsto nE$ define an isomorphism $\text{Pic}(X) \oplus \mathbb{Z} \cong \text{Pic}(X')$.*

Proof. Let $U = X - Y$ and we have $\text{Pic}(X) \cong \text{Pic}(U)$ is similar as $\text{Pic}(X) \xrightarrow{\pi^*} \text{Pic}(X') \rightarrow \text{Pic}(U)$. Hence $\text{Pic}(X) \xrightarrow{\pi^*} \text{Pic}(X') \rightarrow \text{Pic}(X)$ is identity. Consider $\mathbb{Z} \rightarrow \text{Pic}(X') \rightarrow \text{Pic}(X) \rightarrow 0$ is exact, we just need to find a splitting for $\mathbb{Z} \rightarrow \text{Pic}(X')$.

The closed immersion induce $\text{Pic}(X') \rightarrow \text{Pic}(E)$. As E is a projective bundle over Y , then $\text{Pic}(E) \cong \text{Pic}(Y) \oplus \mathbb{Z}$ as regularness by [58] Exercise II.7.9(a). Hence we get

$$f : \mathbb{Z} \rightarrow \text{Pic}(X') \rightarrow \text{Pic}(E) \cong \text{Pic}(Y) \oplus \mathbb{Z} \rightarrow \mathbb{Z}$$

which sends $1 \mapsto \mathcal{O}_{X'}(E) \cong \mathcal{O}_{X'}(-1) \mapsto \mathcal{O}_E(-1) \mapsto -1$. Hence consider $-f$ and we win! \square

Definition A.3.7. *Fix a variety X over a field.*

- (i) *Define the canonical ring $R(X) = \bigoplus_{m \geq 0} H^0(X, mK_X)$, we define the Kodaira dimension as*

$$\kappa(X) := \begin{cases} -\infty, & \text{if } R(X) = \mathbb{C}, \\ \text{trdeg}_{\mathbb{C}} \text{Frac}(R(X)) - 1, & \text{otherwise;} \end{cases}$$

- (ii) *Define X is of general type if $\kappa(X) = \dim X$ (This if and only if K_X is big).*

Definition A.3.8. *About rational, unirational, uniruled. To add.*

Appendix B

Some results of resolution of singularities for surfaces

Theorem B.0.1 (Minimal Resolutions). *Let X be a surface. There exists a unique projective birational morphism $\pi : \tilde{X} \rightarrow X$ from a smooth surface such that every other resolution $Y \rightarrow X$ factors as $Y \rightarrow \tilde{X} \rightarrow X$ (or equivalently such that $K_X \cdot E \geq 0$ for every π -exceptional curve E).*

Proof. See [67] Theorem 2.16. □

Theorem B.0.2 (Embedded Resolutions of Curves in Surfaces). *Let X be a surface and $X_0 \subset X$ be a curve. There is a finite sequence of blow-ups at reduced points of X_0 yielding a projective birational morphism $\tilde{X} \rightarrow X$ such that \tilde{X} is smooth and such that the preimage \tilde{X}_0 of X_0 has set-theoretic normal crossings, i.e. $(\tilde{X}_0)_{red}$ is nodal.*

Proof. See [67] Theorem 1.47. □

Theorem B.0.3 (Castelnuovo's Contraction Theorem). *Let X be a smooth projective surface and E a smooth rational (-1) -curve. Then there is a projective morphism $X \rightarrow Y$ to a smooth surface and a point $y \in Y$ such that $f^{-1}(y) = E$ and $X \setminus E \rightarrow Y \setminus \{y\}$ is an isomorphism.*

Proof. See [67] Theorem 2.14. □

Corollary B.0.4 (Existence of Relative Minimal Models). *A smooth surface X admits a projective birational morphism $X \rightarrow X_{min}$ to a smooth surface such that every projective birational morphism $X_{min} \rightarrow Y$ to a smooth surface is an isomorphism. In particular X_{min} has no smooth rational (-1) -curves.*

Appendix C

Basic theory of algebraic spaces and stacks

C.1 Some basic facts

Theorem C.1.1. (See [75] 8.3.3) Let \mathcal{X}/S be an algebraic stack, then the following statement

- (a) \mathcal{X} is a Deligne-Mumford stack;
- (b) the diagonal $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is formally unramified;
- (c) for any algebraic closed field k and any point $x \in \mathcal{X}(k)$, the group scheme $\underline{\text{Aut}}_x$ is reduced finite k -scheme.

Then (a) \Leftrightarrow (b), and if \mathcal{X} noetherian, then (a) \Leftrightarrow (b) \Leftrightarrow (c).

Theorem C.1.2. Let \mathcal{X} be a smooth noetherian algebraic stack over k and $x \in \mathcal{X}(k)$ be a point with smooth stabilizer. Then

$$\dim_x \mathcal{X} = \dim T_{\mathcal{X},x} - \dim G_x.$$

Theorem C.1.3 (Valuative Criteria). Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of noetherian algebraic stacks. Assume f is of finite type and with separated diagonals. Then consider any DVR and its fraction field K with a 2-commutative diagram

$$\begin{array}{ccc} \text{Spec}K & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \text{Spec}R & \longrightarrow & \mathcal{Y} \end{array}$$

Then

- (1) f is proper if and only if there exists an extension of DVRs $R \rightarrow R'$ and $K \rightarrow K'$ of fraction fields having finite transcendence degree and a lifting unique up to unique isomorphism

$$\begin{array}{ccccc} \text{Spec}K' & \longrightarrow & \text{Spec}K & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & \nearrow \text{dashed} & \downarrow \\ \text{Spec}R' & \longrightarrow & \text{Spec}R & \longrightarrow & \mathcal{Y} \end{array}$$

- (2) f is separated if and only if every two liftings of the first diagram are uniquely isomorphic.
- (3) f is universally closed if for the first diagram, there exists an extension of DVRs $R \rightarrow R'$ and $K \rightarrow K'$ of fraction fields having finite transcendence degree and a lifting as in the second diagram.

Proof. See [1] Theorem 3.8.5 or St 0CLY. □

Proposition C.1.4. Let \mathcal{X} be a Deligne-Mumford stack with an étale cover $X \rightarrow \mathcal{X}$.

- (A) Let $\text{Qcoh}(\mathcal{X})$ be the category of the quasi-coherent sheaves over \mathcal{X} , an object F of it defined as:
 - (A1) A quasi-coherent sheaf F_f on S_{zar} for any $f : S \rightarrow \mathcal{X}$ where S be a scheme;

(A2) An isomorphism $\rho_H : h^*F_g \cong F_f$ for any 2-diagram

$$\begin{array}{ccc} S & \xrightarrow{h} & T \\ f \downarrow & \Downarrow & \swarrow g \\ \mathcal{X} & & \end{array}$$

of schemes;

(A3) For any pair of morphisms $H_1 : f_1 \rightarrow f_2, H_2 : f_2 \rightarrow f_3$ where $f_i : S_i \rightarrow \mathcal{X}$ are schemes, the diagram

$$\begin{array}{ccc} h_1^*(h_2^*(F_{f_3})) & \xrightarrow{\cong} & (h_2 \circ h_1)^*(F_{f_3}) \\ \downarrow h_1^*(\rho_{H_2}) & & \downarrow \rho_{H_2 \circ H_1} \\ h_1^*(F_{f_2}) & \xrightarrow{\rho_{H_1}} & F_{f_1} \end{array}$$

of isomorphisms of sheaves over S_1 commutes.

(B) Let $\text{Eqcoh}(\mathcal{X})$ be the category of the extended quasi-coherent sheaves over \mathcal{X} , an object F of it defined as:

(B1) A quasi-coherent sheaf F_f on S_{zar} for any étale map $f : S \rightarrow \mathcal{X}$ from a scheme;

(B2) An isomorphism $\rho_H : h^*F_g \cong F_f$ for any 2-diagram

$$\begin{array}{ccc} S & \xrightarrow{h} & T \\ f \downarrow & \Downarrow & \swarrow g \\ \mathcal{X} & & \end{array}$$

of étale maps of schemes;

(B3) For any pair of étale morphisms $H_1 : f_1 \rightarrow f_2, H_2 : f_2 \rightarrow f_3$ where $f_i : S_i \rightarrow \mathcal{X}$ are schemes, the diagram

$$\begin{array}{ccc} h_1^*(h_2^*(F_{f_3})) & \xrightarrow{\cong} & (h_2 \circ h_1)^*(F_{f_3}) \\ \downarrow h_1^*(\rho_{H_2}) & & \downarrow \rho_{H_2 \circ H_1} \\ h_1^*(F_{f_2}) & \xrightarrow{\rho_{H_1}} & F_{f_1} \end{array}$$

of isomorphisms of sheaves over S_1 commutes.

(C) Let $\text{Qd}_X(\mathcal{X})$ be the category of the quasi-coherent sheaves over X with descent data related to $X \rightarrow \mathcal{X}$.

Conclusion. Then there are equivalence

$$\text{Qcoh}(\mathcal{X}) \cong \text{Eqcoh}(\mathcal{X}) \cong \text{Qd}_X(\mathcal{X})$$

and their composition, in any one of the three possible orders, is isomorphic to the appropriate identity functor.

Proof. See [8] Proposition XIII.2.9. □

Theorem C.1.5 (Local structure of DM-stacks). *Let \mathcal{X} be a separated Deligne-Mumford stack and $x \in \mathcal{X}(k)$ be a geometric point with stabilizer G_x . Then exists an affine and étale map*

$$f : ([\text{Spec}A/G_x], w) \rightarrow (\mathcal{X}, x)$$

where $w \in (\text{Spec}A)(k)$ such that f induces an isomorphism of the stabilizer groups at w . Moreover, it can be arranged that $f^{-1}(BG_x) \cong BG_w$.

Proof. See [1] Theorem 4.2.1. □

Theorem C.1.6 (Local structure of coarse moduli space). *Let \mathcal{X} be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S . Let $\pi : \mathcal{X} \rightarrow X$ be its coarse moduli space. For any closed point $x \in |\mathcal{X}|$ with geometric stabilizer G_x , there exists a cartesian*

$$\begin{array}{ccc} [\mathrm{Spec}A/G_x] & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}A^{G_x} & \xrightarrow{s} & X \end{array}$$

such that s is an étale neighborhood of $\pi(x) \in |X|$.

Proof. Follows from the construction in the proof of the Keel-Mori theorem (see [1] Theorem 4.3.20). See [1] Corollary 4.3.23. \square

C.2 Miscellany

Theorem C.2.1 (Le Lemme de Gabber). *Let \mathcal{X} be a Deligne-Mumford stack separated and of finite type over a noetherian scheme S . Then there exists a finite, generically étale and surjective morphism $Z \rightarrow \mathcal{X}$ where Z be a scheme.*

Proposition C.2.2. *Let \mathcal{X} be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S . Let $\pi : \mathcal{X} \rightarrow X$ be the coarse moduli space. If \mathcal{L} is a line bundle on \mathcal{X} , then for N sufficiently divisible $\mathcal{L}^{\otimes N}$ descends to X .*

Proof. See [1] Proposition 4.3.37. \square

Proposition C.2.3. *Let G be an algebraic group acting on a scheme H , hence we get a quotient stack $[H/G]$. Then we have $\mathrm{Qcoh}([H/G]) \cong \mathrm{Qcoh}(H, G)$ where the latter is the category of the G -equivariant quasi-coherent sheaf over H .*

Proof. See [8] Proposition XIII.2.19. \square

Corollary C.2.4. *We have a group isomorphism $\mathrm{Pic}([H/G]) \cong \mathrm{Pic}(H, G)$.*

Appendix D

A glimpse of the Moduli of higher dimensional varieties

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