Term exam. in abstract algebra

Taishan College. Shandong Univ.

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Answer the following questions and give all essential details.

1. (i) Let p be a rational prime. Find the splitting field K of

$$X^4 - p \tag{0.1}$$

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and find all the subfields of K.

(ii) State and prove Gauss' lemma. Prove the following theorem. Let R be a UFD with its quotient field K and suppose $f(X) = a_0X^n + a_1X^{n-1} \cdots + a_n \in R[X]$ satisifies the condition: For a prime p,

$$p | a_i, 1 < i < n, \quad p \nmid a_0, \quad p^2 \nmid a_n. \tag{0.2}$$

Then prove that f is irreducible in $K[X]$.

(iii) Find a primitive element of K.

2. (i) State the motivations why you study linear representations of finite groups.

(ii) State Schur's lemma and prove it.

(iii) Prove that irreducible representations of an Abelian group are of degree 1.

(iv) State the direct sum decomposition of an Abelian group G of order $n > 1$. Determine all irreducible representations of G.

3. Let p be a prime. (i) State the definition of the ring $\mathbb{Z}_p$ of p adic integers.

(ii) Find the field K which is obtained from $\mathbb{Q}$ by adjoining all p-th power roots of 1.

(iii) Prove that the Galois group Gal($K/\mathbb{Q}$) is isomorphisc to the unit gorup of $\mathbb{Z}_p$.

Gal(K/Q)