The Mid-term Examination of Abstract Algebra

1. (10) Assume that F is a field, G is a finite subgroup of $(F^*, *)$, prove that G is cyclic.

2. (10) Assume that R is a commutative ring, I_1, I_2 are two ideals of R which satisfy $I_1 + I_2 = R$, prove that $I_1^2 + I_2^2 = R$.

3. (10) Assume that G is a group of order n, p is the minimal prime number dividing n, H is a subgroup of G and $(G : H) = p$, prove that H is a normal subgroup of G.

4. (10) Assume K/F is a finite normal extension, E is an intermediate field. Prove that E/F is normal iff for all automorphisms σ of K over F, $\sigma(E) = E$.

5. (1) (10) Prove that all groups of order 99 are abelian.

   (2) (10) Prove that all groups of order 56 are not simple.

6. (10) Let E be a finite separable extension of F, $[E : F] = n$, $\alpha \in F$, calculate $Tr_F^E(\alpha)$ and $N_F^E(\alpha)$.

7. (10) Assume that A is a commutative ring, $\mathfrak{N} = \{x \in A \mid \exists n \in \mathbb{N}\ s.t.\ x^n = 0\}$ is its nilradical, $SpecA$ is the set of all prime ideals of A, prove that

$$\mathfrak{N} = \bigcap_{p \in SpecA} p.$$

(Hint: To prove $\mathfrak{N} \supseteq \bigcap_{p \in SpecA} p$, you may take $f \in A$ which is not nilpotent and let Σ be the set of all ideals I of A satisfying $\forall n \in \mathbb{N}$, $f^n \notin I$. Then use the Zorn's lemma and prove the maximal element of Σ is prime. Of course, you can also use other methods.)

8. (20) Let $\xi_n = e^{\frac{2\pi i}{n}}$, prove that

   (1) $Gal(\mathbb{Q}(\xi_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^*$,

   (2) $\mathbb{Q}(\xi_n) \cap \mathbb{R} = \mathbb{Q}(\xi_n + \frac{1}{\xi_n})$.

   (3) Determine all the intermediate fields of $\mathbb{Q}(\xi_9)/\mathbb{Q}$.