Homework 3 – A Report on Numerical Differentiation and Integration

Xiaolong Liu, 201900170025

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Shandong University, Taishan College

Abstract

In this report, we use three-point and five-point formula to compute $f'(1)$ where $f(x) = \ln(x^2 + e^x)$ and compare them. Similarly, we use Composite Simpson's Rule and Romberg Integration to compute $\int_0^1 e^x dx$ and compare them.

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1 Introductions

In this report, we use three-point and five-point formula to compute $f'(1)$ where $f(x) = \ln(x^2 + e^x)$ and compare them. Similarly, we use Composite Simpson's Rule and Romberg Integration to compute $\int_0^1 e^x dx$ and compare them.

2 Algorithm Description & Mathematical Derivation

2.1 Numerical Differentiation

Theorem 2.1.1. *Actually, for* $\{x_0, ..., x_n\} \subset I$ *and* $f \in C^{n+1}(I)$*, we have*

$$f'(x_j) = \sum_{k=0}^{n} f(x_k) L'_k(x_j) + \frac{f^{(n+1)}(\xi(x_j))}{(n+1)!} \prod_{k=0, k\neq j}^{n} (x_j - x_k),$$

which is called an $(n+1)$*-point formula*

Actually, we often choose three-point formulas or five-point formulas. As follows (We now only use midpoint case):

Corollary 2.1.2 (XX-Point Midpoint Formula)**.** *(1)(Three-Point) There exists $\xi \in (x_0 - h, x_0 + h)$, we have*

$$f'(x_0) = \frac{1}{2h}(f(x_0+h) - f(x_0-h)) - \frac{h^2}{6} f^{(3)}(\xi);$$

(2)(Five-Point) There exists $\xi \in (x_0 - 2h, x_0 + 2h)$, we have

$$f'(x_0) = \frac{1}{12h}(f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)) + \frac{h^4}{30} f^{(5)}(\xi).$$

2.2 Composite Simpson's Rule on Numerical Integration

Theorem 2.2.1 (Composite Simpson's rule)**.** *Let $f \in C^4[a,b]$ and n be even. Let $h = \frac{b-a}{n}$ and $x_j = a + jh$ for $j = 0, 1, ..., n$. There exists a $\mu \in (a,b)$ for which the Composite Simpson's rule for n subintervals can be written with its error term as*

$$\int_a^b f(x)dx = \frac{h}{3}\left(f(a) + 2\sum_{j=1}^{n/2-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right) - \frac{b-a}{180} h^4 f^{(4)}(\mu).$$

2.3 Romberg Integration on Numerical Integration

Theorem 2.3.1 (Composite Trapezoidal rule)**.** *Let $f \in C^2[a,b]$ and $h = \frac{b-a}{n}$. Let $x_j = a + jh, j = 0, 1, ..., n$. There exists a $\mu \in (a,b)$ for which the Composite Trapezoidal rule for n subintervals can be written with its error term as*

$$\int_a^b f(x)dx = \frac{h}{2}\left(f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right) - \frac{b-a}{12} h^2 f''(\mu).$$

Theorem 2.3.2 (Romberg Integration)**.** *Let $h_k = \frac{b-a}{2^{k-1}}$ and we let $R_{1,1}, R_{2,1}, ...$ as the Composite Trapezoidal rule when $n = 1, 2, 4, 8, 16, ...$ Or we can write as*

$$R_{k,1} = \frac{1}{2}\left(R_{k-1,1} + h_{k-1}\sum_{i=1}^{2^{k-2}} f(a + (2i-1)h_k)\right), k = 2, 3, ..., n,$$

Then we have

$$R_{k,j} = R_{k,j-1} + \frac{1}{4^{j-1}-1}(R_{k,j-1} - R_{k-1,j-1}), k = j, j+1, ...$$

3 Codes and Numerical Experiments

3.1 Numerical Differentiation

Problem 3.1.1. *Given a function $f(x) = \ln(x^2 + e^x)$, we want to use three-midpoint and five-midpoint methods to calculate $f'(1)$ and compare their errors. (We let $h = 0.1$)*

Solution. Consider the following code:

```
>> format long
>> syms x
>> f=@(x) log(x^2+exp(x));
>> f_d1=5*(f(1.1)-f(0.9))

f_d1 =

   1.268915539908338

>> f_d2=(1/1.2)*(f(0.8)-8*f(0.9)+8*f(1.1)-f(1.2))

f_d2 =

   1.268958142011042

>> syms g(x)
>>g(x) = log(x^2+exp(x));
>> Dg=diff(g,x)

Dg(x) =

(2*x + exp(x))/(exp(x) + x^2)

>> DG = double(Dg(1))

DG =

   1.268941421369995

>> abs(f_d1-DG)

ans =

     2.588146165671823e-05

>> abs(f_d2-DG)

ans =

     1.672064104685234e-05
```

Then we the errors of Three-midpoint are bigger than Five-midpoint. ☐

3.2 Numerical Integration

Problem 3.2.1. *Suppose that $f(x) = e^x, x \in [1, 2]$. And we will compute $\int_1^2 e^x dx$ by using composite Simpson's Rule and Romberg integration.*

Solution. We first using composite Simpson's Rule to compute it as follows

```
>> n = input('Input a number:\n');
x=1:1/n:2;
h=1/n;I_11=0;I_12=0;
for j1 = 1 : 1 : n/2-1
    I_11 = I_11+exp(x(1,2*j1));
end
for j2 = 1 : 1 : n/2
    I_12 = I_12+exp(x(1,2*j2-1));
end
I_1=(h/3)*(exp(1)+exp(2)+2*I_11+4*I_12);

Input a number:
4
>> I_1

I_1 =

   3.823992286458485

Input a number:
100
>> I_1

I_1 =

   4.624634522526311
```

We next using Romberg integration. We first we construct a function about Composite Trapezoidal rule:

```
function I_2=CTR(n)
x=1:1/n:2;
h=1/n;I_21=0;
for j1 = 1 : 1 : n-1
    I_21 = I_21+exp(x(1,j1));
end
I_2=(h/2)*(exp(1)+exp(2)+2*I_21);
end
```

Then we have

```
>> n = input('Input a number:\n');
R=zeros(n);
for i = 1 : n
    R(i,1) = CTR(2^(i-1));
end
for i = 2 : n
    for j = 2 : n
        R(i,j)=R(i,j-1)+1/(4^(j-1)-1)*(R(i,j-1)-R(i-1,j-1));
    end
end

Input a number:
```

```
4
>> R(n,n)

ans =

   4.317773074621213

Input a number:
10
>> R(n,n)

ans =

   4.665217247366002
```

The precise solution is:

```
>> syms x
>> int(exp(x),x,1,2)

ans =

exp(2) - exp(1)

>> double(ans)

ans =

   4.670774270471605
```

Then we find that Romberg integration is much precise than composite Simpson's Rule. □

4 Discussions and Conclusions

Actually, the errors of Three-midpoint are bigger than Five-midpoint in numerical differentiation. Romberg integration is much precise than composite Simpson's Rule in numerical integration.

(Probably because I just finished the MATLAB exam (reviewed for a long time), this time I wrote the code very smoothly.)

References

[1] Richard L. Burden, J. Douglas Faires, Annette M. Burden, *Numerical Analysis (Tenth Edition)*, Cengage Learning, 2014.