2022 Final Exam of Analytic Number Theory

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Problem 1. *(i) State the Chebyshev inequality and Mertens' theorem without the proof;*

(ii) Show that

$$\varphi(q) \gg \frac{q}{\log\log q}.$$

Problem 2. *(i) Let $\chi_{\mathbb{P}}$ as $\chi_{\mathbb{P}}(n) = 1$ for n prime and $\chi_{\mathbb{P}}(n) = 0$ for n not a prime. Show that*

$$\chi_{\mathbb{P}} = \mu * \omega;$$

(ii) Prove that

$$\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)};$$

(iii) Prove that

$$\sum_{n\le x} \frac{n}{\varphi(n)} = x \sum_{d|n} \frac{\mu^2(d)}{d\varphi(d)} + O(\log x).$$

Problem 3. *Prove that*

$$\sum_{n\le x,(n,q)=1} 1 = \frac{\varphi(q)}{q} x + O(q^{\varepsilon}),$$

for all $\varepsilon > 0$.

Problem 4. *Let* $\pi(x) = \sum_{p\le x} 1$ *and* $\vartheta(x) = \sum_{p\le x} \log p$, *then show that the following statements are equivalent:*

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right);$$

$$\vartheta(x) = x + O\left(\frac{x}{\log x}\right).$$

Problem 5. *Prove that the equation* $x^2 + 2x - 60 \equiv (\mathrm{mod} 2022)$ *has no integral solutions. (Remark:* 337 *is a prime.)*

Problem 6. *(i) Prove that*

$$\log(n) = \sum_{d|n} \Lambda(d);$$

(ii) Let χ *be a Dirichlet character mod* q, *then assume that* $L(1,\chi) \neq 0$, *show that*

$$\sum_{n\le x} \frac{\chi(n)\Lambda(n)}{n} = O(1).$$

Problem 7. *(i) State the Pólya-Vinogradov inequality without proof;*

(ii) For any $M > N > 0$, *show that*

$$\sum_{N<n\le M} \frac{\chi(n)}{n} \ll \frac{1}{N}\sqrt{q}\log q.$$

Problem 8. *Prove the Dirichlet theorem: giving* $Q > 0$ *and* $x \in \mathbb{R}$, *then exists* $a, q \in \mathbb{Z}$ *with* $(a,q) = 1$ *such that*

$$|qx - a| < \frac{1}{Q}.$$