PDE 复习

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§0. ~~Prel~~ 预备知识.

▷ **Thm 0.1.** (Green – Gauss).

(i) $\vec{u}\in C^1(\bar{U},\mathbb{R}^n)$, then
$$\int_U \operatorname{div}\vec{u}\,dx=\int_{\partial U}\vec{u}\cdot\vec{n}\,ds;$$

(ii) Taking $\vec{u}=(0,\dots,0,\underset{i\text{-th}}{u},0,\dots,0)$
$$\Rightarrow\int_U\frac{\partial u}{\partial x_i}dx=\int_{\partial U}un^i\,ds;$$

(iii). Combine these:
$$\int_U\nabla u\,dx=\int_{\partial U}\nabla u\cdot\vec{n}\,ds.$$

▷ **Thm 0.2.** (分部积分)

设 $u,v\in C^1(\bar{U})$, 则
$$\int_U v\frac{\partial u}{\partial x_i}dx=-\int_U u\frac{\partial v}{\partial x_i}dx+\int_{\partial U}uvn^i\,ds.$$

proof. Use thm 0.1 (ii) to uv. ∎

▷ **Thm 0.3.** (Green). Let $u,v\in C^2(\bar{U})$. Then

(i) $\int_U\Delta u\,dx=\int_{\partial U}\frac{\partial u}{\partial\vec{n}}ds$;

(ii) $\int_U Dv\cdot Du\,dx=-\int_U u\Delta v\,dx+\int_{\partial U}u\frac{\partial v}{\partial\vec{n}}ds$

(iii) $\int_U(u\Delta v-v\Delta u)\,dx=\int_{\partial U}\left(u\frac{\partial v}{\partial\vec{n}}-v\frac{\partial u}{\partial\vec{n}}\right)ds$.

proof. (i) Use Thm 0.1 (i) by $\vec{u}=Du$. ✓

(ii). 用 Thm 0.2 with $v\to v_{x_i}$. ✓

(iii) By (ii) directly. ∎

▷ **Thm 0.4.** (Polar).

(i) $f:\mathbb{R}^n\to\mathbb{R}$ continuous & summable, then $\forall x_0\in\mathbb{R}^n$, we have
$$\int_{\mathbb{R}^n}f\,dx=\int_0^\infty\left(\int_{\partial B(x_0,r)}f\,ds\right)dr;$$

(ii) As special,
$$\frac{d}{dr}\left(\int_{B(x_0,r)}f\,dx\right)=\int_{\partial B(x_0,r)}f\,ds$$

[For more general, Coarea formula:

Let u ⇒ Lipschitz, a.e. $r\in\mathbb{R}$ s.t. $\{x\in\mathbb{R}^n\mid u(x)=r\}$ ⇒ smooth hypersurface of $\mathbb{R}^n$.

Let f continuous & summable
$$\Rightarrow\int_{\mathbb{R}^n}f|Du|\,dx=\int_{\mathbb{R}}\left(\int_{\{u=r\}}f\,ds\right)dr$$

▷ **Thm 0.5** (Moving Regions).

若 $f=f(x,\tau)$ smooth, then
$$\frac{d}{d\tau}\int_{U(\tau)}f\,dx=\int_{\partial U(\tau)}f\vec{v}\cdot\vec{n}\,ds+\int_{U(\tau)}f_\tau\,dx$$
where $\vec{v}$ ⇒ velocity of $\partial U(\tau)$. ◀

▷ **Corollary.** 若 f ⇒ good enough, then
$$\frac{d}{dx}\int_{a(x)}^{b(x)}f(x,t)\,dt=\int_{a(x)}^{b(x)}\frac{\partial f}{\partial x}dt+f(x,b(x))b'(x)-f(x,a(x))a'(x).$$ ✓

§I. 波动方程.

§I.1. 达朗贝尔.

$$\begin{cases}u_{tt}-a^2u_{xx}=f(x,t)\\ u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x)\end{cases}$$

⇓分解

(I) $\begin{cases}u_{tt}-a^2u_{xx}=0\\ u|_{t=0}=\varphi(x),\ u_t|_{t=0}=\psi(x)\end{cases}$ & (II) $\begin{cases}u_{tt}-a^2u_{xx}=f(x,t)\\ u|_{t=0}=u_t|_{t=0}=0.\end{cases}$

▷ **Thm I.1.1.** 达朗贝尔公式: $\varphi\in C^2(\mathbb{R})$, $\psi\in C^1(\mathbb{R})$, 则∃!解:

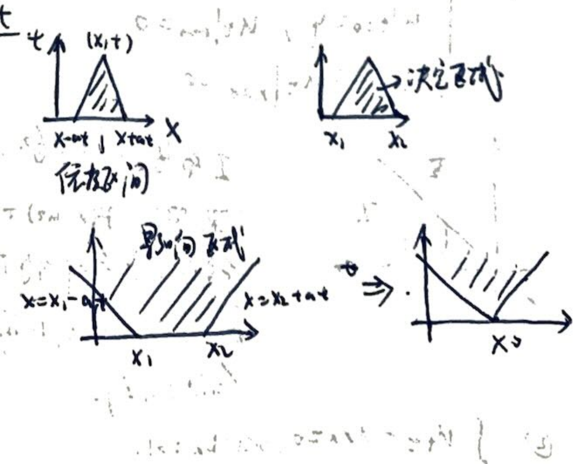
(I) 解为 $u(x,t)=\frac{1}{2}(\varphi(x-at)+\varphi(x+at))+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)\,d\alpha.$

proof. 作 $\xi=x-at$, $\eta=x+at$

$u_{tt}-a^2u_{xx}=0\Rightarrow u_{\xi\eta}=0.$

$\Rightarrow u=\underbrace{F(x-at)}_{\text{右传播波}}+\underbrace{G(x+at)}_{\text{左传播波}}$ ∎

▷ **Fact**



▷ Thm I.1.2. (齐次化原理).

对(III) $\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t) \\ u|_{t=0} = u_t|_{t=0} = 0. \end{cases}$

设 $W(x,t;\tau)$ 为

$$\begin{cases} W_{tt} - a^2 W_{xx} = 0, & (t>\tau) \\ t=\tau: W=0, & W_t = f(x,\tau) \end{cases}$$

的解, 则

$$u(x,t) = \int_0^t W(x,t;\tau)\,d\tau \text{ 为(III)的解.}$$

即 $u(x,t) = \frac{1}{2a}\iint_G f(\xi,\tau)\,d\xi d\tau$,

G为 (x,t) 的依赖区间的决定区域 

「一般齐次化原理:

$W(x,t;\tau)$ 满足

$$\begin{cases} \frac{\partial^m W}{\partial t^m} = LW, & t>\tau>0 \\ W|_{t=\tau} = \cdots = \frac{\partial^{m-2} W}{\partial t^{m-2}}\Big|_{t=\tau} = 0, \\ \frac{\partial^{m-1} W}{\partial t^{m-1}}\Big|_{t=\tau} = f(\tau,x) \end{cases}$$

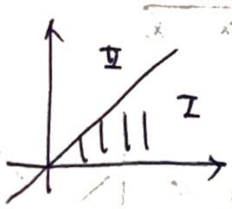
则

$$\begin{cases} \frac{\partial^m u}{\partial t^m} = Lu + f(t,x), & t>0. \\ u|_{t=0} = \cdots = \frac{\partial^{m-1} u}{\partial t^{m-1}}\Big|_{t=0} = 0 \end{cases}$$

的解为 $u(t,x) = \int_0^t W(x,t;\tau)\,d\tau$. ⊥」

[一般带边界的技法 ⇒ 特征线法].

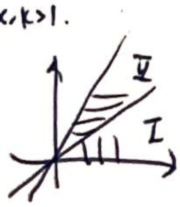
例如① $\begin{cases} u_{tt} - a^2 u_{xx} = 0, & x>0, t>0. \\ u|_{t=0} = \varphi, & u_t|_{t=0} = 0 \\ u_x - k u_t|_{x=0} = 0. \end{cases}$



I 可由达朗贝尔公式.

II 写为 $F(x-at) + G(x+at)$ 代入边界条件得 F&G关系 再代入特征线上的相容条件(与I).

② $\begin{cases} u_{tt} - u_{xx} = 0, & 0<t<kx, k>1. \\ u|_{t=0} = \varphi_0, & x\ge 0 \\ u_t|_{t=0} = \varphi_1, & x\ge 0 \\ u|_{t=kx} = \psi(x) \end{cases}$



类似!

§I.2. 分离变量.

考虑 $\begin{cases} u_{tt} - a^2 u_{xx} = f(x,t) \\ u|_{t=0} = \varphi, \quad u_t|_{t=0} = \psi \\ u|_{x=0} = 0, \quad u|_{x=l} = 0. \end{cases}$ (齐次边界条件).

⇒ (I) $\begin{cases} u_{tt} - a^2 u_{xx} = 0 \\ u|_{t=0} = \varphi, \ u_t|_{t=0} = \psi \\ u|_{x=0} = u|_{x=l} = 0 \end{cases}$ & (II) $\begin{cases} u_{tt} - a^2 u_{xx} = f \\ u|_{t=0} = u_t|_{t=0} = 0 \\ u|_{x=0} = u|_{x=l} = 0. \end{cases}$

解(I). 用 $u(x,t) = X(x)T(t)$

代入 ⇒ $\begin{cases} T'' + \lambda a^2 T = 0 \\ X'' + \lambda X = 0. \end{cases}$

对λ分类讨论:

① $\lambda<0$, $X = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x}$ (不行)

② $\lambda=0$, $X \equiv 0$

③ $\lambda>0$, $X = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$.

⇒ 边界条件 ⇒ λ_k ⇒ $X_k(t)$.

代回 ⇒ $T_k(t)$

⇒ $u(x,t) = \sum_{k=1}^{\infty} X_k(t) T_k(t)$.

代初值 ⇒ ✓

▷ Thm I.2.1. (齐次化原理).

类似, W为 $\begin{cases} W_{tt} - a^2 W_{xx} = 0, & t>\tau \\ t=\tau: W=0, & W_t = f(x,\tau) \\ x=0,l: W=0. \end{cases}$

⇒ $u(x,t) = \int_0^t W(x,t;\tau)\,d\tau$. ✓

▷ Thm I.2.2. 边界齐次化

$$\begin{cases} u_{tt} - a^2 u_{xx} = f \\ u|_{t=0} = \varphi, \ u_t|_{t=0} = \psi \\ u|_{x=0} = \mu_1(t) \\ u|_{x=l} = \mu_2(t). \end{cases}$$

令 $U(x,t) = \mu_1 + \frac{x}{l}(\mu_2 - \mu_1)$

$V = u - U$ ✓

§I.3. 高维波动的 Cauchy 问题.

(I) 为 $\begin{cases} u_{tt}=a^2\Delta u \\ u|_{t=0}=\varphi \\ u_t|_{t=0}=\psi \end{cases}$

边界 ⇒ Dirichlet: $u|_{\partial\Omega}=\mu$

Neumann: $\frac{\partial u}{\partial n}|_{\partial\Omega}=\mu$

Robin: $\left(\frac{\partial u}{\partial n}+\sigma u\right)\Big|_{\partial\Omega}=\mu$. ⊥.

▷ Thm I.3.1. (球平均 ⇒ 三维). [Poisson公式]

对 $\begin{cases} u_{tt}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \\ u|_{t=0}=\varphi \\ u_t|_{t=0}=\psi. \end{cases}$

设 $\varphi=C^3,\ \psi\in C^2$. 则 ∃! 解:

$$u(x,y,z,t)=\frac{\partial}{\partial t}\left(\frac{1}{4\pi a^2 t}\int_{S_{at}^M}\varphi\, dS\right)+\frac{1}{4\pi a^2 t}\int_{S_{at}^M}\psi\, dS$$

其中 S_{at}^M ⇒ $M=(x,y,z)$ 球心, $R=at$ 半径球!

▷ Thm. I.3.2 (降维 ⇒ 二维). [Poisson公式]

对 $\begin{cases} u_{tt}=a^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \\ u|_{t=0}=\varphi,\ u_t|_{t=0}=\psi. \end{cases}$

则 $u(x,y,t)$

$$=\frac{1}{2\pi a}\left(\frac{\partial}{\partial t}\iint_{\Sigma_{at}^M}\frac{\varphi(\xi,\eta)\,d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}+\iint_{\Sigma_{at}^M}\frac{\psi(\xi,\eta)\,d\xi d\eta}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}\right)$$

$$=\frac{1}{2\pi a}\left(\frac{\partial}{\partial t}\int_0^{at}\int_0^{2\pi}\frac{\varphi(x+r\cos\theta,\ y+r\sin\theta)}{\sqrt{(at)^2-r^2}}r\,d\theta\,dr+\int_0^{at}\int_0^{2\pi}\frac{\psi(x+r\cos\theta,\ y+r\sin\theta)}{\sqrt{(at)^2-r^2}}r\,d\theta\,dr\right).$$

$\Sigma_{at}^M:(\xi-x)^2+(\eta-y)^2\le a^2t^2.$

(II) 对 $\begin{cases} u_{tt}=a^2\Delta u+f \\ u|_{t=0}=0 \\ u_t|_{t=0}=0 \end{cases}$

▷ Thm I.3.3. (齐次化).

设 $w(x,y,z,t;\tau)$ 为 $\begin{cases} w_{tt}=a^2\Delta u \\ w|_{t=\tau}=0 \\ w_t|_{t=\tau}=f \end{cases}$

则 $u=\int_0^t w\,d\tau$. 为解!

由 Poisson 公式

$$\Rightarrow w=\frac{1}{4\pi a}\int_{S_{a(t-\tau)}^M}\frac{f(\xi,\eta,\zeta,\tau)}{a(t-\tau)}dS$$

$$\Rightarrow u=\frac{1}{4\pi a^2}\int_0^{at}\iint_{S_r^M}\frac{f(\xi,\eta,\zeta,t-\frac{r}{a})}{r}dS\,dr$$

$$=\frac{1}{4\pi a^2}\iiint_{r\le at}\frac{f(\xi,\eta,\zeta,t-\frac{r}{a})}{r}dV.$$

故对 $\begin{cases} u_{tt}=a^2\Delta u+f \\ u|_{t=0}=\varphi \\ u_t|_{t=0}=\psi. \end{cases}$

$$\Rightarrow u=\frac{\partial}{\partial t}\left(\frac{1}{4\pi a^2 t}\int_{S_{at}^M}\varphi\,dS\right)+\frac{1}{4\pi a^2 t}\int_{S_{at}^M}\psi\,dS+\frac{1}{4\pi a^2}\int_0^{at}\int_{S_r^M}\frac{f(\xi,\eta,\zeta,t-\frac{r}{a})}{r}dS\,dr.$$

T. $\int_0^{\pi}\sin^2x\,dx=\frac{\pi}{2}$; $\int_0^{\pi}\sin^3x\,dx=\frac{4}{3}$;

$\int_0^{\pi}\sin^4x\,dx=\frac{3\pi}{8}$; $\int_0^{\pi}\cos^2x\,dx=\frac{\pi}{2}$;

$\int_0^{\pi}\cos^3x\,dx=0$; $\int_0^{\pi}\cos^4 dx=\frac{3\pi}{8}$. ⊥

! RMK. 球坐标:

$\begin{cases} x=r\sin\theta\cos\varphi & ;\ 0\le\theta\le\pi \\ y=r\sin\theta\sin\varphi & ;\ 0\le\varphi\le 2\pi \\ z=r\cos\theta & ;\ r\ge 0. \end{cases}$

$dx\,dy\,dz=r^2\sin\theta\,dr\,d\theta\,d\varphi.$ ⊥

§Ⅰ.4. 波传播与衰减

▷ Thm Ⅰ.4.1. (后效 & 非后效).

$$\begin{cases} n=1\text{时，有后效 (弥散)} \\ n\equiv 1 \pmod 2,\ n>1\text{时，无后效 (惠更斯原理)} \\ n\equiv 0 \pmod 2\text{ 时，有后效！} \end{cases}$$

▷ Thm Ⅰ.4.2. Cauchy 问题下，

若 φ, ψ 紧支，则

$$\begin{cases} \text{三维} & u = O(t^{-1}) \\ \text{二维} & u = O(t^{-\frac{1}{2}}) \\ \text{一维} & \text{无衰减性！} \end{cases}$$

§Ⅰ.5. 能量 & 波动方程解唯一 & 稳定性.

略，与 §Ⅳ.4 合并！

§Ⅱ 热传导方程

$$\begin{cases} \Rightarrow \text{为 } u_t = a^2\Delta u + f \\ \text{初值为 } u|_{t=0} = \varphi \\ \text{边值为 } \begin{cases} ① \text{ Dirichlet: } u|_{\partial\Omega} = g \\ ② \text{ Neumann: } \frac{\partial u}{\partial n}\big|_{\partial\Omega} = g \\ ③ \text{ Robin: } \left(\frac{\partial u}{\partial n} + \sigma u\right)\big|_{\partial\Omega} = g. \end{cases} \end{cases}$$

§Ⅱ.1. 分离变量法 (一维).

以
$$\begin{cases} u_t - a^2 u_{xx} = 0 \quad (t>0,\ 0<x<l) \\ u|_{t=0} = \varphi(x) \\ u|_{x=0} = 0,\ (u_x + hu)|_{x=l} = 0. \end{cases}$$
为例

① 特解 $u(x,t) = X(x)T(t)$.

② 代入方程得 $\begin{cases} T' + \lambda a^2 T = 0 \\ X'' + \lambda X = 0. \end{cases}$

③ 代入边值 $\begin{cases} X(0) = 0 \\ X'(l) + hX(l) = 0. \end{cases}$

④ 对 $\lambda>0, =0, <0$ 讨论 ODEs

得到 λ_k 与 $X_k(x)$, $T_k(t) \Rightarrow u_k(x,t)$

☆

⑤ 叠加原理、 $u = \sum_{k=1}^{\infty} u_k(x,t)$.

⑥ 代入初值求系数

已说明 $\{\underbrace{\sin\sqrt{\lambda_k}\,x}_{X_k}\}$ 正交.

(由于 $\begin{cases} X_m X_n'' + \lambda_n X_m X_n = 0 \\ X_n X_m'' + \lambda_m X_m X_n = 0 \end{cases}$

⇓ 相减 积分

代入边值 ✓)

$t=0$

则由 $\varphi = \sum_k A_k \sin\sqrt{\lambda_k}\,x$

⇓

$$A_k = \frac{1}{M_k}\int_0^l \varphi(\xi)\sin\sqrt{\lambda_k}\,\xi\, d\xi.$$

§Ⅱ.2. Cauchy 问题

▷ Def. f 在 $\mathbb{R}$ 连续，$f'(x)$ 连续，

f 绝对可积，则

$$F[f](\lambda) = \int_{\mathbb{R}} f(\xi) e^{-i\lambda\xi} d\xi \Rightarrow \text{Fourier 变换}$$

$$F^{-1}[g](x) = \frac{1}{2\pi}\int_{\mathbb{R}} g(\lambda) e^{i\lambda x} d\lambda$$
⇒ Fourier 逆变换.

▷ Prop ① Fourier 变换为线性变换；

② $F[f_1 * f_2] = F[f_1]\cdot F[f_2]$

$(f_1 * f_2 = \int_{\mathbb{R}} f_1(x-t) f_2(t)\,dt)$.

③ $F[f_1\cdot f_2] = \frac{1}{2\pi} F[f_1] * F[f_2]$.

④ 若 f, f' 可 Fourier 变换 且 $\lim_{|x|\to\infty} f = 0$

$\Rightarrow F[f'(x)] = i\lambda F[f(x)]$

⑤ 若 f, xf 可 Fourier 变换，则

$\Rightarrow F[-ixf(x)] = \frac{d}{d\lambda} F[f]$.

Rmk. 高维 Fourier：

$F[f]=g(\lambda_1,\cdots,\lambda_n)=\int_{\mathbb{R}^n} f\cdot e^{-i(x_1\lambda_1+\cdots+x_n\lambda_n)}dx_1\cdots dx_n$

逆变换：

$F^{-1}[g]=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} g\cdot e^{i(x_1\lambda_1+\cdots+x_n\lambda_n)}d\lambda_1\cdots d\lambda_n$

▷ Thm II.2.1. $\begin{cases}u_t=a^2u_{xx}+f\\ u|_{t=0}=\varphi\end{cases}$

分为(I) $\begin{cases}u_t=a^2u_{xx}\\ u|_{t=0}=\varphi\end{cases}$ 与(II) $\begin{cases}u_t=a^2u_{xx}+f\\ u|_{t=0}=0\end{cases}$

(I)：取 对x而 Fourier ⇒ 令 $\begin{cases}F[u]=\tilde{u}\\ F[\varphi]=\tilde{\varphi}\end{cases}$

$\Rightarrow\begin{cases}\frac{d\tilde{u}}{dt}=-a^2\lambda^2\tilde{u}\\ \tilde{u}(\lambda,0)=\tilde{\varphi}(\lambda)\end{cases}$

$\Rightarrow \tilde{u}(\lambda,t)=\tilde{\varphi}(\lambda)e^{-a^2\lambda^2t}$

$F^{-1}[e^{-a^2\lambda^2t}]=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$

$\Rightarrow u=F^{-1}[\tilde{\varphi}\cdot e^{-a^2\lambda^2t}]$

$=\varphi * F^{-1}[e^{-a^2\lambda^2t}]$

$=\frac{1}{2a\sqrt{\pi t}}\int_{\mathbb{R}}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi$. (I)

(II) 用齐次化原理，设 $w(x,t;\tau)$

为 $\begin{cases}w_t=a^2w_{xx},\ t>\tau\\ w|_{t=\tau}=f(x,\tau)\end{cases}$ 的解

$\Rightarrow$ 由(I) 则

$u(x,t)=\int_0^t w(x,t;\tau)\,d\tau$

$=\frac{1}{2a\sqrt{\pi}}\int_0^t\int_{\mathbb{R}}\frac{f(\xi,\tau)}{\sqrt{t-\tau}}e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d\xi d\tau$. (II)

故 (I) & (II) 得

$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{\mathbb{R}}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}d\xi$

$+\frac{1}{2a\sqrt{\pi}}\int_0^t\int_{\mathbb{R}}\frac{f(\xi,\tau)}{\sqrt{t-\tau}}e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}d\xi$.

§II.3. 极值原理、定解问题的唯一性与稳定性

考虑 $u_t=a^2u_{xx}$.

▷ Thm. II.3.1.（极值原理）.

设 u 在 $R_T=\{\alpha\le x\le\beta;\ 0\le t\le T\}$ 连续，内部满足 $u_t=a^2u_{xx}$，令

$\Gamma_T=\{x=\alpha,x=\beta,\ 0\le t\le T\}\cup\{t=0,\ \alpha\le x\le\beta\}$.

则 $\begin{cases}\max\limits_{R_T}u=\max\limits_{\Gamma_T}u\\ \min\limits_{R_T}u=\min\limits_{\Gamma_T}u.\end{cases}$

▷ Thm II.3.2.（推广的极值原理）

Ω 为 $\mathbb{R}^n$ 有界区域，$Q_T=\Omega\times(0,T)$，$\Omega\subseteq\mathbb{R}^n$.

$u\in C^2(Q_T)\cap C^0(\overline{Q_T})$，在 Q_T 内

满足 $u_t-a^2\Delta u\le 0$，则

令 $\Sigma_T=\{t=0,x\in\Omega\}\cup(\partial\Omega)\times(0,T)$

则 $\max\limits_{Q_T}u=\max\limits_{\Sigma_T}u$.

Pf. 反证. 令 $M=\max\limits_{Q_T}u$，$m=\max\limits_{\Sigma_T}u$

设 $M>m$. 设 (x_0,t_0) 上取 M，$x_0\in\Omega$，$t_0\in(0,T)$

作 $V(x,t)=u(x,t)+\frac{M-m}{4\,\mathrm{diam}(\Omega)^2}|x-x_0|^2$.

Σ_T 上 $V<m+\frac{M-m}{4}=\theta M$，$\theta\in(0,1)$

$V(x_0,t_0)=M\Rightarrow V$ 不在 Σ_T 上取最大. 设 (x_1,t_1) 使 V 最大！

$\Rightarrow D_{ii}V\le 0$，$V_t\ge 0$ ($t_1<T$, $V_t=0$; $t_1=T$, $V_t\ge 0$)

$\Rightarrow(V_t-a^2\Delta V)(x_1,t_1)\ge 0$

但 $\Rightarrow(V_t-a^2\Delta V)(x_1,t_1)=u_t-a^2\Delta u-a^2\frac{M-m}{2\,\mathrm{diam}(\Omega)^2}$ 在(x_1,t_1)处 <0，矛盾！ ∎

▷Thm II.3.3. [初边值问题唯一性&稳定性]

(I) $\begin{cases} u_t = a^2 u_{xx} + f \\ u|_{t=0} = \varphi \\ u(\alpha,t) = \mu_1,\ u(\beta,t) = \mu_2 \end{cases}$

的唯一且连续依赖于初边值条件. (极值原理) ☒

(II) $\begin{cases} u_t - a^2 u_{xx} = 0 \\ u|_{t=0} = \varphi \\ u|_{x=0} = \mu_1,\ \left(\frac{\partial u}{\partial x} + hu\right)\Big|_{x=l} = \mu_2,\ h>0. \end{cases}$

RMK: 若为 $u_t - a^2 u_{xx} = f$, 则需令 $v = e^{-\lambda t} u$, $\lambda > 0$

导出 $|u| \le e^{\lambda T} \frac{1}{\lambda} \max\limits_{R_T} (e^{-\lambda t} f)$ 如下:

此时 $\begin{cases} v_t - a^2 v_{xx} + \lambda v = e^{-\lambda t} f \\ v|_{x=0} = e^{-\lambda t}\mu_1,\ \left(\frac{\partial v}{\partial x} + hv\right)\Big|_{x=l} = e^{-\lambda t}\mu_2 \\ v|_{t=0} = \varphi \end{cases}$

若 v 有正最大值 $\Rightarrow v_t \ge 0,\ v_{xx} \le 0,\ v > 0$

$\Rightarrow v = \frac{1}{\lambda}(e^{-\lambda t} f - v_t + a^2 v_{xx}) \le \frac{1}{\lambda} e^{-\lambda t} f$.

唯一性: 即 $\begin{cases} u_t - a^2 u_{xx} = 0 \\ u|_{t=0} = 0 \\ u|_{x=0} = \left(\frac{\partial u}{\partial x} + hu\right)\Big|_{x=l} = 0 \end{cases}$ 只零解!

若不然 ⇒ 由极值原理, 正极大值/负极小值在 Γ_T 取到. 由 $u|_{t=0} = u|_{x=0} = 0$

$\Rightarrow$ 在 $x = l$ 上取到 $\Rightarrow \frac{\partial u}{\partial x} \ge 0,\ hu > 0$, 矛盾!

稳定性. 只看 u 有正极大值. 设在 (x_0,t_0) 处取得 $u(x_0,t_0) > 0$. 由极值原理:

① 若 (x_0,t_0) 在 $t=0$ 或 $x=0$

$\Rightarrow u(x_0,t_0) \le \max\left(\max\limits_{0\le t\le T} \mu_1,\ \max\limits_{0\le t\le T} \varphi\right)$.

② 若在 $x = l$ 处 $\Rightarrow \frac{\partial u}{\partial x} \ge 0$

$\Rightarrow hu \le \mu_2 \Rightarrow u \le \frac{1}{h}\mu_2$

$\Rightarrow u(x_0,t_0) \le \max\limits_{0\le t\le T} \frac{1}{h}\mu_2$.

综上 $u \le \max\left(0,\ \max\limits_t \varphi,\ \max\limits_t \mu_1,\ \max\limits_t \frac{1}{h}\mu_2\right)$

同理 $u \ge \min\left(0,\ \min\limits_t \varphi,\ \min\limits_t \mu_1,\ \min\limits_t \frac{1}{h}\mu_2\right)$.

$\Rightarrow$ 稳定! ☒

(III). $\begin{cases} u_t - a^2 u_{xx} = 0,\ 0<x<l,\ t>0 \\ u|_{t=0} = \varphi \\ u|_{x=0} = \mu_1,\ u_x|_{x=l} = \mu_2. \end{cases}$

令 $\tilde{u} = (l-x+1)u$

变为 $\begin{cases} \tilde{u}_t - a^2 \tilde{u}_{xx} - \frac{2a^2}{l-x+1}\tilde{u}_x - \frac{2a^2}{(l-x+1)^2}\tilde{u} = 0 \\ \tilde{u}|_{t=0} = (l-x+1)\varphi \\ \tilde{u}|_{x=0} = (l+1)\mu_1,\ (\tilde{u}_x + \tilde{u})|_{x=l} = \mu_2 \end{cases}$

令 $v = e^{-\lambda t}\tilde{u}$, $\lambda > 2a^2$

$\Rightarrow \begin{cases} v_t - a^2 v_{xx} - \frac{2a^2}{l-x+1} v_x + \left(\lambda - \frac{2a^2}{(l-x+1)^2}\right) v = 0 \\ v|_{t=0} = (l-x+1)\varphi \\ v|_{x=0} = e^{-\lambda t}(l+1)\mu_1,\ (v_x + v)|_{x=l} = e^{-\lambda t}\mu_2 \end{cases}$

正极大不在内部取到, 否则

$v_t \ge 0,\ v_x = 0,\ v_{xx} \le 0,\ v > 0$

$\lambda = \frac{2a^2}{(l-x+1)^2} > 0$, 矛盾!

时间上讨论即可! ☒

▷Thm II.3.4 [Cauchy问题唯一&稳定.]

$\begin{cases} u_t = a^2 u_{xx} + f \\ u|_{t=0} = \varphi \end{cases}$

若 u 有界 ⇒ 唯一&稳定!

proof. $\forall (x_0,t_0)$, $t_0 > 0$. $|u| \le B$.

设 R_0: $0 \le t \le t_0$, $|x - x_0| \le L$. $L > 0$.

作 $v = \frac{4B}{L^2}\left(\frac{(x-x_0)^2}{2} + a^2 t\right)$

v 满足 $v_t = a^2 v_{xx}$, 且 $\begin{cases} v(x,0) \ge 0 = u(x,0) \\ v(x_0 \pm L, t) \ge 2B \ge u(x_0 \pm L, t) \end{cases}$

极值原理 $\Rightarrow v \ge u$, $\forall$ in R_0.

同理 $u \ge -v \Rightarrow (x_0,t_0)$ 处 $|u| \le \frac{4B}{L^2} a^2 t_0$

$L \to \infty \Rightarrow u(x_0,t_0) = 0 \Rightarrow$ 唯一!

稳定性类似, 设 $|\varphi| \le \eta$, 令 $\tilde{v} = v + \eta$, 同理! ☒

§.II.4. 解的渐近态

Thm II.4.1. [初边值]

例如 $\begin{cases} u_t = a^2 u_{xx} & (t>0)\ (0<x<l). \\ u|_{t=0} = \varphi \\ u|_{x=0} = 0,\ (u_x + hu)|_{x=l} = 0. \end{cases}$

设 $u(x,t) = \sum_{k=1}^{\infty} A_k e^{-a^2 \lambda_k t} \sin\sqrt{\lambda_k}\, x$

又知 $|A_k| \le C_1$ 且 $\lambda_k = O(k^2),\ k\to\infty$.

$$\Downarrow \quad \sum_{k=2}^{\infty} \frac{1}{\lambda_k - \lambda_1} < \infty$$

且 $(\lambda_k - \lambda_1) e^{-a^2(\lambda_k-\lambda_1)t} \le C_2$

故 $|u| \le C_1\left(1 + \sum_{k=2}^{\infty} e^{-a^2(\lambda_k-\lambda_1)t}\right) e^{-a^2\lambda_1 t}$

$$\le C_1\left(1 + \sum_{k=2}^{\infty} (\lambda_k-\lambda_1) e^{-a^2(\lambda_k-\lambda_1)t} \frac{1}{\lambda_k-\lambda_1}\right) e^{-a^2\lambda_1 t}$$

$$\le C_1\left(1 + C_2 \sum_{k=2}^{\infty} \frac{1}{\lambda_k-\lambda_1}\right) e^{-a^2\lambda_1 t}$$

$$\le C e^{-a^2\lambda_1 t}. \quad \blacksquare$$

Thm II.4.2. [Cauchy].

$\begin{cases} u_t = a^2 \Delta u, & x \text{ in } \mathbb{R}^n. \\ u|_{t=0} = \varphi, & \varphi \in L^1(\mathbb{R}^n). \end{cases}$

则

$$u(x,t) = \frac{1}{(2a\sqrt{\pi t})^n} \int_{\mathbb{R}^n} \varphi \cdot \exp\left(-\frac{\sum (x_i-\eta_i)^2}{4a^2 t}\right) d\eta$$

$$\Rightarrow |u| \le \frac{1}{(2a\sqrt{\pi t})^n} \int_{\mathbb{R}^n} |\varphi|\, dy = C \cdot t^{-\frac{n}{2}} \quad \blacksquare$$

§III. 调和方程

$$\Delta u = f$$

边值 $\begin{cases} \text{① Dirichlet 内: } u|_{\partial\Omega} = g \text{ \& 外} \\ \text{② Neumann 内: } \frac{\partial u}{\partial n}\big|_{\partial\Omega} = g \text{ \& 外}\cdot \left(\frac{\partial u}{\partial n_1}\right)\big|_{\partial\Omega} \\ \text{③ Robin: } \left(\frac{\partial u}{\partial n} + \sigma u\right)\big|_{\partial\Omega} = g. \end{cases}$

TRMK. 外问题 $\begin{cases} \text{3维之加 } \lim_{r\to\infty} u = 0 \\ \text{2维之加 } |u| < M,\ |r| > k \end{cases}$

Thm III.0.1. (变分法) $(\Omega \subseteq \mathbb{R}^2)$

例如 $\begin{cases} \Delta u = f \\ u|_{\partial\Omega} = 0 \end{cases}$ (I), 设 $J(u) = \iint_\Omega \left(\frac{1}{2}(\Delta u) + fu\right) dxdy$

~~时若u∈~~ 设 $V_0 = \{v \in C^2(\Omega) \cap C^1(\bar\Omega),\ v|_{\partial\Omega} = 0\}$

则 $\begin{cases} \text{若 } u \in V_0 \text{ 满 } J(u) = \min_{v\in V_0} J(v), \text{ 则 } u \text{ 为(I)解} \\ \text{若 } u \text{ 为(I)解} \Rightarrow J(u) = \min_{v\in V_0} J(v). \end{cases}$

证 $(\Rightarrow)$ $\forall w \in V_0$, 令 $v = u + \lambda w$

$\frac{dJ(v)}{d\lambda}\Big|_{\lambda=0} = 0$ & Green公式 (Thm 0.3(ii))

& w 任意性 $\Rightarrow$ ✓.

$(\Leftarrow)$ $\forall w \in V_0$, $\int_\Omega w(\Delta u - f) = 0$

Green公式 & Let $w = v - u$

$\Rightarrow J(v) \ge J(u)$. ✓ $\blacksquare$

§III.1. Green公式 & 调和函数.

Def. 基本解: $\begin{cases} \dim = 2: & \ln\frac{1}{r_{M_0M}} = G \\ \dim \ge 3: & \frac{1}{r_{M_0M}^{n-2}} = G \end{cases}$

由 Thm 0.3 (iii), 若 $\Delta u = F$, $\Omega \subseteq \mathbb{R}^3$, 则 ★★★

$$\boxed{u(M_0) = -\frac{1}{4\pi}\int_{\partial\Omega}\left(u(M)\frac{\partial}{\partial n}(G) - G\frac{\partial u}{\partial n}\right) dS_M - \frac{1}{4\pi}\int_\Omega FG\, d\Omega_M}$$

对称的!

若 $\Omega\subseteq\mathbb{R}^2$，$\Delta u=0$，则

$$u(M_0)=-\frac{1}{2\pi}\int_{\partial\Omega}\left(u\frac{\partial}{\partial n}G-G\frac{\partial u}{\partial n}\right)ds_M.$$

▷Thm.Ⅲ.1.1 [平均值] u 在 Ω 内 $\Delta u=0$.

$\iff \forall x_0\in\Omega$，设 $B_r(x_0)$ 完全在 Ω 内，则

$$u(x_0)=\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}u\,dy$$
$$=\frac{1}{\omega_n r^{n-1}}\int_{\partial B_r(x_0)}u\,ds$$

其中 ω_n 为单位球体积.

Pf. ($\Rightarrow$) 若 $\Delta u=0$，则

$$0=\int_{B_r}\Delta u\,dx=\int_{\partial B_r}\frac{\partial u}{\partial n}ds,$$

而 $\int_{\partial B_r}\frac{\partial u}{\partial n}ds=\int_{\partial B_r}\frac{\partial u}{\partial r}(x_0+r\omega)ds$

$$=r^{n-1}\int_{|\omega|=1}\frac{\partial u}{\partial r}(x_0+r\omega)\,ds$$
$$=r^{n-1}\frac{\partial}{\partial r}\int_{|\omega|=1}u(x_0+r\omega)\,d\omega$$

$\Rightarrow \int_{|\omega|=1}u(x_0+r\omega)\,d\omega=\text{const}$

令 $r\to 0$ & 中值 ✓

($\Leftarrow$ (用了后面方法)

$\forall$小球 B，$\begin{cases}\Delta v=0, & x\in B\\ v|_{\partial B}=u|_{\partial B}.\end{cases}$

知 Poisson 公式 $\Rightarrow$ v 存在唯一 v.

v 平均值 $\Rightarrow$ so 与 $u-v$

& 极值 $\Rightarrow u=v\Rightarrow \Delta u=0$. ✓ □.

▷Thm.Ⅲ.1.2 [极值] 设 u 非const，$u\in C(\Omega)$ 连续，且满足 $\forall B_R$，$u\le\frac{1}{|\partial B_R|}\int_{\partial B_R}u\,ds$. 则 u 的上(下)确界不在 Ω 内达到. [可设 $\Delta u\ge(\le)0$].

Pf. 反证. 不妨第一种. $\max_{\bar\Omega}u=m$，取 $B_R\subseteq\Omega$

$\partial B_R=S_R\subseteq\Omega$，设 $u(M_0)=m$，S_R 以 M_0 为中心

设 $\exists x_0\in S_R$，$u(x_0)<m$，$\frac{1}{|\partial B_R|}\int_{\partial B_R}u\,ds<\frac{1}{|\partial B_R|}\int_{\partial B_R}m\,ds=m$

但 $u(M_0)=m\le\frac{1}{|\partial B_R|}\int_{\partial B_R}u\,ds$，矛盾! ✓ □

「由此可得 Dirichlet 内外问题唯一性&稳定性」

(证外问题: $u=0$)

▷Thm.Ⅲ.1.3. $\mathbb{R}^n$ 上有界调和 $\Rightarrow$ 常数.

Pf. $\nabla u(x_0)=\frac{1}{|B_R|}\int_{B_R}\nabla u(y)\,dy$

$$=\frac{1}{|B_R|}\int_{\partial B_R(x_0)}u\,\vec n\,dS_R$$

$\Rightarrow |\nabla u(x_0)|\le\frac{n}{R}u(x_0)$..

$R\to\infty \Rightarrow |\nabla u(x_0)|=0\Rightarrow u\equiv\text{const}$. □

§Ⅲ.2. 椭圆方程极值原理

$Lu=a_{ij}D_{ij}u+b_iD_iu+cu$，$x\in\Omega$

这一节的假设：L 为椭圆算子，

记为假设(A) $\begin{cases}\text{即 } 0<\lambda(x)|\xi|^2\le a_{ij}(x)\xi_i\xi_j\le\Lambda(x)|\xi|^2, \\ a_{ij}=a_{ji}.\ \text{且 } a_{ij}\in C(\bar\Omega). \\ \text{且假设 } \frac{|c(x)|}{\lambda(x)},\ \frac{|\vec b(x)|}{\lambda(x)}\le M<\infty. \\ \text{[若 }\lambda\equiv\alpha\text{，且 }\Omega\text{ 有界，则自然成立!]}\end{cases}$

▷Thm.Ⅲ.2.1 [弱极值原理] L 满足(A).

Ω 有界，$u\in C^2(\Omega)\cap C(\bar\Omega)$. ($c\equiv 0$)，$Lu\ge 0$

则 $\max_{\bar\Omega}u=\max_{\partial\Omega}u^+$

(令 $u^+=\max(0,u)$，$u^-=\min(0,u)$.)

Pf. $\forall\varepsilon>0$. 令 $w=u+\varepsilon e^{\alpha x_1}$，$\alpha>0$ 待定.

知 $\alpha>>0$ 时，$Lw=Lu+\varepsilon(a_{11}\alpha^2+b_1\alpha+c)>0$.

若 $x_0\in\Omega$，$\max_{\bar\Omega}w=w(x_0)\ge 0$

$\Rightarrow \partial_i w(x_0)=0$，$(D_{ij}w(x_0))\le 0$

由 $(a_{ij}(x_0))>0 \Rightarrow a_{ij}D_{ij}w(x_0)$
$$=\alpha^T D_{ij}w(x_0)\,\alpha\le 0$$

$\Rightarrow Lw(x_0)\le 0$，矛盾

$$\Rightarrow \sup_{\bar\Omega}w=\sup_{\partial\Omega}(u+\varepsilon e^{\alpha x_1})^+$$
$$\le\sup_{\partial\Omega}u^+ +\varepsilon\sup_{\partial\Omega}e^{\alpha x_1}$$

$\varepsilon\to 0$. ✓ □

▷ **Thm. III.2.2.** [Hopf 极值原理].

L在(不必椭)Ω内(A), $u\in C^2(\Omega)\cap C(\bar{\Omega})$,
$c(x)\le 0$, $Lu(x)\ge 0$, $\forall x\in\Omega$. 设 $x_0\in\partial\Omega$ 使

$$\begin{cases} ① & u(x_0)\ge 0 \\ ② & u(x_0)>u(x),\ \forall x\in\Omega \\ ③ & \Omega \text{ 满足“内球条件”} \end{cases}$$

则 $\forall$ x_0处外向量 $\vec{v}$ 使 $\vec{v}\cdot\vec{n}(x_0)>0$, 有

$$\frac{\partial u}{\partial \vec{v}}(x_0)=\liminf_{t\to 0^+}\frac{1}{t}\left(u(x_0)-u(x_0-t\vec{v})\right)>0.$$

proof. 设 $B=B_R(y)\subseteq\Omega$ 切于 $x_0\in\partial\Omega$.

设 $h(x)=e^{-\alpha|x-y|^2}-e^{-\alpha R^2}$, $\alpha>0$待定

$$Lh=e^{-\alpha|x-y|^2}\left(4\alpha^2 a_{ij}(x_i-y_i)(x_j-y_j)-2\alpha a_{ij}\delta_{ij}-2\alpha b_i(x_i-y_i)\right)+ch$$

$$\ge e^{-\alpha|x-y|^2}\left(4\alpha^2\lambda|x-y|^2-2\alpha\left(a_{ii}+|b||x-y|\right)+c\right)$$

其中 $a_{ij}\xi_i\xi_j\ge\lambda|\xi|^2$.

且 $\lambda^{-1}a_{ij}$, $\lambda^{-1}|b|$, $\lambda^{-1}|c|\le M<\infty$

设 $A=B_R(y)\setminus B_\rho(y)$, $0<\rho<R$.

则 $\alpha>>0 \Rightarrow Lh(x)>0$, $\forall x\in A$.

由 $u<u(x_0)$, 取 $\varepsilon>0$

使 $u(x)-u(x_0)+\varepsilon h(x)\le 0$, $\forall x\in\partial B_\rho(y)$.

($h\equiv 0$, $\forall x\in\partial B_R(y)$)

$\Rightarrow u(x)-u(x_0)+\varepsilon h(x)\le 0$, $\forall x\in\partial A$.

而 $L(u(x)-u(x_0)+\varepsilon h(x))$

$=Lu(x)+\varepsilon Lh(x)-cu(x_0)$

$>-cu(x_0)\ge 0$

由弱极值 $\Rightarrow \underbrace{u(x)-u(x_0)+\varepsilon h(x)}_{}\overset{\equiv v(x)}{}\le 0$, $\forall x\in A$.

故 $0\le\dfrac{v(x_0)-v(x_0-t\vec{v})}{t}=-\dfrac{1}{t}v(x_0-t\vec{v})$

$\le\dfrac{1}{t}\left(u(x_0)-u(x_0-t\vec{v})\right)-\dfrac{\varepsilon}{t}h(x_0-t\vec{v})$

$\Rightarrow \liminf\limits_{t\to 0^+}\dfrac{1}{t}\left(u(x_0)-u(x_0-t\vec{v})\right)$

$\ge-\varepsilon\dfrac{\partial h}{\partial\vec{v}}(x_0)$

$=\varepsilon\cdot 2\alpha e^{-\alpha R^2}(x_0-y)\cdot\vec{v}>0.$ ■

「RMK. 若 $c\equiv 0$, 则不需要 $u(x_0)\ge 0$」

「RMK. 若 $u(x_0)=0$, 则 $c(x)\le 0$ 也不需要.」

▷ **Thm. III.2.3.** [强极值原理]

L在Ω(不必椭)满足(A). $u\in C^2(\Omega)\cap C(\bar{\Omega})$
$c(x)\le 0$, $Lu(x)\ge 0$, $\forall x\in\Omega$. 则若 $u\ne$ const
则 u 无法在Ω内取$\boxed{\text{非负}}$极大值.

[若 $c\equiv 0$, 且 u 内部取极大 $\Rightarrow u\equiv$ const] ⇒ RMK.

Pf. $0\le M=\max\limits_{\bar{\Omega}}u<\infty$. 设 $\Sigma=u^{-1}(M)$.

若 $\Sigma\ne\phi$ 且 $u\ne$ const $\Rightarrow \Omega\setminus\Sigma$ 为开集

取球 $B\subseteq\Omega\setminus\Sigma$ 使 $x_0\in\partial B\cap\Sigma$ 相切

$\Rightarrow u(x)<u(x_0)$, $\forall x\in B$, $u(x_0)=M$.

Hopf原理 $\Rightarrow \dfrac{\partial u}{\partial\vec{n}}(x_0)>0$

但 $x_0\in\Omega\Rightarrow\nabla u(x_0)=0$, 矛盾! ■

▷ **Coro.** [比较原理] L在有界区域Ω满足(A)

$u\in C^2(\Omega)\cap C(\bar{\Omega})$, $c(x)\le 0$, $Lu(x)\ge 0$.

若 $u(x)\le 0$, $\forall x\in\partial\Omega\Rightarrow u(x)\le 0$, $\forall x\in\Omega$. ■

▷ **Coro.** [唯一性] 用Hopf可知
第二边值问题的 $\Delta u=f$ 的解唯一(差一常数),
内外问题. ■

「RMK. 能量法: 对 $\begin{cases}\Delta u=f\\ u|_{\partial\Omega}=g \text{ 或 } \frac{\partial u}{\partial n}|_{\partial\Omega}=g\end{cases}$

考虑 $\begin{cases}\Delta v=0\\ v|_{\partial\Omega}=0 \text{ 或 } \frac{\partial v}{\partial n}|_{\partial\Omega}=0.\end{cases}$

令 $E(v)=\frac{1}{2}\int_\Omega\nabla v\cdot\nabla v\,dx$

$=-\frac{1}{2}\int_\Omega(v\Delta v)\,dx+\frac{1}{2}\int_{\partial\Omega}v\frac{\partial v}{\partial n}\,ds$

$\Rightarrow E(v)=\frac{1}{2}\int_{\partial\Omega}v\frac{\partial v}{\partial n}\,ds$ ✓ ■

§四.3 调和方程 & Green函数

$\Omega\subseteq\mathbb{R}^n$,

Δ的基本解为 $\Gamma(x)=\begin{cases}\frac{1}{2\pi}\log|x-x_0|, & n=2\\ \frac{1}{\omega_n(2-n)}|x-x_0|^{2-n}, & n\geq 3\end{cases}$

可得Green公式:

$$u(x_0)=\int_\Omega \Gamma\Delta u+\int_{\partial\Omega}\left(u\frac{\partial\Gamma}{\partial n}-\Gamma\frac{\partial u}{\partial n}\right)ds.$$

$\boxed{\text{本书}}$ 取 $g(x,x_0)=\begin{cases}\frac{1}{4\pi|x-x_0|}, & n=3\\ \frac{1}{2\pi}\ln\frac{1}{|x-x_0|}, & n=2\end{cases}$

取此函数

故 $u(x_0)=-\int_{\partial\Omega}\left(u\frac{\partial g}{\partial n}-g\frac{\partial u}{\partial n}\right)ds-\int_\Omega g\Delta u\,dx$

Green函数

设 $\boxed{G(x,x_0)=g(x,x_0)-\Phi(x,x_0)}$

其中 $\begin{cases}\Delta\Phi=0 \text{ in } \Omega\\ \Phi|_{\partial\Omega}=g|_{\partial\Omega}\end{cases}$, 故 $\begin{cases}\Delta G=0\\ G|_{\partial\Omega}=0\end{cases}$

若 $\begin{cases}\Delta u=f, x\in\Omega\\ u|_{\partial\Omega}=\varphi\end{cases}$ $\Rightarrow$ $\boxed{u(x_0)=-\int_{\partial\Omega}\varphi\frac{\partial G}{\partial n}ds-\int_\Omega G(x,x_0)f\,dx.}$

▷ prop 1. $G(x,y)=G(y,x)$.

pf. $\forall x\neq y\in\Omega$, 取 $B_r(x), B_r(y)\subseteq\Omega$ 无交.

$\Omega_r=\Omega\setminus B_r(x)\cup B_r(y)$.

设 $u(z)=G(x,z)$, $v(z)=G(y,z)$.

$\Rightarrow \Delta u=\Delta v=0$ in Ω_r.

$\Rightarrow$ Green: $\int_{\partial B_r(x)}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds+\int_{\partial B_r(y)}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds=0$

v在x处 & u在y处光滑!

$\exists K$ s.t. $\left|\int_{\partial B_r(x)}u\frac{\partial v}{\partial n}ds\right|\leq K\int_{\partial B_r(x)}u\,ds$

$=K\int_{\partial B_r(x)}\left(g(x,z)-\Phi(x,z)\right)ds$

$=K\frac{1}{4\pi r}4\pi r^2-K\cdot\Phi^*\cdot 4\pi r^2$

$=Kr-K\Phi^*4\pi r^2$

故 $\lim_{r\to0}\int_{\partial B_r(x)}u\frac{\partial v}{\partial n}ds=0.$

而由调和函数表示公式

$\Rightarrow v(x)=-\int_{\partial B_r(x)}v\frac{\partial u}{\partial n}ds$

$\Rightarrow$ 综上有 $\lim_{r\to0}\int_{\partial B_r(x)}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds=v(x)$

同理 $\lim_{r\to0}\int_{\partial B_r(y)}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds=-u(y)$.

$\Rightarrow u(y)=v(x)\Rightarrow G(x,y)=G(y,x)$. □

▷ prop 2. Ω内有:

$$0<G(M,M_0)<\frac{1}{4\pi|M-M_0|^2}$$

pf. $G(M,M_0)=\frac{1}{4\pi|M-M_0|^2}-\Phi(x,x_0)$

由 $\Phi|_{\partial\Omega}>0$, 极值原理 $\Rightarrow\Phi>0\Rightarrow G<\frac{1}{4\pi r^2}$

而 $\lim_{M\to M_0}G(M,M_0)=+\infty$

$\Rightarrow\exists r>0$ s.t. $\forall z\in B_r(M_0)$, $G(z,M_0)>0$.

而 $G(x,M_0)$ 在 $\Omega\setminus B_r(M_0)$. 由极值原理 $\Rightarrow G(M,M_0)>0$. □

▷ Prop 3. $\int_{\partial\Omega}\frac{\partial G}{\partial n}ds=-1$.

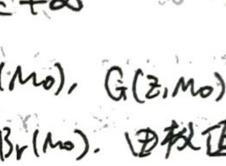
pf. $\begin{cases}\Delta u=0\\ u|_{\partial\Omega}=1\end{cases}$ 有解 $u=-\int_{\partial\Omega}\frac{\partial G}{\partial n}ds$.

而由 $u\equiv1$, 唯一性

$\Rightarrow\int_{\partial\Omega}\frac{\partial G}{\partial n}ds=-1$. □

▷ Thm 四.3.1. (静电源像法) & 球

$r_{OM_0}\cdot r_{OM_1}=R^2$, 取 $\mathbb{R}^n$



$\Rightarrow$ 反演点! $\rho_0=r_{OM_0}$

此时 $\Phi(M,M_0)$

$$\begin{cases}=\frac{1}{4\pi}\frac{R}{\rho_0}\frac{1}{r_{M_1M}}=\frac{1}{4\pi}\frac{R}{|O-M_0|}\frac{1}{|M-M_1|}, & n=3\\ =\frac{1}{2\pi}\ln\frac{R}{\rho_0}\frac{1}{r_{M_1M}}, & n=2\end{cases}$$

$\Rightarrow$ 令 $\gamma = \langle OM_0, OM \rangle$

$$\Rightarrow \begin{cases} \Delta u = 0 & 成为1. \\ u|_{\partial B_R} = f \end{cases}$$

$$u(M_0) = \begin{cases} \frac{1}{4\pi R}\int_{\partial B_R} \frac{R^2-\rho_0^2}{(R^2+\rho_0^2-2R\rho_0\cos\gamma)^{\frac{3}{2}}} f(M)\, dS_M \\ = \frac{R}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(R,\theta,\varphi)\frac{R^2-\rho_0^2}{(R^2+\rho_0^2-2R\rho_0\cos\gamma)^{\frac{3}{2}}}\sin\theta\, d\theta\, d\varphi & ,n=3. \\ \frac{1}{2\pi R}\int_{x^2+y^2=R^2} \frac{R^2-\rho_0^2}{R^2-2R\rho_0\cos\gamma+\rho_0^2} f(\theta)\, ds \\ = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-\rho_0^2)f(\theta)}{R^2+\rho_0^2-2R\rho_0\cos(\theta-\theta_0)}\, d\theta & ,n=2. \end{cases}$$

$\boxed{}$

▷ Thm. III. 3.2 (Harnack 第一定理). Ω 有界.

$\{u_k\}$ 在 Ω 调和函数列, 在 $\bar\Omega$ 上连续.

(I) [第一定理] 若在 $\partial\Omega$ 上一致收敛.

$\Rightarrow$ 也在 $\bar\Omega$ 上一致收敛

且 $\lim_k u_k = u$ 调和!;

(II) [第二定理] 若在 Ω 内某点 P 收敛.

则 Ω 内处处 $\to u$ 调和, 且在 Ω 内闭子域一致. $\boxed{}$

▷ Prop. (Harnack 不等式).

若 u 为非负调和函数, 则在 B_R 内有

$$\frac{R-\rho_0}{(R+\rho_0)^2}R\cdot u(P) \le u(Q) \le \frac{R+\rho_0}{(R-\rho_0)^2}R\cdot u(P)$$

其中 $B_R = B_R(P)$, Q 为 B_R 内 $\forall$ 一个点. $\boxed{}$

▷ Coro. u 在 Ω 内非负调和, $\forall$ 闭子域 $K \subseteq \Omega$

$\exists\, C = C(K) > 0$ 使 $\max_K u \le C \cdot \min_K u$

Pf. $R = \min|K - \partial\Omega|$, K 由 N 个 $\frac{R}{2}$ 球覆盖, $0 \le \rho_0 \le \frac{R}{2}$ 有 $\frac{(R-\rho_0)R}{(R+\rho_0)^2} \ge \frac{2}{9}$, $\frac{(R+\rho_0)R}{(R-\rho_0)^2} \le 6$

Harnack 不等式 $\Rightarrow \frac{2}{9}u(P_1) \le 6\, u(P_2)$ $\boxed{}$

$\Rightarrow$ 取 $C = (27)^N$.

▷ Thm. III. 3.3. (可去奇点)

$\Omega \subseteq \mathbb{R}^n$, u 在 A 邻域内除 A 外为调和函数, 若

$$\begin{cases} u(M) = o\left(\frac{1}{r_{AM}}\right) \ (M\to A). & n=3. \\ u(M) = o\left(\ln\frac{1}{r_{AM}}\right), (M\to A) & n=2. \end{cases}$$

则 $u \Rightarrow$ 延拓至 Ω.

Pf. 统一记 $h_{AM} = \begin{cases} \frac{1}{r_{AM}}, & n=3 \\ \ln\frac{1}{r_{AM}}, & n=2 \end{cases}$

取 $K \Rightarrow A$ 为心, R 为半径的球

设 u_1 满足 $\begin{cases} \Delta u_1 = 0, & 在 K \\ u_1|_{\partial K} = u|_{\partial K} \end{cases}$

记 $w = u - u_1$ 满足

$$\Rightarrow \begin{cases} \Delta w = 0, & 在 K \\ w|_{\partial K} = 0, & w = o(h_{AM}). \end{cases}$$

作 $w_\varepsilon = \begin{cases} \varepsilon\left(h_{AM} - \frac{1}{R}\right) & ,n=3 \\ \varepsilon\left(h_{AM} - \ln\frac{1}{R}\right) & ,n=2 \end{cases}$

$\Rightarrow w_\varepsilon$ 满足 $\begin{cases} \Delta w_\varepsilon = 0, & K \backslash \{A\} \\ w_\varepsilon|_P = 0. \end{cases}$

另外 $\exists\, \delta > 0 \Rightarrow \partial B_\delta$ 上有 $|w| \le w_\varepsilon$ (因 $w = o(h_{AM})$)

由极值 $\Rightarrow \forall M^* \in K\backslash\{A\}$,

$|w(M^*)| \le w_\varepsilon(M^*)$.

$\varepsilon \to 0 \Rightarrow w(M^*) = 0 \Rightarrow w|_{K\backslash\{A\}} \equiv 0$ $\boxed{}$

§IV. 二阶PDE的分类与估计

§IV.1. 分类

形状：$a_{11}u_{xx}+2a_{12}u_{xy}+a_{22}u_{yy}+b_1u_x+b_2u_y+cu=f$.

⇒ 设 $\Delta=a_{12}^2-a_{11}a_{22}$

则 $\begin{cases}\Delta>0\Rightarrow \text{双曲}\\ \Delta=0\Rightarrow\text{抛物}\\ \Delta<0\Rightarrow\text{椭圆}\end{cases}$

标准型：$u_{\xi\xi}+u_{\eta\eta}=Au_\xi+Bu_\eta+Cu+D$.

☆ 特征方程：$\boxed{a_{11}dy^2-2a_{12}dxdy+a_{22}dx^2=0}$

得到 $\frac{dy}{dx}=\frac{a_{12}\pm\sqrt{a_{12}^2-a_{11}a_{22}}}{a_{11}}$.

① $\Delta>0$ ⇒ 得到实曲线 $\varphi_1=C$, $\varphi_2=C$

作 $\begin{cases}\xi=\varphi_1(x,y)\\ \eta=\varphi_2(x,y)\end{cases}$ (可逆)

⇒ $u_{\xi\eta}=Au_\xi+Bu_\eta+Cu+D$.

[还可进一步令 $\xi=\frac{1}{2}(s+t)$, $\eta=\frac{1}{2}(s-t)$]

② $\Delta=0$ ⇒ 只有一族曲线 $\varphi=C$.

作 $\begin{cases}\xi=\varphi(x,y)\\ \eta=\tilde\varphi(x,y)\end{cases}$，其中 $\tilde\varphi$ 任意且与 φ 函数无关！

⇒ $u_{\eta\eta}=Au_\xi+Bu_\eta+Cu+D$

③ $\Delta<0$ ⇒ $\varphi=\varphi_1\pm i\varphi_2=C$

作 $\begin{cases}\xi=\varphi_1\\ \eta=\varphi_2\end{cases}$ 其中 φ_1,φ_2 函数无关

⇒ $u_{\xi\xi}+u_{\eta\eta}=Au_\xi+Bu_\eta+Cu+D$. ◭

RMK. 一般 n 维. 含 $\sum\limits_{i,j}a_{ij}D_{ij}u+\sum\limits_i b_iD_iu+cu=f$ ，且 $a_{ij}=a_{ji}$.

作 $A=(a_{ij})$，则

$\begin{cases}A>0\text{ 或 }A<0\Rightarrow\text{椭圆}\\ A\text{ 退化}\Rightarrow\text{抛物}\\ A\text{ 不退化且不定}\Rightarrow\text{双曲}\end{cases}$

§IV.2. 先验估计

Part I. 最大模估计

▷ Thm. IV.2.1. Ω 有界区域，

L 椭圆算子满足(A)，$u\in C^2(\Omega)\cap C(\bar\Omega)$.

$\begin{cases}Lu=f, & x\in\Omega\\ u|_{\partial\Omega}=\varphi\end{cases}$，$f,\varphi$ 连续.

若 $c(x)\le0$，设 $\Phi=\max\limits_{\partial\Omega}|\varphi|$，$F=\max\limits_{\Omega}|f|$

则 $|u|\le\Phi+C\cdot F$，$C>0$ 为常数

Pf. 设 $\Omega\subseteq\{x: 0<x_1<d\}$

设 $v(x)=\Phi+(e^{\alpha d}-e^{\alpha x_1})F$，$\alpha>0$ 待定

⇒ $v(x)>0$，$\forall x\in\Omega$，且

$Lv=-(a_{11}\alpha^2+b_1\alpha)e^{\alpha x_1}F$
$\quad+c(\Phi+(e^{\alpha d}-e^{\alpha x_1})F)$
$\le-(a_{11}\alpha^2+b_1\alpha)e^{\alpha x_1}F\le-(a_{11}\alpha^2+b_1\alpha)F$
$\le-F$，$\alpha>>0$.

⇒ $\begin{cases}L(\pm u-v)=\pm f-Lv\ge\pm f+F\ge0, & \forall x\in\Omega\\ (\pm u-v)|_{\partial\Omega}\le\Phi-\min\limits_{\partial\Omega}v\le0, & \forall x\in\partial\Omega\end{cases}$

由比较原理 ⇒ $|u|\le v$. ◭

Part II. 能量方法

① 双曲方程：对方程同乘 u_t，用 Green 公式对带 a_{ij} 项凑出能量导数 $\frac{d}{dt}E(t)$，得到形如 $\frac{dE}{dt}\le C\left(E(t)+\int_\Omega f^2dx\right)$

(其中 $\int_\Omega u_tf\le\int_\Omega(u_t^2+f^2)$). [出现 $\int u^2$，设 $E_0=\int u^2$ 即可]

乘 e^{-Ct} 积分 ⇒ $E(t)\le E(0)e^{Ct}+Ce^{Ct}\int_0^t\int_\Omega f^2$ ✓

② 抛物方程：同乘 u，同①理即可！

③ 椭圆方程：同乘 u，再估计得到

$$\int_\Omega(|Du|^2+u^2)dx\le C\int_\Omega f^2.$$ ◭

对波动方程而言：

① 初边值问题.
$$\begin{cases} u_{tt} = a^2(u_{xx}+u_{yy}) + f \\ u|_{t=0} = \varphi,\ u_t|_{t=0} = \psi \\ u|_{\partial\Omega} = 0. \end{cases}$$

$$E(t) = \int_\Omega (u_t^2 + a^2(u_x^2+u_y^2))\,dx\,dy$$

$$\Rightarrow \frac{dE}{dt} = 2\int_\Omega u_t f \le E(t) + \int_\Omega f^2$$

$$\Rightarrow \boxed{E(t) \le C_0\left(E(0) + \int_0^T \int_\Omega f^2\right)}.$$

记 $E_0(t) = \int_\Omega u^2$

$$\Rightarrow \frac{dE_0}{dt} = 2\int_\Omega u u_t \le E_0 + E$$

$$\Rightarrow \boxed{E_0(t) \le e^t E_0(0) + e^t \int_0^t e^{-\tau} E(\tau)\,d\tau}$$

$$\Rightarrow \boxed{\begin{aligned} &E(t) + E_0(t) \\ &\le C\left(E(0) + E_0(0) + \int_0^T \int_\Omega f^2\right). \end{aligned}}$$

② Cauchy 问题.
$$\begin{cases} u_{tt} = a^2 \Delta u \\ u|_{t=0} = \varphi,\ u_t|_{t=0} = \psi. \end{cases}$$

取 $\Omega_t: (x-x_0)^2 + (y-y_0)^2 \le (R-at)^2$

可证明 $\frac{dE_1(\Omega_t)}{dt} \le 0$

其中 $E_1(\Omega_t) = \int_{\Omega_t} (u_t^2 + a^2(u_x^2+u_y^2))\,dx\,dy.$

$$\Rightarrow \boxed{E_1(\Omega_t) \le E_1(\Omega_0)}$$

其中 若记 $E_0(\Omega_t) = \int_{\Omega_t} u^2$,

考虑
$$\begin{cases} u_{tt} = a^2 \Delta u + \underline{\underline{f}} \\ u|_{t=0} = \varphi,\ u_t|_{t=0} = \psi. \end{cases}$$

$$\Rightarrow \frac{dE_1(\Omega_t)}{dt} \le 2\int_{\Omega_t} u_t f \le E_1 + \int_{\Omega_t} f^2$$

$$\Rightarrow \boxed{E_1(\Omega_t) \le e^t E_1(\Omega_0) + \int_0^t \int_{\Omega_t} e^{t-\tau} f^2}.$$

$$\frac{dE_0(\Omega_t)}{dt} = -a\int_{\partial\Omega_t} u^2 + 2\int_{\Omega_t} u u_t$$

$$\le E_0(\Omega_t) + E_1(\Omega_t).$$

$$\Rightarrow \boxed{E_0(\Omega_t) \le e^t E_0(\Omega_0) + \int_0^t e^{t-\tau} E_1(\Omega_\tau)\,d\tau.}$$

Thm 0.5 (Moving Regions).

$$2\int_0^{R-at}\int_0^{2\pi r} u_t(u_{tt} - a^2(u_{xx}+u_{yy}))\,ds\,dr$$
$$+2\int_{\Gamma_t} \left(a^2(u_x u_t \cos(\vec{n},x) + u_y u_t \cos(\vec{n},y)) - \frac{a}{2}(u_t^2 + a^2(u_x^2+u_y^2))\right) ds.$$

$$\|$$

$$\rightarrow \frac{dE_1(\Omega_t)}{dt} = 2\int_0^{R-at}\int_0^{2\pi r} (u_t u_{tt} + a^2(u_x u_{xt} + u_y u_{yt}))\,ds\,dr$$
$$- a\int_{\Gamma_t} (u_t^2 + a^2(u_x^2+u_y^2))\,ds$$