《Stein 复分析 1,2,3,8章选编》

Chapter 1. Preliminaries.

Ex 7. (a) If $|z|=1$, then $\left|\frac{w-z}{1-\bar{w}z}\right| = \left|\frac{w-z}{\bar{z}-\bar{w}}\right| = 1$. Let $f(z) = \frac{w-z}{1-\bar{w}z}$, then f is holomorphic on $\mathbb{D}$ and $|f(z)| \equiv 1$ in $\partial\mathbb{D}$, then $|f(z)|<1$ in Int $\mathbb{D}$. □

(b) (i)(ii)(iii) Trivial. Consider (iv).

$$F\circ F = \frac{w - \frac{w-z}{1-\bar{w}z}}{1-\bar{w}\frac{w-z}{1-\bar{w}z}} = \frac{w-|w|^2z-w+z}{1-\bar{w}z-|w|^2+\bar{w}z} = \frac{z-|w|^2z}{1-|w|^2} = z \Rightarrow \text{bijective.}$$ □

Ex 21. For $|z|<1$ we have:

$$\sum_{k=0}^{\infty}\frac{z^{2^k}}{1-z^{2^{k+1}}} = \sum_{k=0}^{\infty}\frac{z^{2^k}-1+1}{(1+z^{2^k})(1-z^{2^k})} = \sum_{k=0}^{\infty}\left(\frac{1}{1-z^{2^k}} - \frac{1}{1-z^{2^{k+1}}}\right)$$

$$= \frac{1}{1-z} - \lim_{k\to+\infty}\frac{1}{1-z^{2k}} = \frac{1}{1-z} - 1 = \frac{z}{1-z}. \quad \checkmark.$$

$$\sum_{k=0}^{\infty}\frac{2^kz^{2^k}}{1+z^{2^k}} = \sum_{k=0}^{\infty}\frac{2^kz^{2^k}(1+z^{2^k}-2z^{2^k})}{1-z^{2^{k+1}}} = \sum_{k=0}^{\infty}\left(\frac{2^kz^{2^k}}{1-z^{2^k}} - \frac{2^{k+1}z^{2^{k+1}}}{1-z^{2^{k+1}}}\right)$$

$$= \frac{z}{1-z} - \lim_{k\to+\infty}\frac{2^kz^{2^k}}{1-z^{2^k}} = \frac{z}{1-z}. \quad \checkmark.$$ □

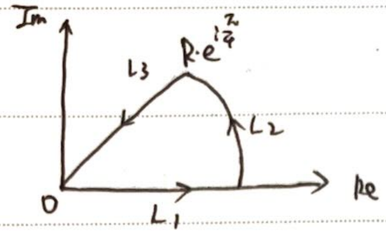
Ex 22. If $\mathbb{Z}_+ = \coprod_{j=1}^{n} S_{a_j,d_j}$, then we have

$$\sum_{k=1}^{\infty}z^k = \sum_{j=1}^{n}\sum_{m=0}^{\infty}z^{a_j+md_j} \quad (|z|<1) \Rightarrow \frac{z}{1-z} = \sum_{j=1}^{n}\frac{z^{a_j}}{1-z^{d_j}}.$$

WLOG let $d_1<\dots<d_n$, $d_n>1$. Let $z_0 = e^{2\pi i\frac{1}{d_n}}$ $\Rightarrow$ z_0 is a pole of the right hand's but not the left hand's! □

Chapter 2. Cauchy's thm and Its applications.

Ex 1. Consider e^{-z^2} and the following path:



Then ~~use~~ Cauchy's thm, we have:

$$\int_0^R e^{-x^2}dx+\int_{L_2}e^{-z^2}dz+\int_{L_3}e^{-z^2}dz=0.$$

Actually, $-\int_{L_3} e^{-z^2}dz=\int_0^R e^{-(te^{i\frac{\pi}{4}})^2}e^{i\frac{\pi}{4}}dt$

$$=e^{\frac{i\pi}{4}}\int_0^R e^{-it^2}dt=e^{\frac{i\pi}{4}}\int_0^R (\cos(t^2)-i\sin(t^2))dt$$

$$=e^{\frac{i\pi}{4}}\int_0^R \cos t^2dt-ie^{\frac{i\pi}{4}}\int_0^R\sin t^2dt.$$

Moreover, $\left|\int_{L_2}e^{-z^2}dz\right|=\left|\int_0^{\frac{\pi}{4}}e^{-(Re^{i\theta})^2}i\cdot Re^{i\theta}d\theta\right|$

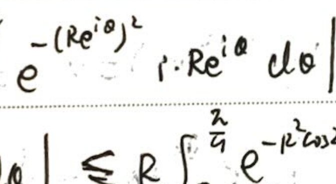
$$=R\left|\int_0^{\frac{\pi}{4}}e^{i\theta-R^2e^{2i\theta}}d\theta\right|\leq R\int_0^{\frac{\pi}{4}}e^{-R^2\cos2\theta}d\theta\leq\frac{\pi}{4}\cdot Re^{-R^2}\to0\ (R\to+\infty).$$

And $\lim_{R\to+\infty}\int_{L_1}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$, then $\Rightarrow\int_0^{+\infty}\sin x^2dx=\int_0^{+\infty}\cos x^2dx=\frac{\sqrt{2\pi}}{4}$. □

Ex 2. Consider $\frac{1}{2i}\int_{\mathbb{R}}\frac{e^{ix}-1}{x}dx=\frac{1}{2i}\int_{\mathbb{R}}\frac{\cos x+i\sin x-1}{x}dx$

$$=\int_0^{+\infty}\frac{\sin x}{x}dx+\frac{1}{2i}\int_{\mathbb{R}}\frac{\cos x-1}{x}dx=\int_0^{+\infty}\frac{\sin x}{x}dx.$$

Then consider the path:



Then use Cauchy's thm $\Rightarrow\int_C\frac{e^{iz}-1}{z}dz=0=\left(\int_{L_1}+\int_{L_2}+\int_{-R}^{-\varepsilon}+\int_{\varepsilon}^{R}\right)\left(\frac{e^{iz}-1}{z}\right)dz.$

Actually, $\int_{L_1}\frac{e^{iz}-1}{z}dz=-\int_0^{\pi}\frac{e^{i\varepsilon e^{i\theta}}-1}{\varepsilon e^{i\theta}}\cdot i\varepsilon e^{i\theta}d\theta=-\int_0^{\pi}i(e^{i\varepsilon e^{i\theta}}-1)d\theta.$

$$=i\pi-i\int_0^{\pi}e^{i\varepsilon\cos\theta-\varepsilon\sin\theta}d\theta\text{, then }\int_{L_1}\to\pi i-\pi i=0\text{ as }\varepsilon\to0^+.$$

Then $\int_{L_2} \frac{e^{iz}-1}{z}dz = \int_0^{\pi} \frac{e^{iRe^{i\theta}}-1}{Re^{i\theta}} \cdot iRe^{i\theta} d\theta = i\int_0^{\pi} (e^{iRe^{i\theta}}-1)d\theta = i\int_0^{\pi} e^{iRe^{i\theta}} d\theta - \pi i$.

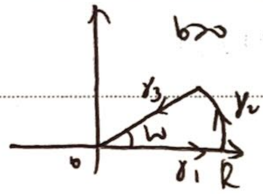
Then $\int_{L_2} \frac{e^{iz}-1}{z}dz \to \pi i \ (R\to+\infty)$. So we have:

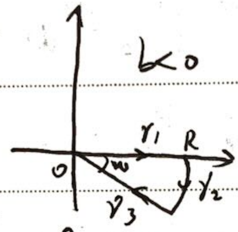
$$\int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{e^{iz}-1}{z}dz = -\left(\int_{L_1}+\int_{L_2}\right)\left(\frac{e^{iz}-1}{z}\right)dz$$, then

$$\int_{\mathbb{R}} \frac{e^{iz}-1}{z}dz = \lim_{\substack{\varepsilon\to 0^+\\ R\to+\infty}} \int_{[-R,-\varepsilon]\cup[\varepsilon,R]} \frac{e^{iz}-1}{z}dz = \pi i$$, then

$$\int_0^{+\infty} \frac{\sin x}{x}dx = \frac{1}{2i}\cdot \pi i = \frac{\pi}{2}.$$ □

Ex3. We have the following contour:





Use Cauchy $\Rightarrow \left(\int_{\gamma_1}+\int_{\gamma_2}+\int_{\gamma_3}\right)(e^{-Az}dz)=0 \Rightarrow \lim_{R\to\infty}\left(\int+\int+\int\right)(\cdots)=0.$

$$\lim_{R\to\infty}\int_{\gamma_1} e^{-Az}dz = \lim_{R\to\infty}\int_0^R e^{-Ax}dx = -\frac{1}{A}\lim_{R\to\infty}\int_0^R de^{-Ax} = \frac{1}{A}.$$

For γ_3, ~~we~~ we have $\int_{\gamma_3} e^{-Az}dz = \int_R^0 e^{-Axe^{i\omega}}dxe^{i\omega} = e^{i\omega}\int_R^0 e^{-Ax\cos\omega - iAx\sin\omega}dx$

$$= e^{i\omega}\int_R^0 e^{-ax}(\cos bx - i\sin bx)\,dx = -e^{i\omega}\int_0^R e^{-ax}\cos bx\,dx + ie^{i\omega}\int_0^R e^{-ax}\sin bx\,dx$$

$$\Rightarrow \lim_{R\to\infty}\int_{\gamma_3} e^{-Az}dz = -\left(\frac{a}{A}\int_0^{\infty} e^{-ax}\cos bx\,dx + \frac{b}{A}\int_0^{\infty} e^{-ax}\sin bx\,dx\right) + i\left(-\frac{b}{A}\int_0^{\infty} e^{-ax}\cos bx\,dx\right.$$

$$\left. + \frac{a}{A}\int_0^{\infty} e^{-ax}\sin bx\,dx\right).$$ Moreover, we have

$$\lim_{R\to\infty}\left|\int_{\gamma_2} e^{-Az}dz\right| \le \lim_{R\to\infty}\int_{\gamma_2}|e^{-Az}||dz| \le \lim_{R\to\infty}\int_{\gamma_2}\frac{1}{e^{AR\cos\omega}}|dz| = 0$$

Use Cauchy $\Rightarrow \int_0^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$,

$$\int_0^{\infty} e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}.$$ □

Ex 5. Let $f = u+iv$, then we have

$$\int_T f(z)dz = \int_T (u+iv)(dx+idy) = \int_T (u+iv)dx + (iu-v)dy$$

$$= \int_{int[T]} \left(i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - i\frac{\partial v}{\partial y} \right) dx dy = 0. \quad \square$$

Ex 6. This is trivial use Riemann extension thm. $\square$

Ex 7. We find that $2f'(0) = \frac{1}{2\pi i}\int_{|\xi|=r} \frac{f(\xi)-f(-\xi)}{\xi^2} d\xi \quad (r\in(0,1))$,

then $2|f'(0)| = \frac{1}{2\pi}\left|\int_{|\xi|=r} \frac{f(\xi)-f(-\xi)}{\xi^2} d\xi\right|$

$$\leq \frac{1}{2\pi}\int_{|\xi|=r} \frac{d}{|\xi|^2} d|\xi| \leq \frac{1}{2\pi}\cdot 2\pi r \cdot \frac{d}{r^2} = \frac{d}{r}.$$

This is right for all $r\in(0,1) \Rightarrow 2|f'(0)| \leq d$. $\square$

Ex 8. For $\eta \geq 0$, we have $|f^{(n)}(z)| \leq \frac{n!\max_{|z-x|\leq\varepsilon}|f(z)|}{\varepsilon^n} \leq \frac{n!\,A(1+|x+\varepsilon|)^\eta}{\varepsilon^n}$

$$\leq \frac{n!\,A(1+|x|+\varepsilon+\varepsilon|x|)^\eta}{\varepsilon^n} = \frac{n!\,A(1+|x|)^\eta(1+\varepsilon)^\eta}{\varepsilon^n} = A_n(1+|x|)^\eta.$$

For $\eta < 0$, we have $|f^{(n)}(z)| \leq \frac{n!\max_{|z-x|\leq\varepsilon}|f(z)|}{\varepsilon^n} \leq \frac{n!\,A(1+|x-\varepsilon|)^\eta}{\varepsilon^n}$

$$\leq \frac{n!\,A(1+|x|-\varepsilon)^\eta}{\varepsilon^n} \leq \frac{n!\,A(1+|x|-\varepsilon-\varepsilon|x|)^\eta}{\varepsilon^n} = \frac{n!\,A(1+|x|)^\eta(1-\varepsilon)^\eta}{\varepsilon^n} = A_n(1+|x|)^\eta. \quad \square$$

Ex 9. $\varphi:\Omega\to\Omega$ and $\exists z_0\in\Omega \Rightarrow \varphi(z_0)=z_0, \varphi'(z_0)=1$. WLOG we let $z_0=0$,

then $\varphi(0)=0$, $\varphi'(0)=1$, then we let $\varphi(0) = z + a_n z^n + O(z^{n+1})$, near 0.

Easy to see that $\varphi_k = \underbrace{\varphi\circ\varphi\circ\cdots\circ\varphi}_{k\text{-tims}} = z + ka_n z^n + O(z^{n+1})$. Since

$|ka_n| \leq \frac{\sup|\varphi_k(z)|}{r^n} \leq \frac{M}{r^n}$, $k\to\infty \Rightarrow |a_n| = 0 \Rightarrow a_n = 0$. $\square$

Ex 12. (a) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$, let $P=-\frac{\partial u}{\partial y}$, $Q=\frac{\partial u}{\partial x}$, $\Rightarrow \frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$.

Let $v=\int_{(x_0,y_0)}^{(x,y)} P\,dx+Q\,dy$, easy to see that $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}$.

Let $f=u+iv$, well done. $\square$

(b) Omitted. $\square$

Ex 13. If $f^{(n)}(z)\not\equiv 0$, $\forall n$, then $f^{(n)}(z)\Rightarrow$ holomorphic. Then $f^{(n)}(z)$ has countable roots, that is, $\#(Z(f^{(n)}))=\aleph_0$. So $\#(\bigcup_n Z(f^{(n)}))=\aleph_0$.

Since $c_n\cdot n!=f^{(n)}(z_0)$, $\forall z_0, \exists n_{z_0}\Rightarrow f^{(n_{z_0})}(z_0)=0 \Rightarrow z_0\in\bigcup_n Z(f^{(n)})$

$\Rightarrow \#(\mathbb{C})\le \#(\bigcup_n Z(f^{(n)}))$, Impossible. $\square$

Ex 14. Let $f(z)=\sum_{n=0}^{\infty}a_n z^n=\frac{C_m}{(z_0-z)^m}+\cdots+\frac{C_1}{z_0-z}+\underbrace{\sum_{n=0}^{\infty}b_n z^n}_{\text{holo.}}$

$$=\sum_{n=0}^{\infty}\left(b_n+\sum_{k=1}^{m}\frac{k(k+1)\cdots(k+n-1)C_k}{n!\,z_0^{n+k}}\right)z^n.$$

So $a_n=b_n+\sum_{k=1}^{m}\frac{k(k+1)\cdots(k+n-1)C_k}{n!\,z_0^{n+k}}$, then $\lim_{n\to\infty}\frac{a_n}{a_{n+1}}=z_0$. $\square$

Ex 15. As Schwarz reflection principle, we can

let $F=\begin{cases} f, & z\in\overline{\mathbb{D}} \\ \frac{1}{\overline{f(\frac{1}{\bar z})}}, & |z|>1\end{cases}$, then F bounded over $\mathbb{C}$,

Use Liouville $\Rightarrow F$ constant! $\square$

Problem 1 (a) Suppose $\theta = \frac{2\pi p}{2^k}$ where $p, k \in \mathbb{N}_+$ and let $z = re^{i\theta}$, then we can choose p, k so that θ can arrive any angle!

Now we fix p, k, then consider

$$|f(re^{i\theta})| = \left|\sum_{n=0}^{\infty} r^{2^n} e^{2\pi i p \cdot 2^{n-k}}\right| = \left|\sum_{n=k+1}^{\infty} (re^{p\cdot 2^{-k}})^{2^n} + \sum_{n=0}^{k} r^{2^n} e^{\frac{2\pi i p}{2^{k-n}}}\right|.$$

So $\lim_{r\to 1^-} |f(re^{i\theta})| = \left|\lim_{r\to 1^-} \sum_{n=k+1}^{\infty} (re^{p\cdot 2^{-k}})^{2^n} + \sum_{n=0}^{k} e^{\frac{2\pi i p}{2^{k-n}}}\right|$

$\exists \delta > 0$ such that $|r-1| < \delta$ with $re^{p\cdot 2^{-k}} > 1$, then we know that $\lim_{r\to 1^-} |f(re^{i\theta})| = \infty$, so well done. □

(b) When $z = e^{i\theta} \Rightarrow \sum_{n=0}^{\infty} 2^{-n\alpha} \cos(2^n x)$ nowhere diff— when $0 < \alpha \leq 1$, so $f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n}$ can't analytically continued when $0 < \alpha \leq 1$. Now we find that $zf'(z) = \sum_{n=0}^{\infty} 2^{n(1-\alpha)} z^{2^n}$ and it can't --- when $0 < -1+\alpha \leq 1$, that is, $1 < \alpha \leq 2$. So f can't -- when $1 < \alpha \leq 2$.

Use induction $\Rightarrow$ f can't -— when $0 < \alpha < \infty$. □

Problem 2. We know that $\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} z^{mn}\right) z^n = \sum_{m,n=1}^{\infty} z^{(m+1)n}$.

Let $k \in \mathbb{N}_+$, then $k = (m+1)n$ have $d(k)$ solutions! This because $m+1 \geq 2$ which represent the divisor of k, and n represent others.

So $\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} z^{(m+1)n} = \sum_{n=1}^{\infty} d(n) z^n$. ✓

Moreover, $|F(r)| = \sum_{n=1}^{\infty} \frac{r^n}{1-r^n} \geq \int_1^{\infty} \frac{r^n}{1-r^n}\, dn = -\frac{1}{\log r} \log\left(\frac{1}{1-r}\right)$.

As $r \to 1$, we have $\log r = r-1 + O((r-1)^2)$, so there are c such that $|F(r)| \geq c\frac{1}{1-r}\log\left(\frac{1}{1-r}\right)$.

Moreover, let $\theta = \frac{2\pi p}{q}$ where $p, q \in \mathbb{N}_+$ and let $z = re^{i\theta}$. WLOG we let $(p,q)=1$, then $z^n = r^n e^{in\theta} = r^n e^{2\pi i n \frac{p}{q}}$. We claim that $|F(re^{i\theta})| \geq C_{p/q} \frac{1}{1-r}\log\left(\frac{1}{1-r}\right)$ as $r \to 1$. Here are :

$$|F(z)| = \left|\sum_{n=1}^{\infty} \frac{z^n}{1-z^n}\right| = \left|\sum_{m=1}^{\infty} \frac{z^{mq}}{1-z^{mq}} + \sum_{q\nmid n} \frac{z^n}{1-z^n}\right| = |A+B|. \quad , z = re^{i\theta}$$

Use the previous conclusion, we have $A \geq C_A \frac{1}{1-r}\log\left(\frac{1}{1-r}\right)$, $A \geq 0$.

Then we consider B: $q\theta = 2\pi p$, so we let k' such that $e^{ik'\theta}$ closest to 1 since $e^{ik'\theta} = e^{ik'\frac{2\pi p}{q}}$, so we have finite choice!

Let $C_B = \min\{1, \inf\{|1-re^{ik'\theta}| \mid 0 \leq r \leq 1\}\}$, so $C_B > 0$, then we have

$$|B| \leq \sum_{q\nmid n} \frac{|z^n|}{|1-z^n|} \leq \sum_{q\nmid n} \frac{|z^n|}{|C_B|} = C_B^{-1} \sum_{q\nmid n} |z|^n \leq C_B^{-1} \sum_{n=1}^{\infty} |z|^n = \frac{C_B^{-1}}{1-r},$$

so $-|B| \geq \frac{1}{r-1} C_B^{-1}$. $A \geq 0$, then we have:

$$|F(re^{i\theta})| \geq A - |B| \geq C_A \frac{1}{1-r}\left(\log\left(\frac{1}{1-r}\right) - C_B^{-1}\right).$$

As $r \to 1$, we can choose a $C_{p/q}$ such that $C_{p/q} \leq C_A + \frac{\frac{C_A}{C_B}}{\log(1-r)}$,

So as $r \to 1$, we have $|F(re^{i\theta})| \geq C_{p/q} \frac{1}{1-r}\log\left(\frac{1}{1-r}\right)$. ✓

So as Problem 1(a), well done. $\square$

Problem 3. (a)(b) Let $\varphi(w) = \varphi(x,y)$ be a C^∞ function with $\varphi \equiv 0$ $(x^2+y^2>1)$ and $\int_{\mathbb{R}^2} \varphi(x,y)\, dV(w) = 1$. Let $\varphi_\varepsilon(z) = \frac{1}{\varepsilon^2}\varphi\left(\frac{z}{\varepsilon}\right)$, then consider $f_\varepsilon(z) = \int_{\mathbb{R}^2} f(z-w)\varphi_\varepsilon(w)\, dV(w)$ $= f * \varphi_\varepsilon$, then $f_\varepsilon \rightrightarrows f$ as $\varepsilon \to 0$ uniformly. Let $\gamma = \partial D$, then use Stokes, we have

$0 = \int_\gamma f_\varepsilon(z)\, dz = \int_D df_\varepsilon \wedge dz = 2i \int_D \bar\partial f_\varepsilon \, dx dy \Rightarrow \bar\partial f_\varepsilon = 0 \Rightarrow f_\varepsilon$ holo $\Rightarrow f$ holo. $\square$

Problem 4. Let $f(z) = \frac{1}{z-z_0}$ where $z_0 \in K^c$ in the bounded component of K^c (This exists by Jordan curve thm). Suppose f can be approximated by polynomials over K, so $\exists p$ s.t. $|f-p| < |z-z_0|$ for all $z \in K$, then we have $|(z-z_0)p(z)-1| < 1$ for all $z \in K$. We know that $(z-z_0)p(z)-1$ is holomorphic over all the interior of ∂K, containing z_0. So use maximal modulus principle $\Rightarrow |(z-z_0)p(z)-1| < 1$ holds in this region. Let $z = z_0 \Rightarrow 1 < 1$ which is impossible

$\square$

Problem 5. (a) If we find an entire function F and an increasing sequence $\{M_n\}$, $M_n \in \mathbb{N}_+$, s.t. $\left|F(z) - P_n(z-M_n)\right| < \frac{1}{n}$ whenever $z \in D_n$, where D_n denotes a disc centered at M_n and of radius n. Then take any entire function $h(z)$, $\exists \{n_k\} \Rightarrow \lim_{k\to\infty} P_{n_k}(z) = h(z)$ in all compact sets of $\mathbb{C}$ uniformly. So $|F(z+M_n) - P_n(z)| < \frac{1}{n}$ for all $z \in B(0,n)$. So for any compact sets K of $\mathbb{C}$, $\exists N \Rightarrow K \subseteq B(0,N)$. $\forall \varepsilon > 0$, then $\exists \tilde{N} > N$ s.t. $\forall n > \tilde{N}$, we have $|F(z+M_n) - h(z)| \leq |F(z+M_n) - P_n(z)| + |P_n(z) - h(z)|$

Let $n \in \{n_k\}$ WLOG $\Rightarrow |F(z+M_n) - h(z)| \leq \frac{1}{n} + \frac{1}{n} < \varepsilon$. ✓ $\square$.

(b). Let $F(z) = \sum_{n=1}^{\infty} u_n(z)$ where $u_n(z) = P_n(z-M_n)e^{-c_n(z-M_n)^2}$ where $c_n, M_n > 0$ and $c_n \to 0$, $M_n \to \infty$. ~~Omitted~~.

$\square$

Chapter 3

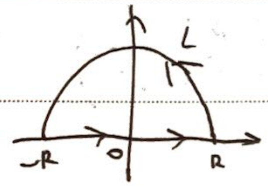
Ex 1. $\sin\pi z = 0 \Leftrightarrow e^{2i\pi z} = 1$, let $z = x+iy$, then

$e^{2i\pi z} = e^{2i\pi x - 2\pi y} = e^{2i\pi x}\cdot e^{-2\pi y} = 1$, so $|e^{-2\pi y}| = 1 \Rightarrow y = 0$.

then $e^{2i\pi x} = 1 \Rightarrow \cos(2\pi x) + i\sin(2\pi x) = 1 \Rightarrow \begin{cases}\cos 2\pi x = 1\\ \sin(2\pi x) = 0\end{cases} \Rightarrow x \in \mathbb{Z} \Rightarrow z \in \mathbb{Z}$.

Then residue of $\frac{1}{\sin\pi z}$ at $n\in\mathbb{Z}$ is $\operatorname{Res}_{z=n}\left(\frac{1}{\sin\pi z}\right) = \lim_{z\to n}\frac{z-n}{\sin\pi z} = (-1)^n\frac{1}{\pi}$. □

Ex 2. Look at the following contour, then two poles $z = e^{\frac{\pi i}{4}}$ & $e^{\frac{3\pi i}{4}}$ in it,



then $\operatorname{Res}_{z=e^{\frac{\pi}{4}i}}\left(\frac{1}{1+z^4}\right) = -\frac{\sqrt{2}+\sqrt{2}i}{8}$,

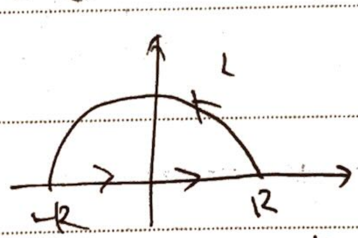
$\operatorname{Res}_{z=e^{\frac{3\pi}{4}i}}\left(\frac{1}{1+z^4}\right) = \frac{\sqrt{2}-\sqrt{2}i}{8}$.

So $\int_C \frac{dz}{1+z^4} = 2\pi i\sum_k \operatorname{Res}\left(\frac{1}{1+z^4}\right) = \frac{\sqrt{2}\pi}{2}$.

Moreover, $\left|\int_L \frac{dz}{1+z^4}\right| \le \int_L \frac{1}{|z|^4-1}\,d|z| \le \frac{\pi R}{R^4-1} \to 0$

So $\int_{\mathbb{R}} \frac{1}{1+x^4}dx = \frac{\pi}{\sqrt{2}}$. □

Ex 3. Consider the following contour, then ~~three~~ poles $z = ia$, then



$\operatorname{Res}_{z=ia}\left(\frac{e^{iz}}{z^2+a^2}\right) = -\frac{ie^{-a}}{2a}$, then

$\int_C \frac{e^{iz}}{z^2+a^2}dz = 2\pi i\cdot\left(-\frac{ie^{-a}}{2a}\right) = \frac{\pi e^{-a}}{a}$.

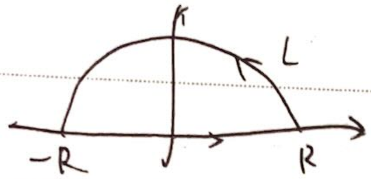
Since $\left|\int_L \frac{e^{iz}}{z^2+a^2}dz\right| = \left|\int_0^\pi \frac{1}{R^2e^{2i\theta}+a^2}e^{iRe^{i\theta}}\cdot iRe^{i\theta}d\theta\right|$.

$|z|>R \Rightarrow \left|\frac{1}{z^2+a^2}\right| < \varepsilon$, then $\hookrightarrow \le R\varepsilon\left|\int_0^\pi e^{iRe^{i\theta}+i\theta}d\theta\right|$

$\le R\varepsilon\int_0^\pi e^{-R\sin\theta}d\theta \le R\varepsilon\int_0^\pi e^{-\frac{2R\theta}{\pi}}d\theta = \pi\varepsilon(1-e^{-R}) \le \pi\varepsilon$.

Then $R\to+\infty \Rightarrow \int_{\mathbb{R}}\frac{\cos z + i\sin z}{z^2+a^2}dz = \frac{\pi e^{-a}}{a} \Rightarrow \int_{\mathbb{R}}\frac{\cos x}{x^2+a^2}dx = \frac{\pi e^{-a}}{a}$. □

Ex4. Consider $\frac{ze^{iz}}{z^2+a^2}$ $(a>0)$, consider the contour:



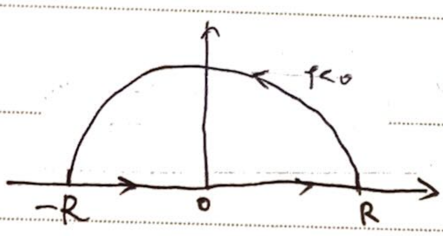
Then $\int_C \frac{ze^{iz}}{z^2+a^2}dz = 2\pi i \operatorname{Res}_{z=ia}\left(\frac{ze^{iz}}{z^2+a^2}\right) = \pi i e^{-a}$.

Easy to see that for $R>R_0$, $\left|\int_L \frac{ze^{iz}}{z^2+a^2}\right| < \varepsilon$.

Then $\int_{-\infty}^{+\infty}\frac{xe^{ix}}{x^2+a^2}dx = \int_{\mathbb{R}}\frac{ix\sin x + x\cos x}{x^2+a^2}dx = \pi i e^{-a}$,

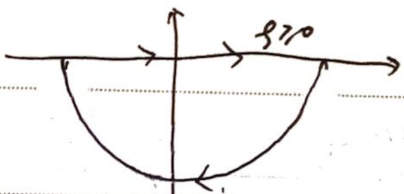
then $\int_{-\infty}^{+\infty}\frac{x\sin x}{a^2+x^2}dx = \pi e^{-a}$. □

Ex5. Consider the following contour:



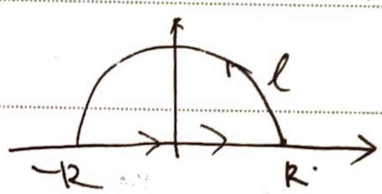
Then $\int_C \frac{e^{-2\pi i z\xi}}{(1+z^2)^2}dz = 2\pi i \operatorname{Res}_{z=i}\left(\frac{e^{-2\pi i z\xi}}{(1+z^2)^2}\right)$

$= \cancel{2\pi i}\, e^{2\pi\xi}\left(\frac{\pi}{2}-\pi^2\xi\right)$.



Similar ~~se~~ easy to see. ! □

Ex.6. Consider the contour:



Then $\int_C \frac{dz}{(1+z^2)^{n+1}} = 2\pi i \operatorname{Res}_{z=i}\left(\frac{1}{(1+z^2)^{n+1}}\right)$

$= \frac{2\pi i}{n!}\left(\frac{1}{(z+i)^{n+1}}\right)^{(n)}(i) = \frac{(-1)^n(n+1)\cdots(2n)}{n!}\frac{(-1)^{2n}}{2^{2n}}\pi$

$= \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\pi$.

~~See~~ Easy to see that $\left|\int_l \frac{dz}{(1+z^2)^{n+1}}\right| \to 0$ $(R\to+\infty)$. ✓ □

Ex7. Put $z=e^{i\theta} \Rightarrow \cos\theta = \frac{z+z^{-1}}{2}$, $d\theta=\frac{dz}{iz}$, then we have,

$$\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2} = \int_{|z|=1}\frac{1}{\left(a+\frac{z+z^{-1}}{2}\right)^2}\frac{dz}{iz} = \cdots = \int_{|z|=1}\frac{-4iz}{(z+a+\sqrt{a^2-1})^2(z+a-\sqrt{a^2-1})^2}dz$$

$= 2\pi i \operatorname{Res}$ ~~*~~, actually, we have:

$$\mathrm{Res} = \left(\frac{-4iz}{(z+a+\sqrt{a^2-1})^2} \right)' \Big|_{z=\sqrt{a^2-1}-a} = \frac{-ai}{(a^2-1)^{\frac{3}{2}}}, \text{ then}$$

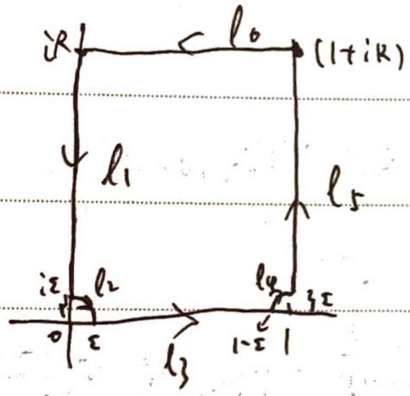
$$LHS = \frac{2\pi a}{(a^2-1)^{\frac{3}{2}}}. \qquad \square$$

Ex. 8. Similar as Ex. 7. Omitted. $\square$

Ex. 9. Consider $f(z) = \log(1-e^{2\pi i z})$, then we know that

$\int_0^1 f(z)dz = \log 2 + \int_0^1 \log(\sin(\pi z))dz$, so we claim

that $\int_0^1 f(z)dz = 0$. Consider the following contour:



Then $\oint_C f(z)dz = 0$.

Actually, $\int_{l_1} + \int_{l_5} = 0$ and $\lim_{R\to\infty} \int_{l_6} f = 0$.

and $\left(\int_{l_2} + \int_{l_4}\right)(f(z)dz) = \varepsilon \log \varepsilon$, then

$\int_0^1 f(z)dz = \lim_{\varepsilon\to 0^+} \left(\int_{l_3} \right) = -\lim_{\varepsilon \to 0^+} \varepsilon \log \varepsilon = 0.$ $\square$

Mathematical analysis:

$I = 2\int_0^{\frac{1}{2}} \log(\sin \pi x) dx$

$\xrightarrow{x \to \frac{1}{2} - x} 2\int_0^{\frac{1}{2}} \log(\cos \pi x) dx$

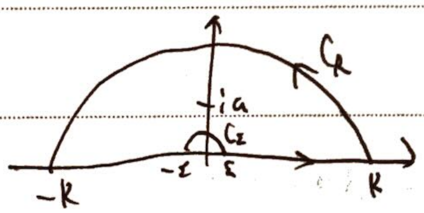
$\Rightarrow 2I = \int_0^{\frac{1}{2}} \log(\sin 2\pi x) dx - \frac{1}{2}\log 2$

$\int_0^{\frac{1}{2}} \log(\sin 2\pi x) dx$

$= \frac{1}{2}\int_0^1 \log \sin \pi x = \frac{1}{2} I$

$\Rightarrow I = -\log 2$. $\square$

Ex. 10. Consider the following contour:



$$\int_C \frac{\log z}{z^2+a^2} dz = 2\pi i \operatorname*{Res}_{z=ia} \left(\frac{\log z}{z^2+a^2} \right)$$

$$= 2\pi i \cdot \left(\frac{\log z}{z+ia} \right) \Big|_{z=ia} = 2\pi i \left(\frac{\log a + \frac{\pi}{2} i}{2ia} \right)$$

$$= \frac{\pi \log a}{a} + \frac{\pi^2 i}{2a}.$$

Easy to see that $\lim_{R\to+\infty} \int_{C_R} = 0 = \lim_{\varepsilon\to 0^+} \int_{C_\varepsilon}$, then

$$\left(\int_{-\infty}^0 + \int_0^{+\infty} \right) \left(\frac{\log z}{z^2+a^2} dz \right) = \frac{\pi \log a}{a} + \frac{\pi^2 i}{2a},$$

Since $\int_{-\infty}^0 \frac{\log z}{z^2+a^2} dz = \int_0^{+\infty} \frac{\log(-x)}{x^2+a^2} dx = \int_0^{+\infty} \frac{\log x + i\pi}{x^2+a^2} dx$,

we have $\Rightarrow 2\int_0^{+\infty} \frac{\log x}{x^2+a^2} dx + i\pi \int_0^{+\infty} \frac{1}{x^2+a^2} dx = \frac{\pi \log a}{a} + \frac{\pi^2 i}{2a}$.

$$\Rightarrow \int_0^{+\infty} \frac{\log x}{x^2+a^2} = \frac{\pi \log a}{2a}. \qquad \square$$

Ex. 11. For $|a|<1$, we find tt there are no poles for $\frac{\log(1-az)}{z}$ in $|z|\le 1$, so $\oint_{|z|=1} \frac{\log(1-az)}{z} dz = 0$. So $\int_0^{2\pi} \log|1-ae^{i\theta}| d\theta = 0$.

For $|a|=1$, we need to show that $\int_0^{2\pi} \log|1-ae^{i\theta}| d\theta = 0$.

Actually, we find that if $a=e^{i\varphi}$, then

$$\int_0^{2\pi} \log|1-e^{i(\theta+\varphi)}| d\theta = \int_{\varphi}^{2\pi+\varphi} \log|1-e^{i\theta}| d\theta = \int_0^{2\pi} \log|1-e^{i\theta}| d\theta.$$

Actually, $|1-e^{i\theta}| = \sqrt{2(1-\cos\theta)} = 2\left|\sin\frac{\theta}{2}\right|$, so we have:

$$\int_0^{2\pi} \log|1-e^{i\theta}| d\theta = 4\int_0^{\frac{\pi}{2}} \log(2\sin\theta) d\theta = 0, \text{ use Ex 9.}$$ □

Ex. 12. Integrating $f(z) = \frac{\pi\cot\pi z}{(u+z)^2}$ over $|z| = R_N = N+\frac{1}{2}$ ($N\in\mathbb{Z}$, $N\ge |u|$)

$$\sum \operatorname{Res}(f(z)) = -\frac{\pi^2}{\sin^2\pi u} + \sum_{k=-N}^{N} \operatorname{Res}_{z=k}(f(z)).$$

Since $f(z) = \frac{\pi\cos\pi z}{(u+z)^2}\cdot\frac{1}{\sin\pi z} = \frac{\pi\cos\pi z}{(u+z)^2}\cdot\frac{1}{\pi(-1)^k\sum_{n=0}^{\infty}(-1)^n\frac{(\pi z-\pi k)^{2n+1}}{(2n+1)!}}\cdot\frac{1}{z-k}$

we have $\operatorname{Res}_{z=k} f(z) = \frac{1}{(u+k)^2}$, so $\sum_{-N\le k\le N} \operatorname{Res}_{z=k}(f) = -\frac{\pi^2}{\sin^2\pi u} + \sum_{k=-N}^{N}\frac{1}{(u+k)^2}$.

So $\int_{|z|=R_N} f(z)dz = 2\pi i\sum_{k=-N}^{N}\frac{1}{(u+k)^2} - 2\pi i\cdot\frac{\pi^2}{\sin^2\pi u}$.

On the other hand, we have ~~et~~:

$$\left|\int_{|z|=R_N} f(z)dz\right| = \left|\int_0^{2\pi}\frac{\pi\cot(\pi R_N e^{i\theta})}{(u+R_N e^{i\theta})^2} R_N i e^{i\theta} d\theta\right|$$

$$\le \int_0^{2\pi}\pi R_N\left|\frac{\cot(\pi R_N e^{i\theta})}{(u+R_N e^{i\theta})^2}\right| d\theta$$

$$\le \int_0^{2\pi}\frac{\pi R_N}{(R_N-u)^2}|\cot(\pi R_N e^{i\theta})| d\theta \to 0.$$

Since $|\cot(\pi R_N e^{i\theta})| = \left|\frac{e^{i\pi e^{i\theta}R_N}+e^{-i\pi e^{i\theta}R_N}}{e^{-i\pi e^{i\theta}R_N}-e^{i\pi e^{i\theta}R_N}}\right| = \left|\frac{e^{2i\pi e^{i\theta}R_N}+1}{e^{2i\pi e^{i\theta}R_N}-1}\right| \le \frac{2}{|e^{2i\pi e^{i\theta}R_N}|-1}+1$,

So $\exists\delta>0$, for $\theta\in(\delta,\pi-\delta),(\pi+\delta,2\pi-\delta)$, then $\exists M_1>0 \Rightarrow |\cot(\pi R_N e^{i\theta})|\le M_1$.

When $\theta\in[-\delta,\delta]\cup[\pi-\delta,\pi+\delta]$, this is also true since $\cot(\pi R_N e^{i0}) = \cot(\pi R_N e^{i\pi}) = 0$. □

Ex 13. Use Laurent expansion ✓. □

Ex.14. Let $f(z_0)=0$ and $z_0 \in U$. Let $g(z)=\frac{1}{f(z)}$.

Use open mapping thm and f is injective $\Rightarrow |f| \geq k > 0$, for $z \in \mathbb{C} \setminus U$.

$\Rightarrow g$ bounded over $\mathbb{C}\setminus U$ and ∞ is removable singular of g.

$\lim_{z\to\infty} g = b$.

① If $b \neq 0 \Rightarrow f$ bounded $\Rightarrow f \equiv C$.

② $b=0 \Rightarrow \lim_{z\to\infty} f = \infty \Rightarrow f$ is polynomial since ∞ is not essential.

by Picard big thm. Use f injective and $0 \Rightarrow f$ linear. $\square$

Ex.15.(a). $\sup_{|z|=R}|f(z)| \leq AR^k + B$, so $|C_{k+1}| \leq \frac{AR^k+B}{R^{k+1}}$

$R \to \infty \Rightarrow C_{k+1} = 0 \Rightarrow C_n = 0$ for all $n \geq k+1 \Rightarrow$

$\Rightarrow f$ is a polynomial with degree $\leq k$. $\square$

(b) $|f| \leq M$. Let $t < \varphi - \theta$ and let $g(z) = f(z)f(ze^{it}) \cdots f(ze^{Nit})$. $(Nt > 2\pi)$

$\forall \varepsilon > 0, \exists r < 1$, $|f| < \varepsilon$ for $r < |z| < 1$, $\theta < \arg z < \varphi$.

$\Rightarrow |g| < M^n \varepsilon$, $\forall r < |z| < 1$. Use MMP $\Rightarrow |g| < M^n \varepsilon$, $\forall z \in \mathbb{D}$.

$\Rightarrow g = 0 \Rightarrow \exists k$, $f(ze^{ikt}) = 0 \Rightarrow f(z) = 0$. $\square$

(c) Put $F(z) = \prod_{k=1}^{n}(z-a_k)$, $|z| \leq 1$. $F(0) = \prod_{k=1}^{n}(-a_k) \Rightarrow |F(0)| = 1$.

MMP $\Rightarrow \exists z_0$, $|z_0| = 1 \Rightarrow |F(z_0)| \geq 1$. $F(a_i) = 0$.

Since F continuous $\Rightarrow \exists |z_0'| = 1 \Rightarrow |F(z_0')| = 1$. $\square$

(We ~~...~~ have a much stronger question.

See our mid-term exam.)

(d). Let $|Re(f)| \leq 6$. $\Rightarrow g = \frac{1}{1+6-f}$ entire and g bounded.

$\Rightarrow \frac{1}{1+6-f} \equiv C \neq 0 \Rightarrow f \equiv 1+6-\frac{1}{C}$. ✓. □.

Ex 16. (a). $\varepsilon_0 = \frac{\min\{|f| : |z|=1\}}{1+\max\{|g| : |z|=1\}} > 0$.

$\forall 0 \leq \varepsilon < \varepsilon_0, |z|=1 \Rightarrow |(f+\varepsilon g) - f| = \varepsilon|g| < |f|$.

Use Rouché $\Rightarrow$ f & $f+\varepsilon g$ ...... □

(b). $z_\varepsilon = \frac{1}{2\pi i} \int_{\partial D} \frac{z(f'+\varepsilon g')}{f+\varepsilon g} dz$. □

Ex 17. (a). If f has no roots in D, then $\left|\frac{1}{f}\right| = 1$ ($|z|=1$), MMP

$\Rightarrow \left|\frac{1}{f(z)}\right| \leq 1$, ($|z| \leq 1$) and $\exists |z_0| < 1$, $\left|\frac{1}{f(z_0)}\right| < 1$.

$\Rightarrow |f(z_0)| > 1$, MMP $\Rightarrow$ impossible

Finally, just need to show that $f = w_0$ always

has solution over D. Let $g_0 = f - w_0$, $|g_0 - f| = |w_0| \leq 1 = |f|$

($|z|=1$), then Rouché $\Rightarrow$ g_0 ...... ✓. □

(b) $h(w) := \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)-w} dz$, by argument principle

$\Rightarrow h(w)$ is number of meeting w $\Rightarrow$ $\mathrm{Im}\, h \subseteq \mathbb{Z}_{\geq 0}$.

But h continuous $\Rightarrow h \equiv C \in \mathbb{N}$. $h(f(z_0)) \geq 1$

$\Rightarrow h \geq 1$ for all $|w| \leq 1$. □.

Ex 19. (a) Let u attains a local maximum at z_0 and let f be a holomorphic function near z_0 and $u = \mathrm{Re}(f)$. Then $f(z_0)$ is the boundary point in the image of the "near z_0". So f is not open which is impossible. □

(b) Trivial by (a). □

Ex. 20. (a) Let $d>0$ and g holo on $U \supseteq \overline{D_d(z)}$, then MVT $\Rightarrow$ $g(z) = \frac{1}{2\pi}\int_0^{2\pi} g(z+\rho e^{i\theta})\,d\theta$, $\forall\, 0<\rho<d$.

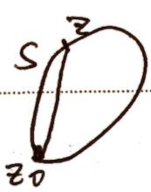
Then $|g(z)|\frac{d^2}{2} \le \frac{1}{2\pi}\int_0^d \int_0^{2\pi} |g(z+\rho e^{i\theta})|\,\rho\, d\theta\, d\rho = \frac{1}{2\pi}\iint_{|w-z|<d} |g(w)|\,dxdy$. Let $g = f^2$, then

$|f(z)| \le \frac{1}{\sqrt{\pi}\, d}\|f\|_{L^2(D_d(z))}$. Take $d = r-s$ $\Rightarrow |f(z)| \le \frac{1}{\sqrt{\pi}(r-s)}\|f\|_{L^2(D_{r-s}(z))} \le \frac{1}{\sqrt{\pi}(r-s)}\|f\|_{L^2(D_r(z_0))}$

Take all $z \in D_s(z_0) \Rightarrow \|f\|_{L^\infty(D_s(z_0))} \le \frac{1}{\sqrt{\pi}(r-s)}\|f\|_{L^2(D_r(z_0))}$. □

(b) Let $K \subset\subset U$, take $d < \mathrm{dist}(K, \mathbb{C}\setminus U)$, then $\|f\|_{L^\infty(K)} \le \frac{1}{\sqrt{\pi}\, d}\|f\|_{L^2(U)}$ ($f: U \to \mathbb{C}$). If $\{f_n\}$ Cauchy sequence resp to $\|\cdot\|_{L^2(U)}$, then it is Cauchy resp $\|\cdot\|_{L^\infty(K)}$ $\Rightarrow \{f_n|_K\}$ convergent uniformly. Such K covers U $\Rightarrow \{f_n\}$ convergence pointwisely. $\forall\, \tilde{K}$ compact $\xRightarrow{\text{Cantor}}$ $\{f_n\}$ convergence uniformly. □

Ex. 21. (a) Let S be a * circle in the $\Omega \subseteq \mathbb{C}$.

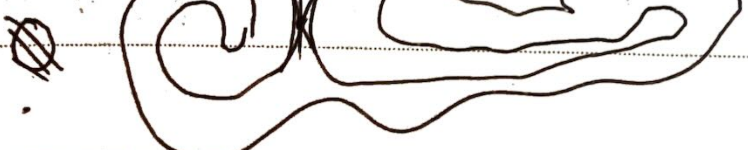


For all $z \in S$ and l_{z,z_0} $\Rightarrow$ line connected z & z_0.

Let l_{z,z_0} be $I \to \Omega$, then from t, $0 \to 1$

$\Rightarrow$ homotopy $\Rightarrow \pi_1 = 0$. □

(b) as (a). □

(c) 

✓ □

Ex 22. First we claim that $\lim_{r\to 1^-}\int_{C_r} f(z)dz = \int_{C_1} f(z)dz$. Actually, we find that for $0<r<1$ we have $\int_{C_r} f(z)dz = r\int_{C_1} f(rz)dz$, then

$$\left|\int_{C_1} f(z)dz - \int_{C_r} f(z)dz\right| = \left|\int_{C_1}(f(z)-f(rz))dz + (1-r)\int_{C_1} f(rz)dz\right|$$
$$\leq \left|\int_{C_1}(f(z)-f(rz))dz\right| + (1-r)\left|\int_{C_1} f(rz)dz\right|.$$

Since f is continuous on $\overline{\mathbb{D}} \Rightarrow |f|\leq M$, & f is uniformly continuous, then

$$\left|\int_{C_1} f - \int_{C_r} f\right| \leq 2\pi \max(|f(z)-f(rz)| : |z|=1) + (1-r)2\pi M.$$

Use this easy to find $\varepsilon \Rightarrow \left|\int_{C_1} f - \int_{C_r} f\right| < \varepsilon$ when $r\to 1$. So $\lim_{r\to 1^-}\int_{C_r} f = \int_{C_1} f$. So $\int_{C_1} f = 0$.

But $\int_{C_1} f = \int_{C_1}\frac{1}{z}dz = 2\pi i$ which is impossible. $\square$

Problem 1. (a) We define $f_n(z) = \frac{z}{n+1}$, then $\frac{1}{n}\notin f(\mathbb{D})$.

$f_n'(0) = \frac{1}{n+1}$. $\square$

(b) Let $f_\varepsilon(z) = \varepsilon(e^{\frac{z}{\varepsilon}}-1)$, then $f_\varepsilon(0)=0$, $f_\varepsilon'(0)=1$.

But $|f_\varepsilon(-1)| = \varepsilon\left|e^{\frac{1}{\varepsilon}}-1\right|$. ~~$\exists\, \delta>0,\ \varepsilon<\delta \Rightarrow |f_\varepsilon(-1)|$~~.

$\Rightarrow |f_\varepsilon(-1)| \leq \varepsilon$ (when ε so small). $\Rightarrow$ well done. $\square$

(c) (This is "Area Theorem") Since h is injective $\Rightarrow h(\{|z|=\rho\})$ divide $\mathbb{C}$ into two parts by Jordan curve thm and $h(D_\rho(0)\setminus\{0\})$ is the area contains ∞. We let the complement of it denoted G_ρ and let $\gamma_\rho = \partial G_\rho$, so we have (in the next page)

$$\text{Area}(G_\rho) = \frac{1}{2i}\int_{G_\rho} d\bar{w}\wedge dw = \frac{1}{2i}\int_{\gamma_\rho} \bar{w}\, dw$$

$$= -\frac{1}{2i}\int_0^{2\pi} \overline{h(\rho e^{i\theta})}\, h'(\rho e^{i\theta})\, i\rho e^{i\theta} d\theta$$

$$= -\frac{\rho}{2}\int_0^{2\pi}\Big(\frac{e^{i\theta}}{\rho}+\sum_{m=0}^{\infty}\overline{c_m}\rho^m e^{-im\theta}\Big)\Big(-\frac{e^{-2i\theta}}{\rho^2}+\sum_{n=1}^{\infty} n c_n \rho^{n-1} e^{i(n-1)\theta}\Big)e^{i\theta}d\theta$$

$$= -\frac{1}{2}\int_0^{2\pi}\Big(\frac{e^{i\theta}}{\rho}+\sum_{m=0}^{\infty}\overline{c_m}\rho^m e^{-im\theta}\Big)\Big(-\frac{e^{-i\theta}}{\rho}+\sum_{n=1}^{\infty} n c_n \rho^{n} e^{in\theta}\Big)d\theta$$

$$= -\pi\Big(\sum_{k=1}^{\infty} k|c_k|^2\rho^{2k}-\rho^{-2}\Big)\geq 0 \Rightarrow \sum_{k=1}^{\infty} k|c_k|^2\rho^{2k}\leq \rho^{-2}.$$

Let $\rho\to 1 \Rightarrow \sum_{n=1}^{\infty} n|c_n|^2 \leq 1$. □

(d). $\frac{f(z)}{z}$ ⇒ nowhere vanishing by f is injective ⇒ $\exists\, \psi$ such that $\psi^2 = \frac{f(z)}{z}$

and $\psi(0)=1$. Let $g(z) = z\psi(z^2)$, then $g(0)=0$, $g'(0)=1$,

easy to see that g is injective. (Just verify). □

(e) $\frac{1}{g(z)} = \sqrt{\frac{1}{f(z^2)}} = \frac{1}{z}\sqrt{\frac{1}{1+a_2z^2+a_3z^4+\cdots}} = \frac{1}{z}+b_0+b_1z+\cdots$

$\Rightarrow 1 = (1+b_0z+b_1z^2+\cdots)^2\cdot(1+a_2z^2+a_3z^4+\cdots)$.

$\Rightarrow b_1 = -\frac{a_2}{2} \Rightarrow \frac{1}{g(z)} = \frac{1}{z}-\frac{a_2}{2}z+\cdots$. g is injective, use (c)

we have $|\frac{a_2}{2}|^2\leq 1 \Rightarrow |a_2|\leq 2$. The equality holds

$\Leftrightarrow a_2 = 2e^{i\theta}$, $b_2=b_3=\cdots=0 \Rightarrow \frac{1}{g(z)} = \frac{1}{z}-e^{i\theta}z$, so

$$f(z^2) = g^2(z) = \frac{z^2}{(1-e^{i\theta}z^2)^2} \Rightarrow f = \frac{z}{(1-e^{i\theta}z)^2}. \quad □$$

(f) Consider $\varphi(z) = \frac{1}{f(z)-w_1} = \frac{z}{1+(c_0-w_1)z+c_1z^2+c_2z^3+\cdots} = b_0+b_1z+\cdots$,

so we have $b_0=0$, $b_1=1$, $b_2=w_1-c_0$, so $\varphi(z) \Rightarrow \varphi(0)=0$,

$\varphi'(0)=1$, φ injective. Use (e) $\Rightarrow |b_2| = |w_1-c_0|\leq 2$.

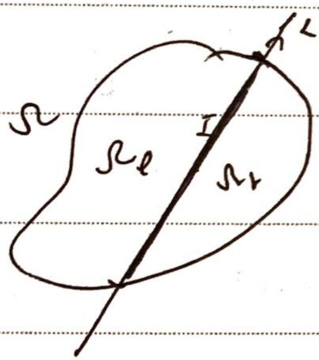
Similar $\Rightarrow |w_2-c_0|\leq 2 \Rightarrow |w_1-w_2|\leq |w_1-c_0|+|w_2-c_0|\leq 4$. □

(g) If f avoids w, then $\frac{1}{f}$ avoids 0 and $\frac{1}{w}$. Use (f) $\Rightarrow \left|\frac{1}{w}-0\right| \leq 4$, so $|w| \geq \frac{1}{4}$, well done! $\square$

Problem 2. Omitted. $\square$

Problem 3. This is just Laurent series, omitted. $\square$

Problem 4. Consider the diagram:



Let $\tilde{\Omega}_\ell = \Omega_\ell \cup I$, $\tilde{\Omega}_r = \Omega_r \cup I$.

So $\tilde{\Omega}_\ell \cup \tilde{\Omega}_r = \Omega$, $\pi_1(I)=0$ since I is an interval. So use van Kampen's thm, we have $\pi_1(\Omega) = \pi_1(\Omega_\ell) * \pi_1(\Omega_r) / N = 0$.

So $\Omega \Rightarrow$ simply connected. $\square$

Chapter 8.

Ex 1. WLOG we let $f(0)=0$. $f'(0)\neq 0$, so $\mathrm{ord}(0)=1$. $\exists r>0$ s.t. $f(z)$ holo on $|z|\leq r$ and $f(z)\neq 0$ on $|z|=r$. Since $|z|=r$ compact $\Rightarrow$ $f(z)\geq m>0$. on $|z|=r$.

$\exists \delta>0$, $\forall |z|<\delta<r \Rightarrow |f(z)|<m$, so if f is not injective over $|z|<\delta$,

then $\exists z_1\neq z_2 \in \{|z|<\delta\}$ s.t. $f(z_1)=f(z_2)=w_0\neq 0$, $|w_0|<m$.

So in $|z|=r \Rightarrow |f(z)|\geq m>w_0\neq 0$, $|w_0|<m$.

Use Rouché $\Rightarrow N(f-w_0, C)=N(f, C)=1$. This is impossible. $\square$

Ex 3. Use prop 1.1 on this chapter, one has $f: U\to V$, $g: V\to U$ with $f\circ g=id$, $g\circ f=id$, then $\pi_1(U)\cong\pi_1(V)=0$. $\square$

Ex 4.

Ex 5. For $f(z)=w \Leftrightarrow z^2+2wz+1=0$, we have $z=-w\pm\sqrt{w^2-1}$.

~~Then~~ With $z=x+iy$, $-\frac{1}{2}(z+\frac{1}{z})=-\frac{1}{2|z|^2}\left(x(|z|^2+1)+y(|z|^2-1)i\right)$

Then $\mathrm{Im}(-\frac{1}{2}(z+\frac{1}{z}))>0 \Leftrightarrow |z|<1$ whenever $y>0$. So f from $\mathbb{D}^+\to\mathbb{H}$.

Just show that f is bijective. We construct its inverse. This inverse given by $g(w)=-w+\sqrt{w^2-1}$, where the holo branch of $\sqrt{(\cdot)}$ on $\mathbb{C}\setminus[-1,1]$

take $-2+\sqrt{3}$ by $w=2$. $\square$

Ex.10. Let $g: \mathbb{H} \to \mathbb{D}$ by $z \mapsto \frac{i-z}{i+z}$ and $F: \mathbb{H} \to \mathbb{D}$ by $F(i) = 0$.

Let $F \circ g^{-1}: \mathbb{D} \to \mathbb{D}$, we find that $F \circ g^{-1}(0) = F(i) = 0$.

Use Schwarz lemma $\Rightarrow |F \circ g^{-1}(z)| \le |z|$ for all $z \in \mathbb{D}$

Since g is bijective $\Rightarrow |F(z)| \le |g(z)| = \left|\frac{z-i}{z+i}\right|$ for all $z \in \mathbb{H}$. □

Ex.11. Take $g(z) := \frac{1}{M} f(Rz)$ with $g: \mathbb{D} \to \mathbb{D}$. Let $\varphi = \frac{z - \frac{f(0)}{M}}{1 - \overline{\left(\frac{f(0)}{M}\right)} \cdot z} : \mathbb{D} \to \mathbb{D}$

with $\varphi \Rightarrow$ conformal, then $\varphi \circ g: \mathbb{D} \to \mathbb{D}$ satisfies $\varphi \circ g(0) = 0$.

Use Schwarz lemma $\Rightarrow |\varphi \circ g(z)| \le |z|$, that is,

~~f(z)~~ $\left|\frac{f(Rz) - f(0)}{M^2 - \overline{f(0)} f(Rz)}\right| \le \frac{|z|}{M}$ for all $z \in \mathbb{D}$.

So $\left|\frac{f(z) - f(0)}{M^2 - \overline{f(0)} f(z)}\right| \le \frac{|z|}{MR}$ for all $z \in D(0, r)$. □

Ex 12. (a) Use Riemann mapping thm $\Rightarrow \exists$ conformal $\varphi: \mathbb{D} \to \mathbb{D}$

by $z_0 \mapsto 0$. We have $f: \mathbb{D} \to \mathbb{D}$ fixing $z_0 \mapsto z_0$, $z_1 \mapsto z_1$, $z_0 \neq z_1$.

Let ~~$z_2 = \varphi^{-1}(z_0)$~~ $\varphi(z_1) = z_2 \neq 0$, then $\varphi \circ f \circ \varphi^{-1}: \mathbb{D} \to \mathbb{D}$

by $0 \mapsto 0$ and $z_2 \mapsto z_2$. Use Schwarz $\Rightarrow \varphi \circ f \circ \varphi^{-1}$ rotation

and $z_2 \mapsto z_2 \Rightarrow \varphi \circ f \circ \varphi^{-1} = id_{\mathbb{D}} \Rightarrow f = id_{\mathbb{D}}$. □

(b) Use $\varphi: \mathbb{H} \to \mathbb{D}$, $z \mapsto \frac{i-z}{i+z}$ is a conformal map,

$$\begin{tikzcd} \mathbb{D} \arrow[r, "f"] & \mathbb{D} \arrow[d, "\varphi^{-1}"] \\ \mathbb{H} \arrow[u, "\varphi"] \arrow[r, "\varphi^{-1} \circ f \circ \varphi"'] & \mathbb{H} \end{tikzcd}$$

just need to show that $\exists\ \mathbb{H} \to \mathbb{H}$ s.t.

no fixed points! consider $g: \mathbb{H} \to \mathbb{H}$ by $z \mapsto z+1$. Well done. □

Ex 13. (a) For $f:\mathbb{D}\to\mathbb{D}$, consider $\varphi_\alpha(z)=\frac{\alpha-z}{1-\bar{\alpha}z}:\mathbb{D}\to\mathbb{D}$ be a conformal map.
Consider $\varphi_{f(w)}\circ f\circ\varphi_w^{-1}$, then $\varphi_{f(w)}\circ f\circ\varphi_w^{-1}: 0\longmapsto 0$, $\mathbb{D}\to\mathbb{D}$. Use Schwarz lemma $\Rightarrow \left|\varphi_{f(w)}\circ f\circ\varphi_w^{-1}(z)\right|\le|z|$, then $\left|(\varphi_{f(w)}\circ f)(z)\right|\le\left|\varphi_w(z)\right|$, that is, $\rho(f(z),f(w))\le\rho(z,w)$.

When $f:\mathbb{D}\to\mathbb{D}$ is an automorphism, then $\varphi_{f(w)}\circ f\circ\varphi_w^{-1}:\mathbb{D}\to\mathbb{D}$ is automorphism taking 0 to 0, so it is a rotation, then $\left|\varphi_{f(w)}\circ f\circ\varphi_w^{-1}\right|=|z|$. □

(b) [Schwarz-Pick lemma]. Use (a) we have $\left|\frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}\right|\le\left|\frac{z-w}{1-\bar{w}z}\right|$. $\forall z,w\in\mathbb{D}$
Then $\left|\frac{f(z)-f(w)}{z-w}\right|\cdot\left|\frac{1}{1-\overline{f(w)}f(z)}\right|\le\frac{1}{|1-\bar{w}z|}$. Let $w\to z$ we have
$\frac{|f'(z)|}{1-|f(z)|^2}\le\frac{1}{1-|z|^2}$, well done. □

Ex 14. ~~Le~~ For $f:\mathbb{H}\to\mathbb{D}$ be a conformal map (equivalence)., then
~~Let~~ let $\varphi_\alpha=\frac{z-\alpha}{z-\bar{\alpha}}$ where

$$\begin{tikzcd} \mathbb{H} \arrow[r, "f"] \arrow[d, "\varphi_{f^{-1}(0)}"'] & \mathbb{D} \\ \mathbb{D} \arrow[ur, "f\circ\varphi^{-1}_{f^{-1}(0)}"'] & \end{tikzcd}$$

easy to see that φ_α be a conformal equivalence. So $f\circ\varphi^{-1}_{f^{-1}(0)}:\mathbb{D}\to\mathbb{D}$ be a conformal equivalence.

So $f\circ\varphi^{-1}_{f^{-1}(0)}=e^{i\theta}\frac{\alpha-z}{1-\bar{\alpha}z}$, that is,

$$f=e^{i\theta}\frac{\alpha-\frac{z-f^{-1}(0)}{z-\overline{f^{-1}(0)}}}{1-\bar{\alpha}\frac{z-f^{-1}(0)}{z-\overline{f^{-1}(0)}}}=e^{i\theta}\frac{\alpha z-\alpha\overline{f^{-1}(0)}-z+f^{-1}(0)}{z-\overline{f^{-1}(0)}-\bar{\alpha}z+\bar{\alpha}f^{-1}(0)}$$

$$=e^{i\theta}\frac{(\alpha-1)z-(\alpha\overline{f^{-1}(0)}-f^{-1}(0))}{-((\bar{\alpha}-1)z-(\bar{\alpha}f^{-1}(0)-\overline{f^{-1}(0)}))}=e^{i\varphi}\frac{z-\beta}{z-\bar{\beta}}\cdot\frac{\alpha-1}{\bar{\alpha}-1}$$

where $\varphi=\theta+\pi$ and $\beta=\frac{\alpha\overline{f^{-1}(0)}-f^{-1}(0)}{\alpha-1}$. Since $\left|\frac{\alpha-1}{\bar{\alpha}-1}\right|=1$, we
let $\frac{\alpha-1}{\bar{\alpha}-1}=e^{i\delta}$, then $f=e^{i(\varphi+\delta)}\cdot\frac{z-\beta}{z-\bar{\beta}}$. □

Problem 4. (a) $J=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $\langle z,w\rangle = w^H J z$.

$\langle Mz, Mw\rangle = \langle z, w\rangle \Leftrightarrow M^H J M = J$ ✓. $\square$

(b) $G=\left\{\varphi_{\frac{b}{\bar{a}}} = \frac{\frac{b}{\bar{a}}-z}{1-\frac{\bar{b}}{a}z} : a,b\in\mathbb{C},\ |a|^2-|b|^2=1\right\}$. Actually, we find that

$$e^{2i\theta}\frac{\alpha-z}{1-\bar{\alpha}z} = \frac{\frac{\alpha e^{i\theta}}{\sqrt{1-|\alpha|^2}} - \frac{e^{i\theta}z}{\sqrt{1-|\alpha|^2}}}{\frac{e^{-i\theta}}{\sqrt{1-|\alpha|^2}} - \frac{\bar{\alpha}e^{-i\theta}z}{\sqrt{1-|\alpha|^2}}} \in G$$

where $\left|\frac{\alpha e^{i\theta}}{\sqrt{1-|\alpha|^2}}\right|^2 - \left|\frac{e^{-i\theta}}{\sqrt{1-|\alpha|^2}}\right|^2 = 1 \Rightarrow G \cong \mathrm{Aut}(\mathbb{D})$.

Consider $\sigma: SU(1,1)\to G$, $\begin{pmatrix}a&b\\\bar{b}&\bar{a}\end{pmatrix}\mapsto \frac{az+b}{\bar{b}z+\bar{a}}$. Easy to check that σ is a group homomorphism. Finally, we claim that $\ker\sigma=\{\pm 1\}\cong\mathbb{Z}/2\mathbb{Z}$. This is easy to see. $\square$