§1 Review for GW-invariants.

→ **Def.** n-marked stable maps & family version. omitted. ◁

⇒ **Thm.** X projective variety, $\beta \in H_2(X,\mathbb{Z})$, then $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is proper DM-stack with projective coarse moduli space $\overline{M}_{g,n}(X,\beta)$.

Moreover, $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,\beta)$ & $\underbrace{\overline{\mathcal{M}}_{g,n}(\mathrm{pt},1)}_{\overline{\mathcal{M}}_{g,n}}$ are smooth stacks. ◁

[Rmk. $2g+n<3 \Rightarrow \overline{\mathcal{M}}_{g,n}(X,\beta)=\emptyset$].

To define Gromov–Witten invariants, first define the virtual classes.

Let X smooth projective always!

⇝ ☆ Construction I. Explicit construction!

Consider smooth quasi-separated algebraic stack $\mathfrak{M}^{pre}_{g,n}$ of prestable curves with separated diagonal & of $\dim = 3g-3+n$. We have forgetful map $F: \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow \mathfrak{M}^{pre}_{g,n}$. Consider universal $\mathfrak{U}^{pre}_{g,n} \longrightarrow \mathfrak{M}^{pre}_{g,n}$ & :

$$\begin{tikzcd} X & \mathfrak{U}^{pre}_{g,n} \times_{\mathfrak{M}^{pre}_{g,n}} \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[l, "f"'] \arrow[r, "\pi"] \arrow[d, "\pi'"] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[d, "F"] \\ & \mathfrak{U}^{pre}_{g,n} \arrow[r] & \mathfrak{M}^{pre}_{g,n} \end{tikzcd}$$

with cycle map f. Then : ($L^\bullet$ for truncated cotangent complex). $T^\bullet_* = (L^\bullet_*)^\vee$

we have morphism: $\mathbb{L}f^* L^\bullet_X \to L^\bullet_{\mathfrak{U}^{pre}_{g,n} \times_{\mathfrak{M}^{pre}_{g,n}} \overline{\mathcal{M}}_{g,n}(X,\beta)} \to L^\bullet_{\pi'} \cong \pi^* L^\bullet_F \cdots (1)$

$$\Rightarrow (\mathbb{R}\pi_* \mathbb{L}f^* T^\bullet_X)^\vee \cong \mathbb{R}\pi_* \mathbb{R}\mathcal{H}om(\mathbb{L}f^* T^\bullet_X, \pi^! \mathcal{O})$$
$$= \mathbb{R}\pi_* (\mathbb{L}f^* L^\bullet_X \otimes^{\mathbb{L}} \omega^\bullet_\pi)$$
$$\xrightarrow{(1)} \mathbb{R}\pi_* (\pi^* L^\bullet_F \otimes^{\mathbb{L}} \omega^\bullet_\pi) \cong L^\bullet_F \otimes^{\mathbb{L}} \mathbb{R}\pi_* \omega^\bullet_\pi$$
$$\cong L^\bullet_F.$$

As X smooth, then $G^\bullet := (\mathbb{R}\pi_* \mathbb{L}f^* T^\bullet_X)^\vee = (\mathbb{R}\pi_* f^* T_X)^\vee$.

→ **prop** [BF97]. $G^\bullet \longrightarrow L^\bullet_F$ as above is a POT of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ related to $\mathfrak{M}^{pre}_{g,n}$. ◁

As we have distinguished triangle :

$$\mathbb{L}F^* L^\bullet_{\mathfrak{M}^{pre}_{g,n}} \to L^\bullet_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \to L^\bullet_F \xrightarrow{+1} \cdots,$$

& $\mathfrak{M}^{pre}_{g,n}$ smooth ⇒ global resolution $\mathbb{L}F^* L^\bullet_{\mathfrak{M}^{pre}_{g,n}} = [A^0 \to A^1]$.

One can show (by [Beh97]) $G^\bullet$ also $G^\bullet = [G^{-1} \to G^0]$.

⇒ we have :

$$\begin{tikzcd} G^\bullet \arrow[r] \arrow[d, "\psi_F"] & A^\bullet[1] \arrow[r] \arrow[d] & E^\bullet[1] \arrow[r] \arrow[d, dashed, "\phi[1]"] & G^\bullet[1] \arrow[d, "\psi_F[1]"] \\ L^\bullet_F \arrow[r] & \mathbb{L}F^* L^\bullet_{\mathfrak{M}^{pre}_{g,n}}[1] \arrow[r] & L^\bullet_{\overline{\mathcal{M}}_{g,n}(X,\beta)}[1] \arrow[r] & L^\bullet_F[1] \end{tikzcd}$$

where $E^\bullet = [G^{-1} \to G^0 \oplus A^0 \to A^1]$.

Then we have $\phi: E^\bullet \to L^\bullet_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$. By 5-lemma & 4-lemma, we get $H^{-1}(\phi)$ surjective & $H^0(\phi)$ bijective & $H^1(\phi): H^1(E^\bullet) \xrightarrow{\cong} H^1(L^\bullet_{\overline{\mathcal{M}}_{g,n}(X,\beta)}) = 0$ bijective ⇒ $H^1(E^\bullet) = 0$. As $G^0 \oplus A^0 \to A^1$ surjective, taking truncation ⇒ get a global resolution of $E^\bullet$ of 2 vector bundles! By [BF97], we get :

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in CH_{vd(\overline{\mathcal{M}}_{g,n}(X,\beta))}(\overline{\mathcal{M}}_{g,n}(X,\beta)).$$

⇝ **prop.** $vd(\overline{\mathcal{M}}_{g,n}(X,\beta)) = \int_\beta c_1(T_X) + (\dim X - 3)(1-g) + n$

proof As construction above, we have:

$$vd(\overline{\mathcal{M}}_{g,n}(X,\beta)) = rk\, G^0 - rk\, G^{-1} + rk(\ker(A^0 \to A^1))$$
$$= \chi((\mathbb{R}\pi_*(f^* T_X))^\vee) + \dim \mathfrak{M}^{pre}_{g,n}$$
$$= rk(\pi_*(f^* T_X)) - rk(\mathbb{R}^1\pi_*(f^* T_X)) + 3g-3+n$$
$$= \chi(C, f^* T_X) + 3g-3+n \quad \text{(for some } [C] \in \mathfrak{M}^{pre}_{g,n}, \text{ as constant} \Rightarrow \text{take smooth one).}$$
$$= \int_C ch(f^* T_X) \cdot td(T_C) + 3g-3+n$$
$$= \int_C (\dim X, c_1(T_X), \cdots) \cdot (1, \tfrac{1}{2} c_1(T_C), \cdots) + 3g-3+n$$
$$= \frac{\dim X}{2} \int_C c_1(T_C) + \int_\beta c_1(T_X) + 3g-3+n$$
$$= \dim X (1-g) + \int_\beta c_1(T_X) + 3g-3+n = \checkmark. \qquad \square$$

Rmk. We can also using $vd = \mathrm{Def}(\overline{\mathcal{M}}_{g,n}(X,\beta)) - \mathrm{Obs}(\overline{\mathcal{M}}_{g,n}(X,\beta))$.

As $T_{\overline{\mathcal{M}}_{g,n}(X,\beta)}|_{(C,f,p_1,\ldots,p_n)} = \mathrm{Ext}^1([\mu^* \Omega_X \to \Omega_C(p_1+\cdots+p_n)], \mathcal{O}_C)$

& Obs lies in $\mathrm{Ext}^2([\mu^* \Omega_X \to \Omega_C(p_1+\cdots+p_n)], \mathcal{O}_C)$

where $\mu: (C,f,p_1,\ldots,p_n) \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ a point.

we can prove this again. ◁

Rmk. When X admits a $\mathbb{C}^*$-action, then the construction above ⇒ $\mathbb{C}^*$-equivariant & get $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in CH^{\mathbb{C}^*}_*(\overline{\mathcal{M}}_{g,n}(X,\beta))$.

◁

⇝ ☆ Construction II. Derived construction!

Derived stack $\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)$ is quasismooth, with inclusion $i: \overline{\mathcal{M}}_{g,n}(X,\beta) \hookrightarrow \mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)$, then

$i^* \mathbb{L}_{\mathbb{R}\overline{\mathcal{M}}_{g,n}(X,\beta)} \to \mathbb{L}_{\overline{\mathcal{M}}_{g,n}(X,\beta)}$ is a POT isomorphic to the one in construction I. ⇒ get $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$.

◁

Finally, we have:

→ **Def.** The Gromov–Witten invariants is

$$\langle GW^X_{g,n,\beta}\rangle(\alpha_1,\dots,\alpha_n) = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} e_1^*(\alpha_1)\cup\dots\cup e_n^*(\alpha_n)$$

with evaluate morphism $e_i : (f, p_1,\dots,p_n)\mapsto f(p_i)$

& $\alpha_1,\dots,\alpha_n \in H^*(X,\mathbb{Q})$ or $CH_*(X)$ if you want. ◁

Let's compute an example.

→ **Example.** Let $V\subseteq \mathbb{P}^4$ smooth quintic 3fold & $d\in\mathbb{Z}_{>0}$. Let ℓ be the line-class, we now compute $\langle GW^V_{0,0,d\ell}\rangle := \int_{[\overline{\mathcal{M}}_{0,0}(V,d\ell)]^{vir}} 1$.

Let V_d is vector bundle of $rk=5d+1$ on $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4,d)$ with fiber over $f:C\to\mathbb{P}^4$ is $H^0(C,f^*\mathcal{O}_{\mathbb{P}^4}(d))$. As $V=Z(s)\subseteq\mathbb{P}^4$ for some $s\in H^0(\mathcal{O}_{\mathbb{P}^4}(d))$, we get $s\longrightarrow$ determine a $\tilde{s}$ of V_d.

& $\overline{\mathcal{M}}_{0,0}(V,d\ell)=Z(\tilde{s})$. Let $i:\overline{\mathcal{M}}_{0,0}(V,d\ell)\hookrightarrow\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4,d)$. So we have

$$[\overline{\mathcal{M}}_{0,0}(V,d\ell)]^{vir} = \underset{\text{refined Gysin.}}{0^!}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4,d)) = 0^*(C_{Z(\tilde{s})/\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4,d)}).$$

$\Rightarrow i_*[\overline{\mathcal{M}}_{0,0}(V,d\ell)]^{vir} = e(V_d)\cap[\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4,d)]$ by [Fulton].

So $$\langle GW^V_{0,0,d\ell}\rangle = \int_{[\overline{\mathcal{M}}_{0,0}(V,d\ell)]^{vir}} 1 = \int_{\overline{\mathcal{M}}_{0,0}(\mathbb{P}^4,d)} e(V_d).$$ ◁

<u>Rmk.</u> We have the following description of GW-invariants:

Consider diagram: $(n+2g\geq 3)$

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[r, "\pi"] & X^n\times\overline{\mathcal{M}}_{g,n} \arrow[r, "p_1"] \arrow[d, "p_2"] & X^n \\ & \overline{\mathcal{M}}_{g,n} & \end{tikzcd}$$

where $\pi:(f;p_1,\dots,p_n)\mapsto(f(p_1),\dots,f(p_n))\times[C]$.

Then we have class $GW^X_{g,n,\beta} = p_{2*}\left(p_1^*(\alpha_1\boxtimes\dots\boxtimes\alpha_n)\cap\pi_*[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}\right)$ in $CH_*(\overline{\mathcal{M}}_{g,n})$.

then
$$\int_{\overline{\mathcal{M}}_{g,n}} GW^X_{g,n,\beta} = \int_{X^n\times\overline{\mathcal{M}}_{g,n}} p_1^*(\alpha_1\boxtimes\dots\boxtimes\alpha_n)\cap\pi_*[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$$
$$= \int_{X^n\times\overline{\mathcal{M}}_{g,n}} \pi_*\left(\pi^*p_1^*(\alpha_1\boxtimes\dots\boxtimes\alpha_n)\cap[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}\right)$$
$$= \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} e_1^*(\alpha_1)\cup\dots\cup e_n^*(\alpha_n)$$
$$= \langle GW^X_{g,n,\beta}\rangle(\alpha_1,\dots,\alpha_n) \quad !$$

$GW^X_{g,n,\beta}$ ⇒ called Gromov–Witten classes. ◁

We state some useful results of Gromov–Witten invariants.

⇒ **Prop.** Let X smooth projective.

(a). $\langle GW^X_{g,n,\beta}\rangle(\alpha_1,\dots,\alpha_n)$ linear in each variables.

(b) If β not effective, then $\langle GW^X_{g,n,\beta}\rangle(\alpha_1,\dots,\alpha_n)=0$.

(c). We have $\langle GW^X_{g,n,\beta}\rangle(\alpha_1,\dots,\alpha_{n-1},[X])=0$ when $n+2g\geq 4$ or $\begin{cases}\beta\neq 0\\ n\geq 1\end{cases}$.

(d). We have $\langle GW^X_{g,n,\beta}\rangle(\underbrace{\alpha_1,\dots,\alpha_{n-1}}_{\text{at } \deg=vd.},\alpha_n) = \int_\beta \alpha_n \langle GW^X_{g,n-1,\beta}\rangle(\alpha_1,\dots,\alpha_{n-1})$ when $n+2g\geq 4$ or $\begin{cases}\beta\neq 0\\ n\geq 1\end{cases}$.

(e) $\langle GW^X_{0,n,0}\rangle(\alpha_1,\dots,\alpha_n) = \begin{cases}\int_X \alpha_1\cup\alpha_2\cup\alpha_3 & ; n=3\\ 0 & ; \text{otherwise.}\end{cases}$

<u>proof.</u> (a) & (b) are trivial. We consider (c), (d), (e).

For (c), note that $\pi_n:\overline{\mathcal{M}}_{g,n}(X,\beta)\to\overline{\mathcal{M}}_{g,n-1}(X,\beta)$ (forget the last marked pts.) exists when $n+2g\geq 4$ or $\begin{cases}\beta\neq 0\\ n\geq 1\end{cases}$. In these cases, easy to show by base-change:

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} = \pi_n^*[\overline{\mathcal{M}}_{g,n-1}(X,\beta)]^{vir}.$$

Then
$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} e_1^*(\alpha_1)\cup\dots\cup e_{n-1}^*(\alpha_{n-1})\cup e_n^*[X]$$
$$= \int_{[\overline{\mathcal{M}}_{g,n-1}(X,\beta)]^{vir}} e_1^*(\alpha_1)\cup\dots\cup e_{n-1}^*(\alpha_{n-1})$$
by dimension reason.

For (d). Similar reason as (c).

For (e). Trivial by dimension reason. □

§2. Some Topological recursive relation.

→ **Def.** Define the gravitational correlator of $\gamma_1,\dots,\gamma_n\in H^*(X)$ & $d_i\in\mathbb{Z}_{\geq 0}$ is

$$\langle\tau_{d_1}(\gamma_1),\dots,\tau_{d_n}(\gamma_n)\rangle_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n \left(\psi_i^{d_i}\cup e_i^*(\gamma_i)\right)$$

where ψ_i ⇒ psi-classes defined by $s_i^*\omega_{\pi_{n+1}}$.

[Rmk]. $\langle\tau_0(\gamma_1),\dots,\tau_0(\gamma_n)\rangle_{g,\beta} = \langle GW^X_{g,n,\beta}\rangle(\gamma_1,\dots,\gamma_n)$. (we will let $\tau_0(\gamma_i)=\gamma_i$).

→ **Def.** Let ω ⇒ complexified Kähler class on X

① Define genus g couplings are

$$\langle\langle\tau_{d_1}(\gamma_1),\dots,\tau_{d_n}(\gamma_n)\rangle\rangle_g := \sum_{k=0}^{\infty}\sum_\beta \frac{1}{k!}\langle\tau_{d_1}(\gamma_1),\dots,\tau_{d_n}(\gamma_n),\underbrace{\gamma,\dots,\gamma}_{k}\rangle_{g,\beta}\,q^\beta$$

where $q^\beta := e^{2\pi i\int_\beta\omega}$ & $\gamma=\sum_i t_iT_i$ where $T_0=1,\dots,T_m$ ⇒ basis of $H^*(X,\mathbb{Q})$.

② Define genus g gravitational Gromov–Witten potential is

$$\Phi_g^{grav}(\gamma) = \sum_{n=0}^{\infty}\sum_{\beta\in H_2(X,\mathbb{Z})}\frac{1}{n!}\langle\underbrace{\gamma,\dots,\gamma}_{n}\rangle_{g,\beta}\,q^\beta$$

where $q^\beta = e^{2\pi i\int_\beta\omega}$.

[Rmk]. $\langle\langle \tau_{d_1}(T_{i_1}), \ldots, \tau_{d_n}(T_{i_n}) \rangle\rangle_g = \dfrac{\partial \Phi_g^{grav}}{\partial t_{d_1}^{i_1} \cdots \partial t_{d_n}^{i_n}} \Big|_{t_d^j = 0, \forall d > 0}$.

$\Rightarrow$ prop (a). [String equation]:

$$\langle \tau_{d_1}(\gamma_1), \ldots, \tau_{d_{n-1}}(\gamma_{n-1}), 1 \rangle$$

$$= \sum_{i=1}^{n-1} \langle \tau_{d_1}(\gamma_1), \ldots, \tau_{d_{i-1}}(\gamma_{i-1}), \tau_{d_{i+1}}(\gamma_{i+1}), \ldots, \tau_{d_{n-1}}(\gamma_{n-1}) \rangle_{g,\beta}.$$

(b). [Dilaton equation].

$$\langle \tau_1(1), \tau_{d_1}(\gamma_1), \ldots, \tau_{d_n}(\gamma_n) \rangle_{g,\beta}$$

$$= (2g-2+n) \langle \tau_{d_1}(\gamma_1), \ldots, \tau_{d_n}(\gamma_n) \rangle_{g,\beta}.$$

(c). [Divisor equation]. Let D divisor.

$$\langle \tau_{d_1}(\gamma_1), \ldots, \tau_{d_{n-1}}(\gamma_{n-1}), D \rangle_{g,\beta} = \left(\int_\beta D \right) \langle \tau_{d_1}(\gamma_1), \ldots, \tau_{d_{n-1}}(\gamma_{n-1}) \rangle_{g,\beta}$$

$$+ \sum_{j=1}^{n-1} \langle \tau_{d_1}(\gamma_1), \ldots, \tau_{d_{j-1}}(\gamma_{j-1}), \tau_{d_j - 1}(D \cup \gamma_j), \tau_{d_{j+1}}(\gamma_{j+1}), \ldots, \tau_{d_{n-1}}(\gamma_{n-1}) \rangle_{g,\beta}.$$

☆ Thm (Topological recursion).