Dualizing Complexes Using Derived Categories

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1 Introduction

The theory of duality is beautiful theory in the commutative algebra and algebraic geometry. In the book [BH98], the authors introduce the duality of Cohen-Macaulay rings. But it can not work for more general rings. In this note, we will introduce the general theory of duality of more general rings using derived category. We will generalize the notion of canonical modules in [BH98] into dualizing complexes and we will see that in the case of Cohen-Macaulay rings, this complex can be concentrated in the one place. So we again have the canonical modules.

Now we give the outline of the note:

- In the section 2, we will give a quike introduction of local cohomology using derived category which is a basic tool.
- In the section 3, we will introduce the basic theory of dualizing complexes of Noetherian rings and introduce the local duality theorem.
- In the section 4, we will consider the special case of Cohen-Macaulay and Gorenstein Rings and to find some special properties. We will also consider the more cases of rings which have dualizing complexes
- In the final section 5, we will give a glimpse of the global theory of dualizing complexes in algebraic geometry.

Note that we will mainly follows the chapter 47 in [Pro24] and Chapter 25 in [GW23]. The basic theory of derived categopries, injective hulls and Matlis duality will be omitted and we refer to [BH98] or the beginning of chapter 47 in [Pro24].

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2 Local Cohomology of Noetherian Rings

Here we will follows section 47.8–47.11 in [Pro24] and summary some results about several definitions of local cohomology using derived categories which we will use later. The reader can begin to read the Section 3 and omit this whole section for now.

More general theory we refer [Har67] or [Gro68] or chapter51 of [Pro24].

2.1 More on Čech Complex and Koszul Complex

Lemma 2.1. *Let* R *be a ring. Let* $\phi : E \to R$ *be an* R*-module map. Let* $e \in E$ *with image* $f = \phi(e)$ *in* R*. Then* $f = de + ed$ *as endomorphisms of* $\mathbf{K}_\bullet(\phi)$*. In particular, multiplication by* f_i *on* $\mathbf{K}_\bullet(f_1, ..., f_r)$ *is homotopic to zero.*

Proof. This is true because $d(ea) = d(e)a - ed(a) = fa - ed(a)$. □

Lemma 2.2. *Let* R *be a ring. Let* $f_1, \ldots, f_r \in R$. *The (extended alternating) Čech complex of* R *is the cochain complex*

$$R \to \bigoplus_{i_0} R_{f_{i_0}} \to \bigoplus_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r}$$

where R *is in degree* 0, *the term* $\bigoplus_{i_0} R_{f_{i_0}}$ *is in degre* 1, *and so on. The maps are defined as follows*

1. *The map* $R \to \bigoplus_{i_0} R_{f_{i_0}}$ *is given by the canonical maps* $R \to R_{f_{i_0}}$.
2. *Given* $1 \leqslant i_0 < \ldots < i_{p+1} \leqslant r$ *and* $0 \leqslant j \leqslant p+1$ *we have the canonical localization map*
$$R_{f_{i_0} \ldots \hat{f}_{i_j} \ldots f_{i_{p+1}}} \to R_{f_{i_0} \ldots f_{i_{p+1}}}$$
3. *The differentials use the canonical maps of (2) with sign* $(-1)^j$.

Then

$$R \to \bigoplus_{i_0} R_{f_{i_0}} \to \bigoplus_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r}$$

is a colimit of the Koszul complexes $\mathbf{K}(R, f_1^n, \ldots, f_r^n)$*; see proof for a precise statement.*

Proof. We have

$$\mathbf{K}(R, f_1^n, \ldots, f_r^n) : 0 \to \wedge^r(R^{\oplus r}) \to \wedge^{r-1}(R^{\oplus r}) \to \ldots \to R^{\oplus r} \to R \to 0$$

with the term $\wedge^r(R^{\oplus r})$ sitting in degree 0. Let $e_1^n, \ldots, e_r^n$ be the standard basis of $R^{\oplus r}$. Then the elements $e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n$ for $1 \leqslant j_1 < \ldots < j_{r-p} \leqslant r$ form a basis for the term in degree p of the Koszul complex. Further, observe that

$$d(e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n) = \sum (-1)^{a+1} f_{j_a}^n e_{j_1}^n \wedge \ldots \wedge \hat{e}_{j_a}^n \wedge \ldots \wedge e_{j_{r-p}}^n.$$

The transition maps of our system

$$\mathbf{K}(R, f_1^n, \ldots, f_r^n) \to \mathbf{K}(R, f_1^{n+1}, \ldots, f_r^{n+1})$$

are given by the rule

$$e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n \longmapsto f_{i_0} \ldots f_{i_{p-1}} e_{j_1}^{n+1} \wedge \ldots \wedge e_{j_{r-p}}^{n+1}$$

where the indices $1 \leqslant i_0 < \ldots < i_{p-1} \leqslant r$ are such that $\{1, \ldots r\} = \{i_0, \ldots, i_{p-1}\} \amalg \{j_1, \ldots, j_{r-p}\}$. We omit the short computation that shows this is compatible with differentials. Observe that the transition maps are always 1 in degree 0 and equal to $f_1 \ldots f_r$ in degree r.

Denote $\mathbf{K}^p(R, f_1^n, \ldots, f_r^n)$ the term of degree p in the Koszul complex. Observe that for any $f \in R$ we have

$$R_f = \varinjlim (R \xrightarrow{f} R \xrightarrow{f} R \to \ldots)$$

Hence we see that in degree p we obtain

$$\varinjlim K^p(R, f_1^n, \ldots f_r^n) = \bigoplus_{1 \leqslant i_0 < \ldots < i_{p-1} \leqslant r} R_{f_{i_0} \ldots f_{i_{p-1}}}$$

Here the element $e_{j_1}^n \wedge \ldots \wedge e_{j_{r-p}}^n$ of the Koszul complex above maps in the colimit to the element $(f_{i_0} \ldots f_{i_{p-1}})^{-n}$ in the summand $R_{f_{i_0} \ldots f_{i_{p-1}}}$ where the indices are chosen such that $\{1, \ldots r\} = \{i_0, \ldots, i_{p-1}\} \amalg \{j_1, \ldots, j_{r-p}\}$. Thus the differential on this complex is given by

$$d(1 \text{ in } R_{f_{i_0} \ldots f_{i_{p-1}}}) = \sum_{i \notin \{i_0, \ldots, i_{p-1}\}} (-1)^{i-t} \text{ in } R_{f_{i_0} \ldots f_{i_t} f_i f_{i_{t+1}} \ldots f_{i_{p-1}}}$$

Thus if we consider the map of complexes given in degree p by the map

$$\bigoplus_{1 \leqslant i_0 < \ldots < i_{p-1} \leqslant r} R_{f_{i_0} \ldots f_{i_{p-1}}} \longrightarrow \bigoplus_{1 \leqslant i_0 < \ldots < i_{p-1} \leqslant r} R_{f_{i_0} \ldots f_{i_{p-1}}}$$

determined by the rule

$$1 \text{ in } R_{f_{i_0} \ldots f_{i_{p-1}}} \longmapsto (-1)^{i_0 + \ldots + i_{p-1} + p} \text{ in } R_{f_{i_0} \ldots f_{i_{p-1}}}$$

then we get an isomorphism of complexes from $\varinjlim \mathbf{K}(R, f_1^n, \ldots, f_r^n)$ to the extended alternating Čech complex defined in this section. We omit the verification that the signs work out. □

2.2 Deriving Torsion

Definition 2.3. *Let* R *be a ring with an ideal* $I \subset \mathsf{R}$*. Fix* $\mathsf{M} \in \mathbf{Mod}_{\mathsf{R}}$*.*

(a) We define

$$\mathsf{M}[I^n] := \{m \in \mathsf{M} : I^n m = 0\}, \quad \mathsf{M}[I^\infty] = \bigoplus_n \mathsf{M}[I^n].$$

(b) We call M *is* I^∞*-torsion if* $\mathsf{M} = \mathsf{M}[I^\infty]$*. We let* I^∞**-torsion** *be the subcategory of* $\mathbf{Mod}_{\mathsf{R}}$ *consist of* I^∞*-torsion modules*

Here we give some easy but important properties of this notion.

Proposition 2.4. *Let* R *be a ring with an ideal* $I \subset \mathsf{R}$*.*

(a) Let $\mathsf{M} \in I^\infty$**-torsion***, then* M *admits a resolution*

$$\cdots \to \mathsf{K}_2 \to \mathsf{K}_1 \to \mathsf{K}_0 \to \mathsf{M} \to 0$$

with each K_i *a direct sum of copies of* R/I^n *for* n *variable. In particular, the category* I^∞**-torsion** *is a Grothendieck abelian category.*

(b) Let I *be a finitely generated ideal of* R*, then for any* $\mathsf{M} \in \mathbf{Mod}_{\mathsf{R}}$ *we have* $(\mathsf{M}/\mathsf{M}[I^\infty])[I] = 0$*.*

(c) Let I *be a finitely generated ideal of* R*, then* I^∞**-torsion** *is a Serre subcategory of the abelian category* $\mathbf{Mod}_{\mathsf{R}}$*, that is, an extension of* I^∞*-torsion modules is* I^∞*-torsion.*

(d) Let I *be a finitely generated ideal of* R *and* $\mathsf{M} \in \mathbf{Mod}_{\mathsf{R}}$*, then we have an exact sequence*

$$0 \to \mathsf{M}[I^\infty] \to \mathsf{M} \to \prod_{\mathfrak{p} \notin V(I)} \mathsf{M}_{\mathfrak{p}}.$$

In particular, we have $\mathsf{M} \in I^\infty$**-torsion** *if and only if* $\operatorname{supp}(\mathsf{M}) \subset V(I)$*. Hence the subcategory* I^∞**-torsion** $\subset \mathbf{Mod}_{\mathsf{R}}$ *depends only on the closed subset* $V(I) \subset \operatorname{Spec}(\mathsf{R})$*.*

Proof. For (a), there is a canonical surjection $\bigoplus_{m \in \mathsf{M}} \mathsf{R}/I^{n_m} \to \mathsf{M} \to 0$ where n_m is the smallest positive integer such that $I^{n_m} \cdot m = 0$. The kernel of the preceding surjection is also an I^∞-torsion module. Proceeding inductively, we construct the desired resolution of M.

For (b), Let $m \in \mathsf{M}$. If m maps to an element of $(\mathsf{M}/\mathsf{M}[I^\infty])[I]$ then $Im \subset \mathsf{M}[I^\infty]$. Write $I = (f_1, ..., f_t)$. Then we see that $f_i m \in \mathsf{M}[I^\infty]$. Thus we see that $I^N m = 0$ for some large $N \gg 0$. Hence m maps to zero in $(\mathsf{M}/\mathsf{M}[I^\infty])$.

For (c), suppose that $0 \to \mathsf{M}' \to \mathsf{M} \to \mathsf{M}'' \to 0$ is a short exact sequence of modules with M' and M'' both I^∞-torsion modules. Then $\mathsf{M}' \subset \mathsf{M}[I^\infty]$ and hence $\mathsf{M}/\mathsf{M}[I^\infty]$ is a quotient of M'' and therefore I^∞-torsion. Combined with (b) this implies that it is zero.

For (d), let $\mathsf{M} \in \mathbf{Mod}_{\mathsf{R}}$ and let $x \in \mathsf{M}$. If $x \in \mathsf{M}[I^\infty]$, then x maps to zero in M_f for all $f \in I$. Hence x maps to zero in $\mathsf{M}_{\mathfrak{p}}$ for all $I \not\subset \mathfrak{p}$. Conversely, if x maps to zero in $\mathsf{M}_{\mathfrak{p}}$ for all $I \not\subset \mathfrak{p}$, then x maps to zero in M_f for all $f \in I$. Hence if $I = (f_1, ..., f_r)$, then $f_i^{n_i} x = 0$ for some $n_i \geqslant 1$. It follows that $x \in \mathsf{M}[I^N]$ for $N = \sum_i n_i$. Thus $\mathsf{M}[I^\infty]$ is the kernel of $\mathsf{M} \to \prod_{\mathfrak{p} \notin Z} \mathsf{M}_{\mathfrak{p}}$. ☐

Definition 2.5 (Fake Local Cohomology)**.** *Let* R *be a ring and let* I *be a finitely generated ideal. By Proposition 2.4(a), the category* I^∞**-torsion** *is a Grothendieck abelian category and hence the derived category* $\mathbf{D}(I^\infty\textbf{-torsion})$ *exists by some homological algebra, as Tag 079Q. Hence we have the derived functor*

$$\mathbf{R}\Gamma_I : \mathbf{D}(\mathsf{R}) \to \mathbf{D}(I^\infty\textbf{-torsion})$$

of $\Gamma_I : \mathbf{Mod}_{\mathsf{R}} \to I^\infty$**-torsion** *given by* $\mathsf{M} \mapsto \mathsf{M}[I^\infty]$ *which is left exact.*

Moreover, we define $H^q_I(\mathsf{K}) := H^q(\mathbf{R}\Gamma_I(\mathsf{K}))$ *for any* $\mathsf{K} \in \mathbf{D}(\mathsf{R})$*.*

Remark 2.6. *Note that this functor does not deserve the name local cohomology unless the ring* R *is Noetherian.*

Now we discuss some basic properties of the functor.

Proposition 2.7. *Let* R *be a ring and let* I *be a finitely generated ideal.*

(a) The functor $\mathbf{R}\Gamma_I$ *is right adjoint to the functor* $\mathbf{D}(I^\infty\textbf{-torsion}) \to \mathbf{D}(\mathsf{R})$*.*

(b) For any object K *of* $\mathbf{D}(\mathsf{R})$ *we have*

$$\mathbf{R}\Gamma_I(\mathsf{K}) = \mathrm{hocolim}\mathbf{R}\,\mathrm{Hom}_{\mathsf{R}}(\mathsf{R}/I^n, \mathsf{K})$$

in $\mathbf{D}(\mathsf{R})$ *and hence*

$$H^q_I(\mathsf{K}) := \mathbf{R}^q\Gamma_I(\mathsf{K}) = \varinjlim \mathrm{Ext}^q_{\mathsf{R}}(\mathsf{R}/I^n, \mathsf{K})$$

as modules for all $q \in \mathbb{Z}$*.*

(c) Let $\mathsf{K}^\bullet$ *be a complex of* A*-modules such that* $f : \mathsf{K}^\bullet \to \mathsf{K}^\bullet$ *is an isomorphism for some* $f \in I$*, i.e.,* $\mathsf{K}^\bullet$ *is a complex of* R_f*-modules. Then* $\mathbf{R}\Gamma_I(\mathsf{K}^\bullet) = 0$*.*

Proof. For (a), this follows from the fact that taking I^∞-torsion submodules is the right adjoint to the inclusion functor $I^\infty\textbf{-torsion} \to \mathbf{Mod}_{\mathsf{R}}$.

For (b), let $\mathsf{J}^\bullet$ be a K-injective resolution of K. Then we have

$$\begin{aligned}\mathbf{R}\Gamma_I(\mathsf{K}) = \Gamma_I(\mathsf{J}^\bullet) = \mathsf{J}^\bullet[I^\infty] &= \varinjlim_n \mathsf{J}^\bullet[I^n] \\ &= \varinjlim_n \mathrm{Hom}_{\mathsf{R}}(\mathsf{R}/I^n, \mathsf{J}^\bullet) = \mathrm{hocolim}\mathbf{R}\,\mathrm{Hom}_{\mathsf{R}}(\mathsf{R}/I^n, \mathsf{K}).\end{aligned}$$

Well done.

For (c), in this case the cohomology modules of $\mathbf{R}\Gamma_I(\mathsf{K}^\bullet)$ are both f^∞-torsion and f acts by automorphisms. Hence the cohomology modules are zero and hence the object is zero. □

In the end of this small section we consider another category. Let R be a ring and let I be a finitely generated ideal. By Proposition 2.4(c), $I^\infty\textbf{-torsion}$ is a Serre subcategory of the abelian category $\mathbf{Mod}_{\mathsf{R}}$. This shows that $I^\infty\textbf{-torsion} \subset \mathbf{Mod}_{\mathsf{R}}$ exact which induce the functor $\mathbf{D}(I^\infty\textbf{-torsion}) \to \mathbf{D}(\mathsf{R})$ which factor through

$$\mathbf{D}(I^\infty\textbf{-torsion}) \to \mathbf{D}_{I^\infty\textbf{-torsion}}(\mathbf{Mod}_{\mathsf{R}}).$$

Proposition 2.8. *Let* R *be a ring and let* I *be a finitely generated ideal. Let* $\mathsf{M}, \mathsf{N} \in I^\infty\textbf{-torsion}$*.*

(a) $\mathrm{Hom}_{\mathbf{D}(\mathsf{R})}(\mathsf{M}, \mathsf{N}) = \mathrm{Hom}_{\mathbf{D}(I^\infty\textbf{-torsion})}(\mathsf{M}, \mathsf{N})$

(b) $\mathrm{Ext}^2_{\mathbf{D}(I^\infty\textbf{-torsion})}(\mathsf{M}, \mathsf{N}) \to \mathrm{Ext}^2_{\mathbf{D}(\mathsf{R})}(\mathsf{M}, \mathsf{N})$ *is not surjective in general. In particular,* $\mathbf{D}(I^\infty\textbf{-torsion}) \to \mathbf{D}_{I^\infty\textbf{-torsion}}(\mathbf{Mod}_{\mathsf{R}})$ *is not an equivalence in general.*

Proof. (a) is trivial and the counterexample of (b) we refer Tag 0A6P. □

Remark 2.9. *However in the Noetherian case this will be true. We will see this later.*

2.3 Basic Theory of Local Cohomology

Now we will introduce some true local cohomologies.

Theorem 2.10 (Real Local Cohomology, I)**.** *Let* R *be a ring and let* $I \subset \mathsf{R}$ *be a finitely generated ideal and* $\mathsf{Z} = \mathsf{V}(I) \subset \mathrm{Spec}(\mathsf{R})$*. There exists a right adjoint* $\mathbf{R}\Gamma_{\mathsf{Z}}$ *to the inclusion functor* $\mathbf{D}_{I^\infty\textbf{-torsion}}(\mathsf{R}) \to \mathbf{D}(\mathsf{R})$*. In fact, if* I *is generated by* $f_1, \ldots, f_r \in \mathsf{R}$*, then we have*

$$\mathbf{R}\Gamma_{\mathsf{Z}}(\mathsf{K}) = \left(\mathsf{R} \to \prod_{i_0} \mathsf{R}_{f_{i_0}} \to \prod_{i_0 < i_1} \mathsf{R}_{f_{i_0} f_{i_1}} \to \ldots \to \mathsf{R}_{f_1 \ldots f_r}\right) \otimes^{\mathbf{L}}_{\mathsf{R}} \mathsf{K}$$

functorially in $\mathsf{K} \in \mathbf{D}(\mathsf{R})$*.*

Proof. Say $I = (f_1, \dots, f_r)$ is an ideal. Let $K^\bullet$ be a complex of R-modules. There is a canonical map of complexes

$$\left(R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0<i_1} R_{f_{i_0} f_{i_1}} \to \dots \to R_{f_1 \dots f_r}\right) \longrightarrow R.$$

from the extended Čech complex to R. Tensoring with $K^\bullet$, taking associated total complex, we get a map

$$\mathrm{Tot}\left(K^\bullet \otimes_R (R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0<i_1} R_{f_{i_0} f_{i_1}} \to \dots \to R_{f_1 \dots f_r})\right) \longrightarrow K^\bullet$$

in $\mathbf{D}(R)$. We claim the cohomology modules of the complex on the left are I^∞-torsion, i.e., the LHS is an object of $D_{I^\infty\text{-}\mathbf{torsion}}(R)$. Namely, we have

$$\left(R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0<i_1} R_{f_{i_0} f_{i_1}} \to \dots \to R_{f_1 \dots f_r}\right) = \varinjlim \mathbf{K}(R, f_1^n, \dots, f_r^n)$$

by Lemma 2.2. Moreover, multiplication by f_i^n on the complex $\mathbf{K}(R, f_1^n, \dots, f_r^n)$ is homotopic to zero by Lemma 2.1. Since

$$H^q(\mathsf{LHS}) = \varinjlim H^q(\mathrm{Tot}(K^\bullet \otimes_A \mathbf{K}(R, f_1^n, \dots, f_r^n)))$$

we obtain our claim. On the other hand, if $K^\bullet$ is an object of $D_{I^\infty\text{-}\mathbf{torsion}}(R)$, then the complexes $K^\bullet \otimes_R R_{f_{i_0} \dots f_{i_p}}$ have vanishing cohomology. Hence in this case the map $\mathsf{LHS} \to K^\bullet$ is an isomorphism in $D(A)$. The construction

$$\mathbf{R}\Gamma_Z(K^\bullet) = \mathrm{Tot}\left(K^\bullet \otimes_R \left(R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0<i_1} R_{f_{i_0} f_{i_1}} \to \dots \to R_{f_1 \dots f_r}\right)\right)$$

is functorial in $K^\bullet$ and defines an exact functor $\mathbf{D}(R) \to \mathbf{D}_{I^\infty\text{-}\mathbf{torsion}}(R)$ between triangulated categories. It follows formally from the existence of the natural transformation $\mathbf{R}\Gamma_Z \to \mathrm{id}$ given above and the fact that this evaluates to an isomorphism on $K^\bullet$ in the subcategory, that $\mathbf{R}\Gamma_Z$ is the desired right adjoint. □

Hence now we have the functor

$$\mathbf{R}\Gamma_Z : \mathbf{D}(R) \to \mathbf{D}_{I^\infty\text{-}\mathbf{torsion}}(R).$$

As we have seen, we construct the functor using Čech complex. Is there some relation between this and the functor in algebraic geometry?

Definition 2.11 (Real Local Cohomology, II)**.** *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. Consider the functor $\Gamma_Z : \mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_X(X))$ given by*

$$\Gamma_Z(\mathscr{F}) := \{s \in \Gamma(X, \mathscr{F}) : \mathrm{supp}(s) \subset Z\}.$$

Using K-injective resolutions, we obtain the right derived functor

$$\mathbf{R}\Gamma_Z(X, -) : \mathbf{D}(\mathscr{O}_X) \to \mathbf{D}(\mathscr{O}_X(X)).$$

The group $H^q_Z(X, K) = H^q(\mathbf{R}\Gamma_Z(X, K))$ the cohomology module with support in Z.

We now show that they are the same! Indeed, we can use Čech complex to rebuild $\mathbf{R}\Gamma_Z(X, -)$, and then we can connected to $\mathbf{R}\Gamma_Z : \mathbf{D}(R) \to \mathbf{D}_{I^\infty\text{-}\mathbf{torsion}}(R)$ as before.

Proposition 2.12. *Let R be a ring and let I be a finitely generated ideal. Set $Z = V(I) \subset X = \mathrm{Spec}(R)$. For $K \in \mathbf{D}(A)$ corresponding to $\widetilde{K} \in \mathbf{D}_{\mathrm{QCoh}}(\mathscr{O}_X)$, there is a functorial isomorphism*

$$\mathbf{R}\Gamma_Z(K) = \mathbf{R}\Gamma_Z(X, \widetilde{K}).$$

Proof. Note that there exists a distinguished triangle

$$\mathbf{R}\Gamma_Z(X, \widetilde{K}) \to \mathbf{R}\Gamma(X, \widetilde{K}) \to \mathbf{R}\Gamma(U, \widetilde{K}) \to \mathbf{R}\Gamma_Z(X, \widetilde{K})[1]$$

where $U = X \setminus Z$. We know that $\mathbf{R}\Gamma(X, \widetilde{K}) = K$. Say $I = (f_1, \ldots, f_r)$. Then we obtain a finite affine open covering $\mathcal{U} : U = D(f_1) \cup \ldots \cup D(f_r)$. As affine schemes are separated, the alternating Čech complex $\mathrm{Tot}(\widehat{\mathcal{C}}^\bullet_{alt}(\mathcal{U}, \widetilde{K^\bullet}))$ computes $R\Gamma(U, \widetilde{K})$ where $K^\bullet$ is any complex of R-modules representing K. Working through the definitions we find

$$\mathbf{R}\Gamma(U, \widetilde{K}) = \mathrm{Tot}\left(K^\bullet \otimes_R (\prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r})\right)$$

It is clear that $K^\bullet = \mathbf{R}\Gamma(X, \widetilde{K^\bullet}) \to \mathbf{R}\Gamma(U, \widetilde{K}^\bullet)$ is induced by the diagonal map from A into $\prod R_{f_i}$. Hence we conclude that

$$\mathbf{R}\Gamma_Z(X, \mathcal{F}^\bullet) = \mathrm{Tot}\left(K^\bullet \otimes_R \left(R \to \prod_{i_0} R_{f_{i_0}} \to \prod_{i_0 < i_1} R_{f_{i_0} f_{i_1}} \to \ldots \to R_{f_1 \ldots f_r}\right)\right)$$

Well dominate. □

Nex we will introduce the noetherian case and compare the fake local cohomology and the real cohomology.

Proposition 2.13. *Let R be a Noetherian ring and let $I \subset R$ be an ideal.*

1. *The adjunction $\mathbf{R}\Gamma_I(K) \to K$ is an isomorphism for $K \in \mathbf{D}_{I^\infty\text{-}\mathbf{torsion}}(R)$.*
2. *The functor $\mathbf{D}(I^\infty\text{-}\mathbf{torsion}) \to \mathbf{D}_{I^\infty\text{-}\mathbf{torsion}}(R)$ is an equivalence.*
3. $\mathbf{R}\Gamma_I(K) = \mathbf{R}\Gamma_Z(K)$ *for $K \in \mathbf{D}(R)$.*

Proof. Boring proof, we refer Tag 0955. □

So in the Noetherian case (so in the whole theory we consider) we will use $\mathbf{R}_I(-)$.

Finally we will consider local cohomology and completion which will used in local duality theorem. We will consider noetherian to avoid the derived completion.

Proposition 2.14. *Let A be a Noetherian ring and let I be an ideal. For an object $K \in \mathbf{D}(A)$ we have*

$$\mathbf{R}\Gamma_Z(K^\wedge) = \mathbf{R}\Gamma_Z(K) \quad \textit{and} \quad (\mathbf{R}\Gamma_Z(K))^\wedge = K^\wedge.$$

Hence the functors $\mathbf{R}\Gamma_Z$ and $^\wedge$ define quasi-inverse equivalences of categories $\mathbf{D}_{I^\infty\text{-}torsion}(A) \leftrightarrow \mathbf{D}_{comp}(A, I)$.

Proof. See Tag 0A6W. □

2.4 Local Cohomology and Depth

In this small section we will introduce a result about the depth and local cohomology. Note that $\mathrm{depth}_I(M)$ here is the grade $\mathrm{Grade}(I, M)$ in [BH98].

Theorem 2.15. *Let R be a Noetherian ring, let $I \subset R$ be an ideal, and let M be a finite A-module such that $IM \neq M$. Then the following integers are equal:*

(1) $\mathrm{depth}_I(M)$,

(2) the smallest integer i such that $\mathrm{Ext}^i_A(A/I, M)$ is nonzero, and

(3) the smallest integer i such that $H^i_I(M)$ is nonzero.

Moreover, we have $\mathrm{Ext}^i_A(N, M) = 0$ *for* $i < \mathrm{depth}_I(M)$ *for any finite* A*-module* N *annihilated by a power of* I.

Proof. We prove the equality of (1) and (2) by induction on $\mathrm{depth}_I(M)$ which is allowed since $\mathrm{depth}_I(M) < \infty$ now.

If $\mathrm{depth}_I(M) = 0$, then I is contained in the union of the associated primes of M. By prime avoidance we see that $I \subset \mathfrak{p}$ for some associated prime $\mathfrak{p}$. Hence $\mathrm{Hom}_A(A/I, M)$ is nonzero. Thus equality holds in this case.

Assume that $\mathrm{depth}_I(M) > 0$. Let $f \in I$ be a nonzerodivisor on M such that $\mathrm{depth}_I(M/fM) = \mathrm{depth}_I(M) - 1$. Consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

and the associated long exact sequence for $\mathrm{Ext}^*_A(A/I, -)$. Note that $\mathrm{Ext}^i_A(A/I, M)$ is a finite A/I-module. Hence we obtain

$$\mathrm{Hom}_A(A/I, M/fM) = \mathrm{Ext}^1_A(A/I, M)$$

and short exact sequences

$$0 \to \mathrm{Ext}^i_A(A/I, M) \to \mathrm{Ext}^i_A(A/I, M/fM) \to \mathrm{Ext}^{i+1}_A(A/I, M) \to 0$$

Thus the equality of (1) and (2) by induction.

Observe that $\mathrm{depth}_I(M) = \mathrm{depth}_{I^n}(M)$ for all $n \geqslant 1$ for example by the fact that the sequence $(f_1, ..., f_r)$ is regular if and only if $(f_1^{e_1}, \ldots, f_r^{e_r})$ is regular for any fixed $e_i > 0$ (see Tag 07DV for the proof). Hence by the equality of (1) and (2) we see that $\mathrm{Ext}^i_A(A/I^n, M) = 0$ for all n and $i < \mathrm{depth}_I(M)$. Let N be a finite A-module annihilated by a power of I. Then we can choose a short exact sequence

$$0 \to N' \to (A/I^n)^{\oplus m} \to N \to 0$$

for some $n, m \geqslant 0$. Then $\mathrm{Hom}_A(N, M) \subset \mathrm{Hom}_A((A/I^n)^{\oplus m}, M)$ and $\mathrm{Ext}^i_A(N, M) \subset \mathrm{Ext}^{i-1}_A(N', M)$ for $i < \mathrm{depth}_I(M)$. Thus a simply induction argument shows that the final statement of the lemma holds.

Finally, we prove that (3) is equal to (1) and (2). We have $H^p_I(M) = \varinjlim \mathrm{Ext}^p_A(A/I^n, M)$ by Proposition 2.7(b). Thus we see that $H^i_I(M) = 0$ for $i < \mathrm{depth}_I(M)$. For $i = \mathrm{depth}_I(M)$, using the vanishing of $\mathrm{Ext}^{i-1}_A(I/I^n, M)$ we see that the map $\mathrm{Ext}^i_A(A/I, M) \to H^i_I(M)$ is injective which proves nonvanishing in the correct degree. □

3 Dualizing Complexes

In this section we will consider the general theory of dualizing complexes of Noetherian rings.

3.1 General Properties of Dualizing Complexes

Definition 3.1. *Let* A *be a Noetherian ring. A* dualizing complex *is a complex of* A*-modules* $\omega^\bullet_A$ *such that*

(a) $\omega^\bullet_A$ *has finite injective dimension.*

(b) $H^i(\omega^\bullet_A)$ *is a finite* A*-module for all* i.

(c) $A \to \mathbf{R}\mathrm{Hom}_A(\omega^\bullet_A, \omega^\bullet_A)$ *is a quasi-isomorphism.*

Next we consider some most basic properties.

Proposition 3.2. *Let* A *be a Noetherian ring. If* $\omega^\bullet_A$ *is a dualizing complex, then the functor*

$$D : K \longmapsto \mathbf{R}\mathrm{Hom}_A(K, \omega^\bullet_A)$$

is an anti-equivalence $\mathbf{D}_{Coh}(A) \to \mathbf{D}_{Coh}(A)$ *which exchanges* $\mathbf{D}^+_{Coh}(A)$ *and* $\mathbf{D}^-_{Coh}(A)$ *and induces an anti-equivalence* $\mathbf{D}^b_{Coh}(A) \to \mathbf{D}^b_{Coh}(A)$. *Moreover* $D \circ D$ *is isomorphic to the identity functor.*

Proof. Note that $\mathbf{R}\operatorname{Hom}_A(K, \omega_A^\bullet) \in \mathbf{D}_{\mathrm{Coh}}(A)$ follows from the fact that $\omega_A^\bullet$ has finite injective dimension and consider the exact triangle $\tau_{\leqslant n}K \to K \to \tau_{\geqslant n+1}K \to \tau_{\leqslant n}K[1]$.

Now we know that there is a canonical morphism

$$K = \mathbf{R}\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet) \otimes_A^{\mathbf{L}} K \longrightarrow \mathbf{R}\operatorname{Hom}_A(R\operatorname{Hom}_A(K, \omega_A^\bullet), \omega_A^\bullet)$$

by taking K-injective resolutions. We can show that this is an isomorphism when K is pseudo-coherent (see Tag 0A68, since we can choose finite projective modules as resolution). Consider exact triangle $\tau_{\leqslant n}K \to K \to \tau_{\geqslant n+1}K \to \tau_{\leqslant n}K[1]$ again, then this is an isomorphism for $\tau_{\leqslant n}K$. So it suffices to show that both $\tau_{\geqslant n+1}K$ and $\mathbf{R}\operatorname{Hom}_A(\mathbf{R}\operatorname{Hom}_A(\tau_{\geqslant n+1}K, \omega_A^\bullet), \omega_A^\bullet)$ have vanishing cohomology in degrees $\leqslant n - c$ for some c. But $\omega_A^\bullet$ has finite injective dimension, this is trivial. □

Let R be a ring. Recall that an object L of $\mathbf{D}(R)$ is invertible if it is an invertible object for the symmetric monoidal structure on $\mathbf{D}(R)$ given by derived tensor product.

Proposition 3.3. *Let A be a Noetherian ring.*

(a) Let $F : \mathbf{D}^b_{Coh}(A) \to \mathbf{D}^b_{Coh}(A)$ be an A-linear equivalence of categories. Then $F(A)$ is an invertible object of $\mathrm{D}(A)$.

(b) [Uniqueness] If $\omega_A^\bullet$ and $(\omega_A')^\bullet$ are dualizing complexes, then $(\omega_A')^\bullet$ is quasi-isomorphic to $\omega_A^\bullet \otimes_A^{\mathbf{L}} L$ for some invertible object L of $\mathbf{D}(A)$.

(c) Let $B = S^{-1}A$ be a localization. If $\omega_A^\bullet$ is a dualizing complex, then $\omega_A^\bullet \otimes_A B$ is a dualizing complex for B.

(d) Let $f_1, \ldots, f_n \in A$ generate the unit ideal. If $\omega_A^\bullet$ is a complex of A-modules such that $(\omega_A^\bullet)_{f_i}$ is a dualizing complex for A_{f_i} for all i, then $\omega_A^\bullet$ is a dualizing complex for A.

(e) Let $A \to B$ be a finite ring map of Noetherian rings. Let $\omega_A^\bullet$ be a dualizing complex. Then $R\operatorname{Hom}(B, \omega_A^\bullet)$ is a dualizing complex for B. In particular, this is right for any surjective ring map.

Proof. For (a), this is not about the dualizing complex and complicated. We refer Tag 0A7E.

For (b), By Proposition 3.2 and (a) the functor

$$K \mapsto \mathbf{R}\operatorname{Hom}_A(\mathbf{R}\operatorname{Hom}_A(K, \omega_A^\bullet), (\omega_A')^\bullet)$$

maps A to an invertible object L. In other words, there is an isomorphism

$$L \longrightarrow \mathbf{R}\operatorname{Hom}_A(\omega_A^\bullet, (\omega_A')^\bullet)$$

Since L has finite tor dimension, this means that we can apply the similar proof in Proposition 3.2 to see that

$$\mathbf{R}\operatorname{Hom}_A(\omega_A^\bullet, (\omega_A')^\bullet) \otimes_A^{\mathbf{L}} K \longrightarrow \mathbf{R}\operatorname{Hom}_A(\mathbf{R}\operatorname{Hom}_A(K, \omega_A^\bullet), (\omega_A')^\bullet)$$

is an isomorphism for K in $\mathbf{D}^b_{Coh}(A)$. In particular, setting $K = \omega_A^\bullet$ finishes the proof.

For (c), let $\omega_A^\bullet \to I^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded complex of injectives. Then $S^{-1}I^\bullet$ is a bounded complex of injective $B = S^{-1}A$-modules representing $\omega_A^\bullet \otimes_A B$. Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension. Since $H^i(\omega_A^\bullet \otimes_A B) = H^i(\omega_A^\bullet) \otimes_A B$ by flatness of $A \to B$ we see that $\omega_A^\bullet \otimes_A B$ has finite cohomology modules. Finally, the map

$$B \longrightarrow R\operatorname{Hom}_A(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B)$$

is a quasi-isomorphism as internal hom commutes with flat base change in this case (Tag 0A6A). Well done.

For (d), consider the double complex

$$\prod_{i_0} (\omega_A^\bullet)_{f_{i_0}} \to \prod_{i_0 < i_1} (\omega_A^\bullet)_{f_{i_0} f_{i_1}} \to \ldots$$

The associated total complex is quasi-isomorphic to $\omega_A^\bullet$ for example by the descent theory.

By assumption the complexes $(\omega_A^\bullet)_{f_i}$ have finite injective dimension as complexes of A_{f_i}-modules. This implies that each of the complexes $(\omega_A^\bullet)_{f_{i_0}\dots f_{i_p}}$, $p>0$ has finite injective dimension over $A_{f_{i_0}\dots f_{i_p}}$. This in turn implies that each of the complexes $(\omega_A^\bullet)_{f_{i_0}\dots f_{i_p}}$, $p>0$ has finite injective dimension over A as injectivity can be descent via flat maps. Hence $\omega_A^\bullet$ has finite injective dimension as a complex of A-modules (as it can be represented by a complex endowed with a finite filtration whose graded parts have finite injective dimension). Since $H^n(\omega_A^\bullet)_{f_i}$ is a finite A_{f_i} module for each i we see that $H^i(\omega_A^\bullet)$ is a finite A-module. Finally, the (derived) base change of the map $A \to \mathbf{R}\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ to A_{f_i} is the map $A_{f_i} \to \mathbf{R}\operatorname{Hom}_A((\omega_A^\bullet)_{f_i}, (\omega_A^\bullet)_{f_i})$. Hence we deduce that $A \to \mathbf{R}\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet)$ is an isomorphism and the proof is complete.

For (e), let $\omega_A^\bullet \to I^\bullet$ be a quasi-isomorphism with $I^\bullet$ a bounded complex of injectives. Then $\operatorname{Hom}_A(B, I^\bullet)$ is a bounded complex of injective B-modules representing $\mathbf{R}\operatorname{Hom}(B, \omega_A^\bullet)$. Thus $\mathbf{R}\operatorname{Hom}(B, \omega_A^\bullet)$ has finite injective dimension. By spectral sequence $\operatorname{Ext}_A^p(B, H^q(\omega_A^\bullet)) \Rightarrow \operatorname{Ext}_A^{p+q}(B, \omega_A^\bullet)$ that $\omega_A^\bullet$ is an object of $\mathbf{D}_{Coh}(B)$. Finally, we compute

$$\operatorname{Hom}_{\mathbf{D}(B)}(\mathbf{R}\operatorname{Hom}(B, \omega_A^\bullet), \mathbf{R}\operatorname{Hom}(B, \omega_A^\bullet)) = \operatorname{Hom}_{\mathbf{D}(A)}(\mathbf{R}\operatorname{Hom}(B, \omega_A^\bullet), \omega_A^\bullet) = B$$

and for $n \neq 0$ we compute

$$\operatorname{Hom}_{\mathbf{D}(B)}(\mathbf{R}\operatorname{Hom}(B, \omega_A^\bullet), \mathbf{R}\operatorname{Hom}(B, \omega_A^\bullet)[n]) = \operatorname{Hom}_{\mathbf{D}(A)}(\mathbf{R}\operatorname{Hom}(B, \omega_A^\bullet), \omega_A^\bullet[n]) = 0$$

which proves the last property of a dualizing complex. In the displayed equations, the second equality holds by Proposition 3.2. □

Note that not all rings have dualizing complexes, here we give some way to construct new rings with this property from the old one.

Proposition 3.4. *Let A be a Noetherian ring which has a dualizing complex $\omega_A^\bullet$.*

(a) Then $\omega_A^\bullet \otimes_A A[x]$ is a dualizing complex for $A[x]$.

(b) Then any A-algebra essentially of finite type over A has a dualizing complex.

(c) Let $\mathfrak{m} \subset A$ be a maximal ideal and set $\kappa = A/\mathfrak{m}$. Then $\mathbf{R}\operatorname{Hom}_A(\kappa, \omega_A^\bullet) \cong \kappa[n]$ for some $n \in \mathbb{Z}$.

Proof. For (a), set $B = A[x]$ and $\omega_B^\bullet = \omega_A^\bullet \otimes_A B$. We know that in this case $\omega_B^\bullet$ has finite injective dimension. Since $H^i(\omega_B^\bullet) = H^i(\omega_A^\bullet) \otimes_A B$ by flatness of $A \to B$ we see that $\omega_A^\bullet \otimes_A B$ has finite cohomology modules. Finally, the map

$$B \longrightarrow R\operatorname{Hom}_B(\omega_B^\bullet, \omega_B^\bullet)$$

is a quasi-isomorphism as formation of internal hom commutes with flat base change in this case. Well done.

For (b), this follows from a combination of (a) and Proposition 3.3(c)(e).

For (c), this is true because $\mathbf{R}\operatorname{Hom}_A(\kappa, \omega_A^\bullet)$ is a dualizing complex over κ by Proposition 3.3(e), because dualizing complexes over κ are unique up to shifts (Proposition 3.3(b)), and because κ is a dualizing complex over κ. □

3.2 Dualizing Complexes over Local Rings

Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. By Proposition 3.3(b) we know that if R has a dualizing complex, then it is unique up to an invertible objects. Now we define a canonical choice this these complexes:

Definition 3.5. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Then a dualizing complex $\omega_A^\bullet$ is said to be normalized, if*

$$\mathbf{R}\operatorname{Hom}_A(\kappa, \omega_A^\bullet) \cong \kappa[0]$$

by Proposition 3.4(c).

Here we consider some basic results of normalizeddualizing complexes.

Lemma 3.6. *Let $(A, \mathfrak{m}, \kappa) \to (B, \mathfrak{m}', \kappa')$ be a finite local map of Noetherian local rings. Let $\omega_A^\bullet$ be a normalized dualizing complex. Then $\omega_B^\bullet = \mathbf{R}\operatorname{Hom}(B, \omega_A^\bullet)$ is a normalized dualizing complex for B. In particular, this is right for surjective morphism.*

Proof. By Proposition 3.3(e) the complex $\omega_B^\bullet$ is dualizing for B. We have

$$\mathbf{R}\operatorname{Hom}_B(\kappa', \omega_B^\bullet) = \mathbf{R}\operatorname{Hom}_B(\kappa', \mathbf{R}\operatorname{Hom}(B, \omega_A^\bullet)) = \mathbf{R}\operatorname{Hom}_A(\kappa', \omega_A^\bullet)$$

since $\mathbf{R}\operatorname{Hom}$ is right adjoint to the restriction functor. Since κ' is isomorphic to a finite direct sum of copies of κ as an A-module and since $\omega_A^\bullet$ is normalized, we see that this complex only has cohomology placed in degree 0. Thus $\omega_B^\bullet$ is a normalized dualizing complex as well. □

Lemma 3.7. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let F be an A-linear self-equivalence of the category of finite length A-modules. Then F is isomorphic to the identity functor.*

Proof. Since κ is the unique simple object of the category we have $F(\kappa) \cong \kappa$. Since our category is abelian, we find that F is exact. Hence $F(E)$ has the same length as E for all finite length modules E. Since $\operatorname{Hom}(E, \kappa) = \operatorname{Hom}(F(E), F(\kappa)) \cong \operatorname{Hom}(F(E), \kappa)$ we conclude from Nakayama's lemma that E and $F(E)$ have the same number of generators. Hence $F(A/\mathfrak{m}^n)$ is a cyclic A-module. Pick a generator $e \in F(A/\mathfrak{m}^n)$. Since F is A-linear we conclude that $\mathfrak{m}^n e = 0$. The map $A/\mathfrak{m}^n \to F(A/\mathfrak{m}^n)$ has to be an isomorphism as the lengths are equal. Pick an element

$$e \in \varprojlim_n F(A/\mathfrak{m}^n)$$

which maps to a generator for all n (small argument omitted). Then we obtain a system of isomorphisms $A/\mathfrak{m}^n \to F(A/\mathfrak{m}^n)$ compatible with all A-module maps $A/\mathfrak{m}^n \to A/\mathfrak{m}^{n'}$ (by A-linearity of F again). Since any finite length module is a cokernel of a map between direct sums of cyclic modules, we obtain the isomorphism of the lemma. □

Lemma 3.8. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $E = E(\kappa)$ be an injective hull of κ. Then there exists a functorial isomorphism*

$$\mathbf{R}\operatorname{Hom}_A(N, \omega_A^\bullet) = \operatorname{Hom}_A(N, E)[0]$$

for N running through the finite length A-modules.

Proof. By induction on the length of N we see that $\mathbf{R}\operatorname{Hom}_A(N, \omega_A^\bullet)$ is a module of finite length sitting in degree 0. Thus $\mathbf{R}\operatorname{Hom}_A(-, \omega_A^\bullet)$ induces an anti-equivalence on the category of finite length modules. Since the same is true for $\operatorname{Hom}_A(-, E)$ by Matlis duality we see that

$$N \longmapsto \operatorname{Hom}_A(\mathbf{R}\operatorname{Hom}_A(N, \omega_A^\bullet), E)$$

is an equivalence as in Lemma 3.7. Hence it is isomorphic to the identity functor. Since $\operatorname{Hom}_A(-, E)$ applied twice is the identity by Matlis duality again, we obtain the statement of the lemma. □

Lemma 3.9. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let M be a finite A-module and let $d = \dim(\operatorname{supp}(M))$. Then*

(a) if $\operatorname{Ext}_A^i(M, \omega_A^\bullet)$ is nonzero, then $i \in \{-d, \ldots, 0\}$,

(b) the dimension of the support of $\operatorname{Ext}_A^i(M, \omega_A^\bullet)$ is at most $-i$,

(c) $\operatorname{depth}(M)$ is the smallest integer $\delta \geqslant 0$ such that $\operatorname{Ext}_A^{-\delta}(M, \omega_A^\bullet) \neq 0$.

Proof. We prove this by induction on d. If $d = 0$, this follows from Lemma 3.8 and Matlis duality which guarantees that $\operatorname{Hom}_A(M, E)$ is nonzero if M is nonzero.

Assume the result holds for modules with support of dimension $< d$ and that M has depth > 0. Choose an $f \in \mathfrak{m}$ which is a nonzerodivisor on M and consider the short exact sequence

$$0 \to M \to M \to M/fM \to 0$$

Since $\dim(\mathrm{supp}(M/fM)) = d-1$ we may apply the induction hypothesis. Writing $E^i = \mathrm{Ext}^i_A(M, \omega_A^\bullet)$ and $F^i = \mathrm{Ext}^i_A(M/fM, \omega_A^\bullet)$ we obtain a long exact sequence

$$\ldots \to F^i \to E^i \xrightarrow{f} E^i \to F^{i+1} \to \ldots$$

By induction $E^i/fE^i = 0$ for $i+1 \notin \{-\dim(\mathrm{supp}(M/fM)), \ldots, -\mathrm{depth}(M/fM)\}$. By Nakayama's lemma we conclude $E^i = 0$ for $i \notin \{-\dim(\mathrm{supp}(M)), \ldots, -\mathrm{depth}(M)\}$. Moreover, in the boundary case $i = -\mathrm{depth}(M)$ we deduce that E^i is nonzero as F^{i+1} is nonzero by induction. Since $E^i/fE^i \subset F^{i+1}$ we get

$$\dim(\mathrm{supp}(F^{i+1})) \geqslant \dim(\mathrm{supp}(E^i/fE^i)) \geqslant \dim(\mathrm{supp}(E^i)) - 1$$

we also obtain the dimension estimate (b).

If M has depth 0 and $d > 0$ we let $N = M[\mathfrak{m}^\infty]$ and set $M' = M/N$. Then M' has depth > 0 and $\dim(\mathrm{supp}(M')) = d$. Thus we know the result for M' and since $R\,\mathrm{Hom}_A(N, \omega_A^\bullet) = \mathrm{Hom}_A(N, E)$ (Lemma 3.8) the long exact cohomology sequence of Ext's implies the result for M. □

Remark 3.10. *Let $(A, \mathfrak{m})$ and $\omega_A^\bullet$ be as in Lemma 3.9, we see that $\omega_A^\bullet$ has injective-amplitude in $[-d, 0]$ because part (c) of that lemma applies. In particular, for any A-module M (not necessarily finite) we have $\mathrm{Ext}^i_A(M, \omega_A^\bullet) = 0$ for $i \notin \{-d, \ldots, 0\}$.*

Some easy but important corollaries:

Corollary 3.11. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Then $\mathrm{depth}(A)$ is the smallest integer $\delta \geqslant 0$ such that $H^{-\delta}(\omega_A^\bullet) \neq 0$.*

Proof. Immediate from Lemma 3.9(c). □

Corollary 3.12. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. If $\dim(A) = 0$, then $\omega_A^\bullet \cong E[0]$ where E is an injective hull of the residue field.*

Proof. Immediate from Lemma 3.8. □

Corollary 3.13. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex. Let $I \subset \mathfrak{m}$ be an ideal of finite length. Set $B = A/I$. Then there is a distinguished triangle*

$$\omega_B^\bullet \to \omega_A^\bullet \to \mathrm{Hom}_A(I, E)[0] \to \omega_B^\bullet[1]$$

in $\mathbf{D}(A)$ where E is an injective hull of κ and $\omega_B^\bullet$ is a normalized dualizing complex for B.

Proof. Use the short exact sequence $0 \to I \to A \to B \to 0$ and Lemmas 3.8 and 3.6. □

Corollary 3.14. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $f \in \mathfrak{m}$ be a nonzerodivisor. Set $B = A/(f)$. Then there is a distinguished triangle*

$$\omega_B^\bullet \to \omega_A^\bullet \to \omega_A^\bullet \to \omega_B^\bullet[1]$$

in $\mathbf{D}(A)$ where $\omega_B^\bullet$ is a normalized dualizing complex for B.

Proof. Use the short exact sequence $0 \to A \to A \to B \to 0$ and Lemma 3.6. □

Lemma 3.15. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $\mathfrak{p}$ be a minimal prime of A with $\dim(A/\mathfrak{p}) = e$. Then $H^i(\omega_A^\bullet)_\mathfrak{p}$ is nonzero if and only if $i = -e$.*

Proof. Since $A_\mathfrak{p}$ has dimension zero, there exists an integer $n > 0$ such that $\mathfrak{p}^n A_\mathfrak{p}$ is zero. Set $B = A/\mathfrak{p}^n$ and $\omega_B^\bullet = R\,\mathrm{Hom}_A(B, \omega_A^\bullet)$. Since $B_\mathfrak{p} = A_\mathfrak{p}$ we see that

$$(\omega_B^\bullet)_\mathfrak{p} = R\,\mathrm{Hom}_A(B, \omega_A^\bullet) \otimes_A^{\mathbf{L}} A_\mathfrak{p} = R\,\mathrm{Hom}_{A_\mathfrak{p}}(B_\mathfrak{p}, (\omega_A^\bullet)_\mathfrak{p}) = (\omega_A^\bullet)_\mathfrak{p}$$

By Lemma 3.6 we may replace A by B. After doing so, we see that $\dim(A) = e$. Then we see that $H^i(\omega_A^\bullet)_\mathfrak{p}$ can only be nonzero if $i = -e$ by Lemma 3.9 parts (1) and (2). On the other hand, since $(\omega_A^\bullet)_\mathfrak{p}$ is a dualizing complex for the nonzero ring $A_\mathfrak{p}$ (Proposition 3.3(c)) we see that the remaining module has to be nonzero. □

In the end of this section, we will consider some dimension theory of noetherian local rings with dualizing complex.

Lemma 3.16. *Let A be a Noetherian ring. Let $\mathfrak{p}$ be a minimal prime of A. Then $H^i(\omega_A^\bullet)_\mathfrak{p}$ is nonzero for exactly one i.*

Proof. The complex $\omega_A^\bullet \otimes_A A_\mathfrak{p}$ is a dualizing complex for $A_\mathfrak{p}$ (Proposition 3.3(c)). The dimension of $A_\mathfrak{p}$ is zero as $\mathfrak{p}$ is minimal. Hence the result follows from Corollary 3.12. □

This lemmas shows that generically the cohomology of dualizing complex lying over the only place.

Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. Proposition 3.4(c) allows us to define a function

$$\delta = \delta_{\omega_A^\bullet} : \mathrm{Spec}(A) \longrightarrow \mathbb{Z}$$

by mapping $\mathfrak{p}$ to the integer $\delta(\mathfrak{p})$ which is the unique integer such that

$$(\omega_A^\bullet)_\mathfrak{p}[-\delta(\mathfrak{p})]$$

is a normalized dualizing complex over the Noetherian local ring $A_\mathfrak{p}$.

Corollary 3.17. *Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. Let $A \to B$ be a surjective ring map and let $\omega_B^\bullet = \mathbf{R}\,\mathrm{Hom}(B, \omega_A^\bullet)$ be the dualizing complex for B. Then we have*

$$\delta_{\omega_B^\bullet} = \delta_{\omega_A^\bullet}|_{\mathrm{Spec}(B)}$$

Proof. This follows from the definition of the functions and Lemma 3.6. □

Lemma 3.18. *Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex. Then the function $\delta = \delta_{\omega_A^\bullet}$ satsifies that for any specialization $x \rightsquigarrow y$ and $x \neq y$ in $\mathrm{Spec}(A)$, we have $\delta(x) > \delta(y)$, and if $x \rightsquigarrow y$ is immediate, then $\delta(x) = \delta(y) + 1$.*

Proof. Let $\mathfrak{p} \subset \mathfrak{q}$ be an immediate specialization. We have to show that $\delta(\mathfrak{p}) = \delta(\mathfrak{q}) + 1$. We may replace A by $A/\mathfrak{p}$, the complex $\omega_A^\bullet$ by $\omega_{A/\mathfrak{p}}^\bullet = R\,\mathrm{Hom}(A/\mathfrak{p}, \omega_A^\bullet)$, the prime $\mathfrak{p}$ by (0), and the prime $\mathfrak{q}$ by $\mathfrak{q}/\mathfrak{p}$, see Corollary 3.17. Thus we may assume that A is a domain, $\mathfrak{p} = (0)$, and $\mathfrak{q}$ is a prime ideal of height 1.

Then $H^i(\omega_A^\bullet)_{(0)}$ is nonzero for exactly one i, say i_0, by Lemma 3.16. In fact $i_0 = -\delta((0))$ because $(\omega_A^\bullet)_{(0)}[-\delta((0))]$ is a normalized dualizing complex over the field $A_{(0)}$.

On the other hand $(\omega_A^\bullet)_\mathfrak{q}[-\delta(\mathfrak{q})]$ is a normalized dualizing complex for $A_\mathfrak{q}$. By Lemma 3.15 we see that

$$H^e((\omega_A^\bullet)_\mathfrak{q}[-\delta(\mathfrak{q})])_{(0)} = H^{e-\delta(\mathfrak{q})}(\omega_A^\bullet)_{(0)}$$

is nonzero only for $e = -\dim(A_\mathfrak{q}) = -1$. We conclude

$$-\delta((0)) = -1 - \delta(\mathfrak{q})$$

as desired. □

By the argument of point-set topology, one can show that if A has that kind of function δ, then $\mathrm{Spec}(A)$ is catenary (see Tag 02IA) Now we using these we have the following interesting facts.

Proposition 3.19. *Let A be a Noetherian ring and let $\omega_A^\bullet$ be a dualizing complex.*

(a) Then A is universally catenary of finite dimension.

(b) Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let $d = \dim(A)$ and $\omega_A = H^{-d}(\omega_A^\bullet)$. Then

1. *the support of ω_A is the union of the irreducible components of $\mathrm{Spec}(A)$ of dimension d,*
2. *ω_A satisfies (S_2).*

Proof. For (a), this is because we have the function

$$\delta = \delta_{\omega_A^\bullet} : \mathrm{Spec}(A) \longrightarrow \mathbb{Z}$$

and by Lemma 3.18, it is catenary as we have seen. Hence by Proposition 3.4(b) that A is universally catenary.

Because any dualizing complex $\omega_A^\bullet$ is in $D^b_{Coh}(A)$ the values of the function $\delta_{\omega_A^\bullet}$ in minimal primes are bounded by Lemma 3.16. On the other hand, for a maximal ideal $\mathfrak{m}$ with residue field κ the integer $i = -\delta(\mathfrak{m})$ is the unique integer such that $\mathrm{Ext}^i_A(\kappa, \omega_A^\bullet)$ is nonzero by Proposition 3.4(c). Since $\omega_A^\bullet$ has finite injective dimension these values are bounded too. Since the dimension of A is the maximal value of $\delta(\mathfrak{p}) - \delta(\mathfrak{m})$ where $\mathfrak{p} \subset \mathfrak{m}$ are a pair consisting of a minimal prime and a maximal prime we find that the dimension of Spec(A) is bounded. This finish (a).

For (b), we will use Lemma 3.9 without further mention. By Lemma 3.15 the support of ω_A contains the irreducible components of dimension d. Let $\mathfrak{p} \subset A$ be a prime. By Lemma 3.18 the complex $(\omega_A^\bullet)_\mathfrak{p}[-\dim(A/\mathfrak{p})]$ is a normalized dualizing complex for $A_\mathfrak{p}$. Hence if $\dim(A/\mathfrak{p}) + \dim(A_\mathfrak{p}) < d$, then $(\omega_A)_\mathfrak{p} = 0$. This proves the support of ω_A is the union of the irreducible components of dimension d, because the complement of this union is exactly the primes $\mathfrak{p}$ of A for which $\dim(A/\mathfrak{p}) + \dim(A_\mathfrak{p}) < d$ as A is catenary by (a). On the other hand, if $\dim(A/\mathfrak{p}) + \dim(A_\mathfrak{p}) = d$, then

$$(\omega_A)_\mathfrak{p} = H^{-\dim(A_\mathfrak{p})}((\omega_A^\bullet)_\mathfrak{p}[-\dim(A/\mathfrak{p})])$$

Hence in order to prove ω_A has (S_2) it suffices to show that the depth of ω_A is at least $\min(\dim(A), 2)$. We prove this by induction on $\dim(A)$. The case $\dim(A) = 0$ is trivial.

Assume $\mathrm{depth}(A) > 0$. Choose a nonzerodivisor $f \in \mathfrak{m}$ and set $B = A/fA$. Then $\dim(B) = \dim(A) - 1$ and we may apply the induction hypothesis to B. By Corollary 3.14 we see that multiplication by f is injective on ω_A and we get $\omega_A/f\omega_A \subset \omega_B$. This proves the depth of ω_A is at least 1. If $\dim(A) > 1$, then $\dim(B) > 0$ and ω_B has depth > 0. Hence ω_A has depth > 1 and we conclude in this case.

Assume $\dim(A) > 0$ and $\mathrm{depth}(A) = 0$. Let $I = A[\mathfrak{m}^\infty]$ and set $B = A/I$. Then B has depth $\geqslant 1$ and $\omega_A = \omega_B$ by Corollary 3.13. Since we proved the result for ω_B above the proof is done. □

3.3 Grothendieck's Local Duality Theorem

Lemma 3.20. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let $Z = V(\mathfrak{m}) \subset \mathrm{Spec}(A)$. Then $E = R^0\Gamma_Z(\omega_A^\bullet)$ is an injective hull of κ and $R\Gamma_Z(\omega_A^\bullet) = E[0]$.*

Proof. As our ring is noetherian, we have $\mathbf{R}\Gamma_\mathfrak{m} = R\Gamma_Z$. Thus

$$\mathbf{R}\Gamma_Z(\omega_A^\bullet) = \mathbf{R}\Gamma_\mathfrak{m}(\omega_A^\bullet) = \mathrm{hocolim}\,\mathbf{R}\,\mathrm{Hom}_A(A/\mathfrak{m}^n, \omega_A^\bullet)$$

by Proposition 2.7(b). Let E' be an injective hull of the residue field. By Lemma 3.8 we can find isomorphisms

$$\mathbf{R}\,\mathrm{Hom}_A(A/\mathfrak{m}^n, \omega_A^\bullet) \cong \mathrm{Hom}_A(A/\mathfrak{m}^n, E')[0]$$

compatible with transition maps. Since $E' = \bigcup E'[\mathfrak{m}^n] = \varinjlim \mathrm{Hom}_A(A/\mathfrak{m}^n, E')$ as we seen in the course we conclude that $E \cong E'$ and that all other cohomology groups of the complex $\mathbf{R}\Gamma_Z(\omega_A^\bullet)$ are zero. □

Here is the main result of this section.

Theorem 3.21 (Grothendieck's Local Duality Theorem)**.** *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let E be an injective hull of the residue field. Let $Z = V(\mathfrak{m}) \subset \mathrm{Spec}(A)$. Denote $^\wedge$ derived completion with respect to $\mathfrak{m}$. Then*

$$\mathbf{R}\,\mathrm{Hom}_A(K, \omega_A^\bullet)^\wedge \cong \mathbf{R}\,\mathrm{Hom}_A(\mathbf{R}\Gamma_Z(K), E[0])$$

for K in $D(A)$.

Proof. Observe that $E[0] \cong R\Gamma_Z(\omega_A^\bullet)$ by Lemma 3.20. Now completion on the left hand side goes inside, thus we have to prove

$$\mathbf{R}\,\mathrm{Hom}_A(K^\wedge, (\omega_A^\bullet)^\wedge) = \mathbf{R}\,\mathrm{Hom}_A(R\Gamma_Z(K), R\Gamma_Z(\omega_A^\bullet))$$

This follows from the equivalence between $D_{\mathsf{comp}}(A, \mathfrak{m})$ and $D_{\mathfrak{m}^\infty\text{-torsion}}(A)$ given in Proposition 2.14. □

Here is a special but useful case of the theorem above.

Corollary 3.22. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $\omega_A^\bullet$ be a normalized dualizing complex. Let E be an injective hull of the residue field. Let $K \in D_{Coh}(A)$. Then*

$$\mathrm{Ext}_A^{-i}(K, \omega_A^\bullet)^\wedge = \mathrm{Hom}_A(H_{\mathfrak{m}}^i(K), E)$$

where $^\wedge$ denotes $\mathfrak{m}$-adic completion.

4 Cohen-Macaulay, Gorenstein Rings and More

4.1 Cohen-Macaulay Rings

Recall that by Corollary 3.11, if $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$, then we have

$$\mathrm{depth}(A) = \min\{\delta \geqslant 0 : H^{-\delta}(\omega_A^\bullet) \neq 0\}.$$

Proposition 4.1. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$. Let M be a finite A-module. The following are equivalent*

(a) *M is Cohen-Macaulay,*

(b) *$\mathrm{Ext}_A^i(M, \omega_A^\bullet)$ is nonzero for a single i,*

(c) *$\mathrm{Ext}_A^{-i}(M, \omega_A^\bullet)$ is zero for $i \neq \dim(\mathrm{supp}(M))$.*

Denote CM_d the category of finite Cohen-Macaulay A-modules of depth d. Then $M \mapsto \mathrm{Ext}_A^{-d}(M, \omega_A^\bullet)$ defines an anti-auto-equivalence of CM_d.

Proof. We will use the results of Lemma 3.9 without further mention. Fix a finite module M. If M is Cohen-Macaulay, then only $\mathrm{Ext}_A^{-d}(M, \omega_A^\bullet)$ can be nonzero, hence (a) ⇒ (c). The implication (3c) ⇒ (b) is immediate. Assume (b) and let $N = \mathrm{Ext}_A^{-\delta}(M, \omega_A^\bullet)$ be the nonzero Ext where $\delta = \mathrm{depth}(M)$. Then, since

$$M[0] = \mathrm{R}\,\mathrm{Hom}_A(\mathrm{R}\,\mathrm{Hom}_A(M, \omega_A^\bullet), \omega_A^\bullet) = \mathrm{R}\,\mathrm{Hom}_A(N[\delta], \omega_A^\bullet)$$

(Proposition 3.2) we conclude that $M = \mathrm{Ext}_A^{-\delta}(N, \omega_A^\bullet)$. Thus $\delta \geqslant \dim(\mathrm{Supp}(M))$. However, since we also know that $\delta \leqslant \dim(\mathrm{Supp}(M))$, we conclude that M is Cohen-Macaulay.

To prove the final statement, it suffices to show that $N = \mathrm{Ext}_A^{-d}(M, \omega_A^\bullet)$ is in CM_d for M in CM_d. Above we have seen that $M[0] = \mathbf{R}\,\mathrm{Hom}_A(N[d], \omega_A^\bullet)$ and this proves the desired result by the equivalence of (a) and (c). □

Apply this to the ring itself, we have:

Corollary 4.2. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring with normalized dualizing complex $\omega_A^\bullet$ and dualizing module $\omega_A = H^{-\dim(A)}(\omega_A^\bullet)$. The following are equivalent*

1. *A is Cohen-Macaulay,*
2. *$\omega_A^\bullet$ is concentrated in a single degree, and*
3. *$\omega_A^\bullet = \omega_A[\dim(A)]$.*

In this case ω_A is a maximal Cohen-Macaulay module.

As we may replace A by the localization at a prime, we have:

Corollary 4.3. *Let A be a Noetherian ring. If there exists a finite A-module ω_A such that $\omega_A[0]$ is a dualizing complex, then A is Cohen-Macaulay.*

Proposition 4.4 (Opneness of CM locus)**.** *Let* A *be a Noetherian ring with dualizing complex* $\omega_A^\bullet$*. Let* M *be a finite* A*-module. Then*

$$U = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_\mathfrak{p} \text{ is Cohen-Macaulay}\}$$

is an open subset of $\mathrm{Spec}(A)$ *whose intersection with* $\mathrm{supp}(M)$ *is dense.*

Proof. If $\mathfrak{p}$ is a generic point of $\mathrm{supp}(M)$, then $\mathrm{depth}(M_\mathfrak{p}) = \dim(M_\mathfrak{p}) = 0$ and hence $\mathfrak{p} \in U$. This proves denseness. If $\mathfrak{p} \in U$, then we see that

$$\mathbf{R}\,\mathrm{Hom}_A(M, \omega_A^\bullet)_\mathfrak{p} = \mathbf{R}\,\mathrm{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, (\omega_A^\bullet)_\mathfrak{p})$$

has a unique nonzero cohomology module, say in degree i_0, by Proposition 4.1. Since $\mathrm{R}\,\mathrm{Hom}_A(M, \omega_A^\bullet)$ has only a finite number of nonzero cohomology modules H^i and since each of these is a finite A-module, we can find an $f \in A$, $f \notin \mathfrak{p}$ such that $(H^i)_f = 0$ for $i \neq i_0$. Then $\mathbf{R}\,\mathrm{Hom}_A(M, \omega_A^\bullet)_f$ has a unique nonzero cohomology module and reversing the arguments just given we find that $D(f) \subset U$. □

Hence apply this to A we know that

$$U = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_\mathfrak{p} \text{ is Cohen-Macaulay}\}$$

is a dense open subset of $\mathrm{Spec}(A)$.

4.2 Gorenstein Rings

In this section we will consider a ring with best behaver with its dualizing complex.

Definition 4.5. *(a) Let* A *be a Noetherian local ring. We say* A *is* Gorenstein *if* $A[0]$ *is a dualizing complex for* A*.*

(b) Let A *be a Noetherian ring. We say* A *is* Gorenstein *if* $A_\mathfrak{p}$ *is Gorenstein for every prime* $\mathfrak{p}$ *of* A*.*

This seems different as one in [BH98], but we will see later they are equivalent. Here we consider some corollaries.

Corollary 4.6. *A Gorenstein ring is Cohen-Macaulay.*

Proof. Follows from Corollary 4.2. □

Corollary 4.7. *A regular local ring is Gorenstein. A regular ring is Gorenstein.*

Proof. Let A be a regular ring of finite dimension d. Then A has finite global dimension d. Hence $\mathrm{Ext}_A^{d+1}(M, A) = 0$ for all A-modules M. Thus A has finite injective dimension as an A-module. It follows that $A[0]$ is a dualizing complex, hence A is Gorenstein. □

Next we will give some conditions and properties of Gorenstein rings. Note that when A is local Gorenstein, we have dualizing complex $A[0]$, but in the non-local case, this may not true.

Proposition 4.8. *Let* A *be a Noetherian ring.*

(1) If A *has a dualizing complex* $\omega_A^\bullet$*, then*

  (a) A *is Gorenstein* $\Leftrightarrow$ $\omega_A^\bullet$ *is an invertible object of* $D(A)$*;*

  (b) $\{\mathfrak{p} \in \mathrm{Spec}(A) \mid A_\mathfrak{p} \text{ is Gorenstein}\}$ *is an open subset.*

(2) If A *is Gorenstein, then* A *has a dualizing complex if and only if* $A[0]$ *is a dualizing complex.*

Proof. For (1)(b), assume that $A_\mathfrak{p}$ is Gorenstein. Let n_x be the unique integer such that $H^{n_x}((\omega_A^\bullet)_\mathfrak{p})$ is nonzero and isomorphic to $A_\mathfrak{p}$. Since $\omega_A^\bullet$ is in $D^b_{Coh}(A)$ there are finitely many nonzero finite A-modules $H^i(\omega_A^\bullet)$. Thus there exists some $f \in A$, $f \notin \mathfrak{p}$ such that only $H^{n_x}((\omega_A^\bullet)_f)$ is nonzero and generated by 1 element over A_f. Since dualizing complexes are faithful (by definition) we conclude that $A_f \cong H^{n_x}((\omega_A^\bullet)_f)$. In this way we see that $A_\mathfrak{q}$ is Gorenstein for every $\mathfrak{q} \in D(f)$. This proves that the set in (1)(b) is open.

For (1)(a), the implication $\Leftarrow$ follows from the fact after localization. The implication $\Rightarrow$ follows from the discussion in the previous paragraph, where we showed that if $A_\mathfrak{p}$ is Gorenstein, then for some $f \in A$, $f \notin \mathfrak{p}$ the complex $(\omega_A^\bullet)_f$ has only one nonzero cohomology module which is invertible.

For (2), if $A[0]$ is a dualizing complex then A is Gorenstein by part (1). Conversely, we see that part (1) shows that $\omega_A^\bullet$ is locally isomorphic to a shift of A. Since being a dualizing complex is local by Proposition 3.3(d) the result is clear. □

Proposition 4.9. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Then A is Gorenstein if and only if $\mathrm{Ext}^i_A(\kappa, A)$ is zero for $i \gg 0$.*

Proof. Observe that $A[0]$ is a dualizing complex for A if and only if A has finite injective dimension as an A-module (follows immediately from Definition). Thus the lemma follows from the fact that $M \in \mathbf{Mod}_R$ has finite injective dimension if and only if $\mathrm{Ext}^i_A(\kappa, M)$ is zero for $i \gg 0$ (Tag 0AVJ). □

This proposition shows our definition is the same as one in [BH98].

Proposition 4.10. *Let $(A, \mathfrak{m}, \kappa)$ be a Noetherian local ring. Let $f \in \mathfrak{m}$ be a nonzerodivisor. Set $B = A/(f)$. Then A is Gorenstein if and only if B is Gorenstein.*

Proof. If A is Gorenstein, then B is Gorenstein by Corollary 3.14.

Conversely, suppose that B is Gorenstein. Then $\mathrm{Ext}^i_B(\kappa, B)$ is zero for $i \gg 0$ by Proposition 4.9. Recall that $\mathbf{R}\,\mathrm{Hom}(B, -) : D(A) \to D(B)$ is a right adjoint to restriction. Hence

$$\mathbf{R}\,\mathrm{Hom}_A(\kappa, A) = \mathbf{R}\,\mathrm{Hom}_B(\kappa, \mathbf{R}\,\mathrm{Hom}(B, A)) = \mathbf{R}\,\mathrm{Hom}_B(\kappa, B[1])$$

The final equality by direct computation for $\mathbf{R}\,\mathrm{Hom}(B, A)$. Thus we see that $\mathrm{Ext}^i_A(\kappa, A)$ is zero for $i \gg 0$ and A is Gorenstein by Proposition 4.9 again. □

Proposition 4.11. *Let $A \to B$ be a flat local homomorphism of Noetherian local rings. The following are equivalent*

(1) B is Gorenstein;

(2) A and $B/\mathfrak{m}_A B$ are Gorenstein.

Proof. Below we will use without further mention that a local Gorenstein ring has finite injective dimension by Proposition 4.9. By flat base-change we have

$$\mathrm{Ext}^i_A(\kappa_A, A) \otimes_A B = \mathrm{Ext}^i_B(B/\mathfrak{m}_A B, B)$$

for all i.

Assume (2). Using that $\mathbf{R}\,\mathrm{Hom}(B/\mathfrak{m}_A B, -) : \mathbf{D}(B) \to \mathbf{D}(B/\mathfrak{m}_A B)$ is a right adjoint to restriction we obtain

$$\mathbf{R}\,\mathrm{Hom}_B(\kappa_B, B) = \mathbf{R}\,\mathrm{Hom}_{B/\mathfrak{m}_A B}(\kappa_B, \mathbf{R}\,\mathrm{Hom}(B/\mathfrak{m}_A B, B))$$

The cohomology modules of $\mathbf{R}\,\mathrm{Hom}(B/\mathfrak{m}_A B, B)$ are the modules $\mathrm{Ext}^i_B(B/\mathfrak{m}_A B, B) = \mathrm{Ext}^i_A(\kappa_A, A) \otimes_A B$. Since A is Gorenstein, we conclude only a finite number of these are nonzero and each is isomorphic to a direct sum of copies of $B/\mathfrak{m}_A B$. Hence since $B/\mathfrak{m}_A B$ is Gorenstein we conclude that $\mathbf{R}\,\mathrm{Hom}_B(B/\mathfrak{m}_B, B)$ has only a finite number of nonzero cohomology modules. Hence B is Gorenstein.

Assume (1). Since B has finite injective dimension, $\mathrm{Ext}^i_B(B/\mathfrak{m}_A B, B)$ is 0 for $i \gg 0$. Since $A \to B$ is faithfully flat we conclude that $\mathrm{Ext}^i_A(\kappa_A, A)$ is 0 for $i \gg 0$. We conclude that A is Gorenstein. This implies that $\mathrm{Ext}^i_A(\kappa_A, A)$ is nonzero for exactly one i, namely for $i = \dim(A)$, and $\mathrm{Ext}^{\dim(A)}_A(\kappa_A, A) \cong \kappa_A$ (see Lemma 3.6, Corollary 4.2, and Corollary 4.6). Thus we see that $\mathrm{Ext}^i_B(B/\mathfrak{m}_A B, B)$ is zero except for one i, namely $i = \dim(A)$ and $\mathrm{Ext}^{\dim(A)}_B(B/\mathfrak{m}_A B, B) \cong B/\mathfrak{m}_A B$. Thus $B/\mathfrak{m}_A B$ is Gorenstein by Lemma 3.6. □

4.3 More Rings with Dualizing Complexes

In this section, we will see which kind of Noetherian rings have dualizing complexes.

Lemma 4.12. *Let A be a Noetherian ring and let $I \subset A$ be an ideal. Let $\omega_A^\bullet$ be a dualizing complex. Then $\omega_A^\bullet \otimes_A A^\wedge$ is a dualizing complex on the I-adic completion $A^\wedge$.*

Proof. We just show the following fact:

- Let $A \to B$ be a flat map of Noetherian rings. Let $I \subset A$ be an ideal such that $A/I = B/IB$ and such that IB is contained in the Jacobson radical of B. Let $\omega_A^\bullet$ be a dualizing complex. Then $\omega_A^\bullet \otimes_A B$ is a dualizing complex for B.

Indeed, it is clear that $\omega_A^\bullet \otimes_A B$ is in $\mathbf{D}^b_{Coh}(B)$. By base-change we see that

$$\mathbf{R}\operatorname{Hom}_B(K \otimes_A B, \omega_A^\bullet \otimes_A B) = \mathbf{R}\operatorname{Hom}_A(K, \omega_A^\bullet) \otimes_A B$$

for any $K \in \mathbf{D}^b_{Coh}(A)$. For any ideal $IB \subset J \subset B$ there is a unique ideal $I \subset J' \subset A$ such that $A/J' \otimes_A B = B/J$. Thus $\omega_A^\bullet \otimes_A B$ has finite injective dimension (see Tag 0DW2). Finally, we also have

$$\mathbf{R}\operatorname{Hom}_B(\omega_A^\bullet \otimes_A B, \omega_A^\bullet \otimes_A B) = \mathbf{R}\operatorname{Hom}_A(\omega_A^\bullet, \omega_A^\bullet) \otimes_A B = A \otimes_A B = B$$

as desired. □

Corollary 4.13. *The following types of rings have a dualizing complex:*

(1) Gorenstein local rings (such as, regular local rings),

(2) Noetherian complete local rings,

(3) Dedekind domains,

(4) any ring which is obtained from one of the rings above by taking an algebra essentially of finite type, or by taking an ideal-adic completion.

Proof. (1) follows from definition and Corollary 4.7.

(2) follows from Proposition 3.3(e) and the Cohen structure theorem that any complete Noetherian local ring is the quotient of a regular local ring.

For (3), let A be a Dedekind domain. Then every ideal I is a finite projective A-module. Thus every A-module has finite injective dimension at most 1 since we can check $\operatorname{Ext}^i(A/I, K)$ by directly computation. It follows easily that $A[0]$ is a dualizing complex.

(4) follows from Proposition 3.4(b) and Lemma 4.12. □

So now we have many rings with dualizing complexes. Note also that there are many rings without dualizing complexes since by Proposition 3.19, any Noetherian local rings with dualizing complex is universally catenary of finite dimension. So we may wonder that is there some equivalent condition of the ring with dualizing complexes? Yes we have!

Theorem 4.14 (Sharp's conjecture; Kawasaki 2002)**.** *A Noetherian ring has a dualizing complex if and only if it is a quotient of a finite dimensional Gorenstein ring.*

Proof. See the Corollary 1.2 in [Kaw02] for the proof. □

5 A Glimpse of Duality in Algebraic Geometry

Here we first introduce some history of Grothendieck duality in algebraic geometry. For a modern introduction of duality theory and global dualizing complexes we refer Chapter 25 in [GW23].

Our aim is to find a right adjoint functor $f^!$ of $\mathbf{R}f_*$ where $f : X \to Y$ be a proper morphism of schemes (actually this defined as $f^\times$ for proper morphisms and $f^!$ for separated morphisms using Nagata's compactification). The projective case was found many years ago (see 3.5 in [Illar] for arguments). But the proper case is harder. The first discussion of this is in the book [Har66] by the communication of Grothendieck and Hartshorne using residue theory. But there are several mistakes in it and is corrected in [Con00]. Another discussion is in the appendix of [Har66] due to Pierre Deligne and [Ver69] due to Jean-Louis Verdier (see also [LH09]).

As the words of Neeman, these two methods are both not good enough since the first one is very complicated and the second one is hard to compute them. So the modern theory due to Neeman is [Nee96]. See also Chapter 25 in [GW23] or Chapter 48 in [Pro24].

Here we assume our schemes are all Noetherian (we have more general results as in Chapter 25 in [GW23] again).

Theorem 5.1 (Grothendieck Duality). *Let $f : X \to S$ be a proper morphism of schemes with $E \in \mathbf{D}_{\mathsf{Qcoh}}(X)$ and $F \in \mathbf{D}^+_{\mathsf{Qcoh}}(S)$, then we have*

$$\mathbf{R}f_*\mathbf{R}\mathscr{H}om_X(E, f^!F) \cong \mathbf{R}\mathscr{H}om_X(\mathbf{R}f_*E, F).$$

Moreover, we have the following properties.

(a) $(-)^!$ is stable under composition. It satisfies flat base-change (more generally, tor-independent base-change, see Theorem 25.31 in [GW23]).

(b) When f is also of finite Tor-dimension (see Remark 25.39 in [GW23] for more informations), then

$$f^!F \cong \mathbf{L}f^*F \otimes^{\mathbf{L}} f^!\mathscr{O}_S = \mathbf{L}f^*F \otimes^{\mathbf{L}} \omega^\bullet_{X/S}$$

where $\omega^\bullet_{X/S} := f^!\mathscr{O}_S$ is relative dualizing complex.

(c) Let $f : X \xrightarrow{g} Y \xrightarrow{h} S$ are proper morphisms, then $\omega^\bullet_X := f^!\omega^\bullet_S$ and $\omega^\bullet_Y := h^!\omega^\bullet_S$ are again dualizing complexes and if g is of finite Tor-dimension, then

$$\omega^\bullet_X \cong \mathbf{L}f^*\omega^\bullet_Y \otimes^{\mathbf{L}} \omega^\bullet_{X/Y}.$$

(d) When $f : X \to S$ is smooth proper of relative dimension n, then for any $F \in \mathbf{D}^+_{\mathsf{Qcoh}}(S)$ we have

$$f^!F \cong \mathbf{L}f^*F \otimes^{\mathbf{L}} \Omega^n_{X/S}[n].$$

(e) When $i : Z \hookrightarrow X$ be a closed immersion, then for all $K \in \mathbf{D}^+_{\mathsf{Qcoh}}(X)$ we have

$$i^!K \cong R(i^*\mathscr{H}om_{\mathscr{O}_X}(i_*\mathscr{O}_Z, K)).$$

As an application, we have the following Serre duality:

Corollary 5.2 (Serre Duality). *Let X be a proper scheme over a field k. Denote by $f : X \to \operatorname{Spec} k$ the structure morphism. Then $f^!\mathscr{O}_{\operatorname{Spec} k} = \omega^\bullet_X$ be the dualizing complex. Moreover, we have the following:*

(a) $\omega^\bullet_X \in \mathbf{D}^b_{\mathsf{Coh}}(X)$ and $H^i(\omega^\bullet_X) = 0$ for all $i \notin [-\dim(X), 0]$.

(b) For $K \in \mathbf{D}_{\mathsf{Qcoh}}(X)$, there are isomorphisms for all $i \in \mathbb{Z}$

$$\operatorname{Ext}^i_X(K, \omega^\bullet_X) \cong H^{-i}(X, K)^\vee.$$

When X is Cohen-Macaulay, we have $\operatorname{Ext}^i_X(K, \omega_X[\dim X]) \cong H^{-i}(X, K)^\vee$ and $\operatorname{Ext}^{\dim X - i}_X(\mathscr{F}, \omega_X) \cong H^i(X, \mathscr{F})^\vee$ for $\mathscr{F} \in \operatorname{Qcoh}(X)$.

(c) If K is a perfect complex on X (e.g., a finite locally free $\mathscr{O}_X$-module), then there are functorial isomorphisms of finite-dimension k-vector spaces for all i

$$H^i(X, \omega^\bullet_X \otimes^{\mathbf{L}}_X K^\vee) \cong H^{-i}(X, K)^\vee.$$

If X is Cohen-Macaulay, we have $H^{\dim X - i}(X, \omega_X \otimes_{\mathscr{O}_X} \mathscr{E}^\vee) \cong H^i(X, \mathscr{E})$ for vector bundles $\mathscr{E}$.

More recently, Clausen and Scholze have sketched a new approach to duality for coherent sheaves using their theory of condensed mathematics [Sch19]. In this setting, the category of schemes is embedded fully faithfully in the category of discrete adic spaces. In this category, every discrete adic space and every morphism of discrete adic spaces has a functorial "compactification" which simplifies the definition of $f^!$ if f is "sufficiently finite". One even obtains a full six-functor formalism if one restricts to "sufficiently finite" morphisms for the functors $f_!$ and $f^!$.

References

[BH98] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay Rings*. Cambridge University Press, 1998.

[Con00] Brian Conrad. *Grothendieck duality and base change*. Springer, 2000.

[Gro68] Alexander Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2)*. North-Holland Publishing Co., 1968.

[GW23] Ulrich Görtz and Torsten Wedhorn. *Algebraic Geometry II: Cohomology of Schemes*. Springer Spektrum Wiesbaden, 2023.

[Har66] Robin Hartshorne. *Residues and Duality*. Springer-Verlag, 1966.

[Har67] Robin Hartshorne. *Local Cohomology*. Springer, 1967.

[Illar] Luc Illusie. *Topics in Algebraic Geometry*. `http://staff.ustc.edu.cn/~yiouyang/Illusie.pdf`, Unkown Year.

[Kaw02] Takesi Kawasaki. On arithmetic macaulayfication of noetherian rings. *Trans. Amer. Math. Soc.*, 354(1):123–149, 2002.

[LH09] Joseph Lipman and Mitsuyasu Hashimoto. *Foundations of Grothendieck duality for diagrams of schemes*. Springer, 2009.

[Nee96] Amnon Neeman. The grothendieck duality theorem via bousfield's techniques and brown representability. *Journal of the American Mathematical Society*, 9(1):205–236, 1996.

[Pro24] Stacks Project. The stacks project. *https://stacks.math.columbia.edu/*, 2024.

[Sch19] Peter Scholze. Lectures on condensed mathematics. *https://www.math.uni-bonn.de/people/scholze/Condensed.pdf*, 2019.

[Ver69] Jean-Louis Verdier. Base change for twisted inverse image of coherent sheaves. *Algebraic geometry*, pages 393–408, 1969.