Modern Theory of Moduli Spaces and Stability

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Contents

Preface

First, we will mainly follows the paper [8] and the Chapter 6 in the book draft [4] which gives the general theory of moduli theory in the modern way.

Second, we will use these to construct the moduli space of semistable sheaves and complexes with Bridgeland stability and so on.

Chapter 1

General Theory of Good Moduli Space

Here we will introduce some basic background about good moduli theory and the theory of Θ-complete and S-completedue to J. Alper in [6] and [8]. These will play an important role in our foundamental theory.

We will give the main properties, theorems and their motivations and some idea of proofs. For the detailed proof we refer reader to the original paper [6][8] or the book draft [4] of J. Alper.

1.1 Properties of Good Moduli Spaces

As we all know, in the modern construction of the moduli space of stable curves follows from the following way:

(a) Construct the stack $\overline{\mathscr{M}}_{g,n}$ and show that it is a Deligne-Mumford stack;

(b) show the stable-reduction of stable curves and find that $\overline{\mathscr{M}}_{g,n}$ is proper;

(c) use Keel-Mori theorem to construct the coarse moduli space $\overline{\mathscr{M}}_{g,n} \to \overline{M}_{g,n}$ and show that it is projective.

But in our case, we can not use Keel-Mori theorem to the moduli stack of semistable sheaves because the inertia stack $\mathscr{I}_{\mathscr{X}} \to \mathscr{X}$ is not finite. In order to this the similar modern way (instead of GIT-construction), J.Alper developed a nice similar (but much more complicated) theory to solve this problem – the theory of good moduli space ([6] and [8]) for linear reductive groups and the theory of adequate moduli spaces ([5]) for geometric reductive groups.

For now, the theory of good moduli space plays a central role in the construction of moduli spaces, such as the Hassett-Mori program $\overline{\mathscr{M}}_{g,n}(\alpha) \to \overline{\mathbb{M}}(\alpha)$ in [SecondHMP], moduli stack of semistable sheaves $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$, moduli of $\mathcal{G}$-torsors and K-moduli stack $\mathscr{X}_{n,V}^{\mathrm{Kss}}$ which aim to construct a good moduli space of Fano varieties (see the [book draft] due to C. Xu).

Of course we will just introduce some of them and there are many beautiful results we will not introduce, such as the Section 6.6 and 6.7 in [4] which gave us many applications and examples.

Definition 1.1.1 (Good moduli space)**.** *For an algebraic stack $\mathscr{X}$, its good moduli space is an algebraic space X together with a qcqs morphism $\pi : \mathscr{X} \to X$ such that*

(i) the natural map $\mathscr{O}_X \to \pi_*\mathscr{O}_{\mathscr{X}}$ is an isomorphism;

(ii) the functor $\pi_* : \mathrm{QCoh}(\mathscr{X}) \to \mathrm{QCoh}(X)$ is exact.

Note that the condition in (ii) is called cohomologically affine.

The definition of good moduli space is inspired from the GIT-quotient of linear reductive group G (that is, $V \mapsto V^G$ is exact. Hence G is linear reductive if and only if $\mathbf{B}G$ is cohomologically affine)

$$[X/G] \dashrightarrow [X^{\mathrm{ss}}/G] \to X /\!\!/ G = \operatorname{Proj} \bigoplus_{d\geq 0} \Gamma(X, \mathscr{O}_X(d))^G.$$

Or locally, the map $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$. Of coarse, a tame coarse moduli space is a good moduli space by the local structure of coarse moduli spaces.

Here we state several basic properties of cohomologically affine morphisms.

Lemma 1.1.2. *Consider a cartesian*

$$\begin{tikzcd}
\mathscr{X}' \arrow[r, "g'"] \arrow[d, "\pi'"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{X} \arrow[d, "\pi"] \\
\mathscr{Y}' \arrow[r, "g"] & \mathscr{Y}
\end{tikzcd}$$

of algebraic stacks, then:

(i) If g is faithfully flat and π' is cohomologically affine, then π is cohomologically affine.

(ii) If Y has quasi-affine diagonal and π is cohomologically affine, then π' is cohomologically affine.

If we consider a cartesian

$$\begin{tikzcd}
\mathscr{X}' \arrow[r, "g'"] \arrow[d, "\pi'"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{X} \arrow[d, "\pi"] \\
X' \arrow[r, "g"] & X
\end{tikzcd}$$

of algebraic stacks where X, X' are quasi-separated algebraic spaces, then:

(iii) *If g is faithfully flat and π' is a good moduli space, then π is a good moduli space.*

(iv) *If π is a good moduli space, so is π'.*

(v) *Let π is a good moduli space. For $\mathscr{F} \in \mathrm{QCoh}(X)$ and $\mathscr{G} \in \mathrm{QCoh}(X)$, the adjunction map $\pi_*\mathscr{F} \otimes \mathscr{G} \cong \pi_*(\mathscr{F} \otimes \pi^*\mathscr{G})$ is an isomorphism. In particular, the adjunction map $\mathscr{G} \cong \pi_*\pi^*\mathscr{G}$ is an isomorphism.*

(vi) *For $\mathscr{F} \in \mathrm{QCoh}(X)$, then $g^*\pi_*\mathscr{F} \cong \pi'_*(g')^*\mathscr{F}$.*

(vii) *For a quasi-coherent sheaf of ideals $\mathscr{J} \subset \mathscr{O}_X$, the natural map $\mathscr{J} \cong \pi_*(\pi^{-1}\mathscr{J} \cdot \mathscr{O}_{\mathscr{X}})$ is an isomorphism.*

If $\pi : \mathscr{X} \to X$ be a good moduli space with X quasi-separated, then

(viii) *If $\mathscr{A}$ is a quasi-coherent sheaf of $\mathscr{O}_{\mathscr{X}}$-algebras, then $\underline{\mathrm{Spec}}_{\mathscr{X}}\mathscr{A} \to \underline{\mathrm{Spec}}_X \pi_*\mathscr{A}$ is a good moduli space.*

(ix) *If $\mathscr{Z} \subset \mathscr{X}$ is a closed substack and $\mathrm{Im}\mathscr{Z} \subset X$ is the scheme-theoretic image, then $\mathscr{Z} \to \mathrm{Im}\mathscr{Z}$ is a good moduli space.*

Proof. See section 4 in fundamental paper [6]. □

Now some important properties of good moduli spaces and give some comments. Actually these are similar as the properties of GIT.

Theorem 1.1.3. *Let $\pi : \mathscr{X} \to X$ be a good moduli space where $\mathscr{X}$ is a quasi-separated algebraic stack defined over an algebraic space S. Then*

(i) *π is surjective and universally closed (and universally submersive);*

(ii) *for closed substacks $\mathscr{Z}_1, \mathscr{Z}_2 \subset \mathscr{X}$, we have $\mathrm{Im}(\mathscr{Z}_1 \cap \mathscr{Z}_2) = \mathrm{Im}(\mathscr{Z}_1) \cap \mathrm{Im}(\mathscr{Z}_2)$. For geometric points $x_1, x_2 \in \mathscr{X}(k)$, $\pi(x_1) = \pi(x_2) \in \mathscr{X}(k)$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_1\}} \neq \emptyset$ in $|\mathscr{X} \times_S k|$. In particular, π induces a bijection between closed points in $\mathscr{X}$ and closed points in X;*

(iii) *if $\mathscr{X}$ is noetherian, so is X. If $\mathscr{X}$ is of finite type over S and S is noetherian, then X is of finite type over S and π_* preserves coherence;*

(iv) *If X is noetherian, then π is universal for maps to algebraic spaces.*

Proof. Here we give some idea. The proof we refer the Theorem 4.16 in [6].

For (i), by Lemma 1.1.2 (iv) we know that $\mathscr{X} \times_X \mathrm{Spec}\, k \to \mathrm{Spec}\, k$ is good moduli space. Hence $\Gamma(\mathscr{X} \times_X \mathrm{Spec}\, k, \mathscr{O}_{\mathscr{X} \times_X \mathrm{Spec}\, k}) = k$ and $|\mathscr{X} \times_X \mathrm{Spec}\, k| \neq \emptyset$. Hence π is surjective. Again by Lemm 1.1.2 (ix) we know that if $\mathscr{Z} \subset \mathscr{X}$ is a closed substack and $\mathrm{Im}\mathscr{Z} \subset X$ is the scheme-theoretic image, then $\mathscr{Z} \to \mathrm{Im}\mathscr{Z}$ is a good moduli space. Hence it is surjective and hence π is closed. By Lemma 1.1.2 (iv) we know that it is universally closed.

For (ii), let ideal sheaves be $\mathscr{I}_1, \mathscr{I}_2$, then by the exactness of π_* we have

$$\begin{tikzcd}
 & & \pi_*\mathscr{I}_2 \arrow[d] \arrow[rd, two heads] & & \\
0 \arrow[r] & \pi_*\mathscr{I}_1 \arrow[r] & \pi_*(\mathscr{I}_1 + \mathscr{I}_2) \arrow[r, two heads] & \pi_*\mathscr{I}_2/\pi_*(\mathscr{I}_1 \cap \mathscr{I}_2) \arrow[r] & 0
\end{tikzcd}$$

Hence the inclution $\pi_*(\mathscr{I}_1 + \mathscr{I}_2) \to \pi_*(\mathscr{I}_1 + \mathscr{I}_2)$ is surjective.

For (iii), X is noetherian follows from Lemma 1.1.2 (vii). We omit others and (iv). □

There is an interesting result which we will use it:

Proposition 1.1.4. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a cohomologically affine morphism of algebraic stacks where $\mathscr{Y}$ has quasi-affine diagonal. If f is representable (that is, $\mathscr{I}_{\mathscr{X}/\mathscr{Y}} \to \mathscr{X}$ is trivial, or equivalently, Δ_{Δ_f} is an isomorphism), then f is affine.*

Proof. Trivial by faithfully flat descent and Serre's Criterion. □

1.2 Luna's Results and Étale Local Structure of Algebraic Stacks

1.2.1 Luna's Fundamental Lemma and Luna's Étale Slice Theorem

Luna's results are classical and you can find them even in [45].

Theorem 1.2.1 (Luna's Fundamental Lemma)**.** *Consider a commutative diagram:*

$$\begin{tikzcd}
\mathscr{X}' \arrow[r, "f"] \arrow[d, "\pi'"'] & \mathscr{X} \arrow[d, "\pi"'] \\
X' \arrow[r, "g"] & X
\end{tikzcd}$$

where f is a separated and representable morphism of noetherian algebraic stacks, each with affine diagonal, and where π and π' are good moduli spaces. Let $x' \in \mathscr{X}'$ be a point such that

- *(a) f is étale at x';*
- *(b) f induces an isomorphism of stabilizer groups at x', and*
- *(c) $x' \in \mathscr{X}'$ and $x = f(x') \in \mathscr{X}$ are closed points.*

Then there is an open neighborhood $U' \subset X'$ of $\pi'(x')$ such that $U' \to X$ is étale and such that $U' \times_X \mathscr{X} \cong (\pi')^{-1}(U')$.

Sketch. Using limit-argument, we may let $X = \operatorname{Spec} A$, where A is a strictly henselian local ring. After shrink $\mathscr{X}'$, we may let f is étale. Then by Zariski main theorem we get $\mathscr{X}' \to \widetilde{\mathscr{X}} = \underline{\operatorname{Spec}}_{\mathscr{X}} \mathscr{A} \to \mathscr{X}$. Hence $\widetilde{\mathscr{X}} \to \widetilde{X} = \underline{\operatorname{Spec}}_{\mathscr{X}} \pi_* \mathscr{A}$ is a good moduli space with $\widetilde{X} \to X$ finite. Hence we can let $\widetilde{X} = \coprod_i \operatorname{Spec} A_i$ of henselian local rings. Take $U' = \operatorname{Spec} A_i$ contains image of x'. Well done. □

Hence we have an very important corollary we will use:

Corollary 1.2.2. *With the same hypotheses, suppose that f is étale and that for all closed points $x' \in \mathscr{X}'$ we have*

(a) $f(x')$ closed;

(b) f induces an isomorphism of stabilizer groups at x'.

Then $g : X' \to X$ étale and that commutative diagram is cartesian.

This is our main motivation to define the Θ-completeness and S-completeness. We will discuss this deeply later.

Next we introduce Luna's étale slice theorem which was motivated the étale local structure of algebraic stacks.

Lemma 1.2.3 (Luna Map)**.** *Let G be a linearly reductive group over an algebraically closed field k and let X be an affine scheme of finite type over k with an action of G. If $x \in X(k)$ has linearly reductive stabilizer G_x, there exists a G_x-equivariant morphism (Luna map)*

$$f : X \to T_{X,x} := \underline{\operatorname{Spec}} \operatorname{Sym} \mathfrak{m}_x / \mathfrak{m}_x^2$$

sending x to the origin. If X is smooth at x, then f is étale at x.

Proof. Letting $X = \operatorname{Spec} A$, then $\mathfrak{m}_x$ and $\mathfrak{m}_x/\mathfrak{m}_x^2$ are G_x-representations and we see that G_x acts naturally on the tangent space $T_{X,x} := \underline{\operatorname{Spec}} \operatorname{Sym} \mathfrak{m}_x/\mathfrak{m}_x^2$. Since G_x is linearly reductive, the surjection $\mathfrak{m}_x \to \mathfrak{m}_x/\mathfrak{m}_x^2$ of G_x-representations has a section $\mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathfrak{m}_x$. This induces a G_x-equivariant ring map $\operatorname{Sym} \mathfrak{m}_x/\mathfrak{m}_x^2 \to A$ and thus a G_x-equivariant morphism $f : X \to T_{X,x}$ sending x to the origin. If x is smooth, then since f induces an isomorphism of tangent spaces at x, we conclude that f is étale at x. □

Theorem 1.2.4 (Luna's Étale Slice Theorem)**.** *Let G be a linearly reductive group over an algebraically closed field k and let X be an affine scheme of finite type over k with an action of G. If $x \in X(k)$ has linearly reductive stabilizer G_x, then there exists a G_x-invariant, locally closed, and affine subscheme $W \subset X$ such that the induced map*

$$[W/G_x] \to [X/G]$$

is affine étale. If in addition $Gx \subset X$ *closed (then by Matsushima's theorem* G_x *is linearly reductive), then there is a cartesian*

$$\begin{tikzcd}
{[W/G_x]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & {[X/G]} \arrow[d] \\
W /\!\!/ G_x \arrow[r] & X /\!\!/ G
\end{tikzcd}$$

where $W /\!\!/ G_x \to X /\!\!/ G$ *is also étale.*

Moreover, if $x \in X$ *is a smooth point and if we denote by* $N_x = T_{X,x}/T_{Gx,x}$ *the normal space to the orbit, then it can be arranged that there is an* G_x*-invariant étale morphism* $W \to N_x$ *which is the pullback of an étale map* $W /\!\!/ G_x \to N_x /\!\!/ G_x$ *of GIT quotients.*

Proof. Pick a finite G-representation V and a G-equivariant closed immersion $X \subset \mathbb{A}(V)$. Then using this we can reduce to the case where $x \in X$ is smooth.

Hence we have Luna map $f : X \to T_{X,x}$ is G_x-equivariant and étale at x. The subspace $T_{Gx,x} \subset T_{X,x}$ is G_x-invariant and again since G_x is linearly reductive, the surjection $T_{X,x} \to N_x$ has a section $N_x \to T_{X,x}$. We define W as

$$\begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & N_x \arrow[d] \\
X \arrow[r, "f"] & T_{X,x}
\end{tikzcd}$$

Then $[W/G_x] \to [X/G]$ and $[W/G_x] \to [N_x/G_x]$ induce an isomorphism of tangent spaces and stabilizer groups at w, they are both étale at x. Hence we have commutative diagram

$$\begin{tikzcd}
{[N_x/G_x]} \arrow[d] & {[W/G_x]} \arrow[l] \arrow[r] \arrow[d] & {[X/G]} \arrow[d] \\
N_x /\!\!/ G_x & W /\!\!/ G_x \arrow[l] \arrow[r] & X /\!\!/ G
\end{tikzcd}$$

Hence using Luna's fundamental lemma 1.2.1 twice and well done. □

1.2.2 Coherent Tannaka Duality and Coherent Completeness

Here we introduce some very importent results aiming to extend to morphisms.

Theorem 1.2.5 (Coherent Tannaka Duality)**.** *For noetherian algebraic stacks* $\mathscr{X}$ *and* $\mathscr{Y}$ *with affine diagonal, the functor*

$$\mathrm{MOR}(\mathscr{X}, \mathscr{Y}) \to \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathscr{Y}), \mathrm{Coh}(\mathscr{X})), \quad f \mapsto f^*$$

is an equivalence of categories where the latter category denote the right exact additive tensor functors $\mathrm{Coh}(\mathscr{Y}) \to \mathrm{Coh}(\mathscr{X})$ *of symmetric monoidal abelian categories where morphisms are tensor natural transformations.*

Proof. This follows from a nice observation of Lurie in [39]. For the proof we refer [4] Theorem 6.4.1. □

Definition 1.2.6. *A noetherian algebraic stack* $\mathscr{X}$ *is coherently complete along a closed substack* $\mathscr{X}_0$ *if the natural functor*

$$\mathrm{Coh}(\mathscr{X}) \to \varprojlim \mathrm{Coh}(\mathscr{X}_n), \quad F \mapsto (F_n)$$

is an equivalence of categories, where $\mathscr{X}_n$ *denotes the* n*-th nilpotent thickening of* $\mathscr{X}_0$*.*

Remark 1.2.7. *(i) This motivated by the Grothendieck's Existence Theorem asserts that if* X *is a proper scheme over a complete local ring* $(R, \mathfrak{m})$ *and* $X_0 = X \times_R R/\mathfrak{m}$*, then* X *is coherently complete along* X_0*.*

Actually this is right even for proper algebraic stack over some I*-adically complete noetherian ring. We refer [49].*

(ii) Let k *be an algebraically closed field and* R *be a complete noetherian local* k*-algebra with residue field* k*. Let* G *be a linearly reductive group over* k *acting on an affine scheme* $\mathrm{Spec}\, A$ *of finite type over* R*. Suppose that* $A^G = R$ *and that there is a* G*-fixed* k*-point* $x \in \mathrm{Spec}\, A$*. Then* $[\mathrm{Spec}\, A/G]$ *is coherently complete along the closed substack* $\mathbf{B}G$ *defined by* x*. See the Theorem 6.4.11 in [4] for the proof.*

We will use the follows corollary many times:

Corollary 1.2.8. *Let* $\mathscr{X}$ *and* $\mathscr{Y}$ *be noetherian algebraic stacks with affine diagonal. Suppose that* $\mathscr{X}$ *is coherently complete along* $\mathscr{X}_0$*. Then there is an equivalence of categories*

$$\mathrm{MOR}(\mathscr{X}, \mathscr{Y}) \to \varprojlim \mathrm{MOR}(\mathscr{X}_n, \mathscr{Y}), \quad f \mapsto (f_n).$$

Proof. This is directly:

$$\begin{aligned} \mathrm{MOR}(\mathscr{X}, \mathscr{Y}) &\cong \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathscr{Y}), \mathrm{Coh}(\mathscr{X})) \\ &\cong \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathscr{Y}), \varprojlim \mathrm{Coh}(\mathscr{X}_n)) \\ &\cong \varprojlim \mathrm{MOR}^{\otimes}(\mathrm{Coh}(\mathscr{Y}), \mathrm{Coh}(\mathscr{X}_n)) \\ &\cong \varprojlim \mathrm{MOR}(\mathscr{X}_n, \mathscr{Y}) \end{aligned}$$

and well done. □

1.2.3 Some Deformation Theory

Proposition 1.2.9. *Consider a commutative diagram*

$$\begin{tikzcd}
\mathscr{W} \arrow[r, "f"] \arrow[d, hook] & \mathscr{X} \arrow[d] \\
\mathscr{W}' \arrow[r] \arrow[ur, dashed] & \mathscr{Y}
\end{tikzcd}$$

of noetherian algebraic stacks with affine diagonal where $\mathscr{X} \to \mathscr{Y}$ is smooth and affine- and $\mathscr{W} \to \mathscr{W}'$ is a closed immersion defined by a square-zero sheaf of ideals $\mathscr{J}$. If $\mathscr{W}$ is cohomologically affine, there exists a lift in the above diagram.

Proof. As the case of schemes, the set of liftings is a torsor under $\mathrm{Hom}(f^*\Omega_{\mathscr{X}/\mathscr{Y}}, \mathscr{J})$. Hence let $\mathscr{F} := f^*\Omega^\vee_{\mathscr{X}/\mathscr{Y}} \otimes \mathscr{J}$. Consider

$$\begin{tikzcd}
(U/\mathscr{W})^2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & U \arrow[r] \arrow[d, hook] \arrow[dr, phantom, very near start, "\ulcorner"] & \mathscr{W} \arrow[r] \arrow[d, hook] & \mathscr{X} \arrow[d] \\
(U'/\mathscr{W}')^2 \arrow[r, shift left] \arrow[r, shift right] & U' \arrow[r] \arrow[urr, "f'_U"] & \mathscr{W}' \arrow[r] \arrow[ur, dashed, "f'"'] & \mathscr{Y}
\end{tikzcd}$$

where $(U/\mathscr{W})^2 = U \times_{\mathscr{W}} U$ is affine. Because $\mathscr{X} \to \mathscr{Y}$ is representable, to check that f'_U descends to a morphism f', we need to arrange that $f'_U \circ p_1 = f'_U \circ p_2$. As $f'_U \circ p_1 - f'_U \circ p_2 \in \Gamma((U/\mathscr{W})^2, q_2^*\mathscr{F})$, this follows from the $\mathscr{W}$ is cohomologically affine and exact sequences. Omitted and see [4] Proposition 6.5.8.

There is another way, one can show that the obstruction to this deformation problem lies in $\mathrm{Ext}^1_{\mathscr{O}_{\mathscr{W}}}(f^*\Omega_{\mathscr{X}/\mathscr{Y}}, \mathscr{J}) \cong H^1(\mathscr{W}, \mathscr{F})$ which vanishes since $\mathscr{W}$ is cohomologically affine. □

Proposition 1.2.10. *Let $\mathscr{W} \to \mathscr{W}'$ be a closed immersion of algebraic stacks of finite type over k with affine diagonal defined by a square-zero sheaf of ideals $\mathscr{J}$. Let G be an affine algebraic group over k. If $\mathscr{W}$ is cohomologically affine, then every principal G-bundle $\mathscr{P} \to \mathscr{W}$ extends to a principal G-bundle $\mathscr{P}' \to \mathscr{W}'$.*

Proof. Similar as the proof above and we need to take $\mathscr{F} = \mathfrak{g} \otimes \mathscr{J}$ from the deformation theory of principal G-bundles in [4] D.2.9.

There is also another way. Note that this is equivalent to the deformation of $f : \mathscr{W} \to \mathbf{B}G$ to $\mathscr{W}' \to \mathbf{B}G$ which is the same problem in Proposition 1.2.9 to $\mathbf{B}G \to \operatorname{Spec} k$ which is not affine. See the arguments in Remark 6.5.11 in [4], we can see the obstruction lies in $H^2(\mathscr{W}, \mathfrak{g} \otimes \mathscr{J})$ which vanishes since $\mathscr{W}$ is cohomologically affine. □

Remark 1.2.11. *All these results are the special case in Theorem 1.5 in [48].*

1.2.4 Étale Local Structure of Algebraic Stacks

There is a fundamental theorem about algebraic stacks as follows:

Theorem 1.2.12 (Minimal Presentations)**.** *Let $\mathscr{X}$ be a noetherian algebraic stack and let $x \in |\mathscr{X}|$ be a finite type point with smooth stabilizer G_x. Then there exists a scheme U with a closed point $u \in U$ and a smooth morphism $(U, u) \to (\mathscr{X}, x)$ of relative dimension $\dim G_x$ such that the diagram*

$$\begin{tikzcd}
\operatorname{Spec} \kappa(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & U \arrow[d] \\
\mathcal{G}_x \arrow[r, hook] & \mathscr{X}
\end{tikzcd}$$

is cartesian.

Proof. This is easy in Theorem 3.6.1 in [4]. Let $(U, u) \to (\mathscr{X}, x)$ be a smooth morphism of relative dimension n, hence we have

$$\begin{tikzcd}
O(u) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & U \arrow[d] \\
\mathcal{G}_x \arrow[r, hook] & \mathscr{X}
\end{tikzcd}$$

As $\dim \mathcal{G}_x = -\dim G_x$, then $O(u)$ is a regular scheme of dimension $c := n - \dim G_x$. By Nakayama's lemma, we pick a reguler sequence $f_1, ..., f_c \in \mathscr{O}_U$ and consider $W = V(f_1, ..., f_c)$ and then $W \cap O(u) = \operatorname{Spec} \kappa(u)$. By the local criterion for flatness and smooth descent to $U \times_{\mathscr{X}} U \rightrightarrows \mathscr{X}$, we know that $W \to \mathscr{X}$ is flat. Checking on the fibers we can conclude the result. $\square$

Before giving the statement of the étale local structure of algebraic stacks, we will give a useful criteria for morphisms to be closed immersions or isomorphisms.

Lemma 1.2.13. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a representable morphism of algebraic stacks of finite type over an algebraically closed field k with affine diagonal. Assume that $|\mathscr{X}| = \{x\}$ and $|\mathscr{Y}| = \{y\}$ consist of a single point and that f induces an isomorphism of residue gerbes $\mathscr{X}_0 := \mathcal{G}_x = \mathbf{B}G_x$ with $\mathscr{Y}_0 := \mathcal{G}_y = \mathbf{B}G_y$. Let $\mathfrak{m}_x, \mathfrak{m}_y$ be the ideal sheaves defining them, and let $f_1 : \mathscr{X}_1 \to \mathscr{Y}_1$ be the induced morphism between the first nilpotent thickenings.*

(i) If f_1 is a closed immersion, then so is f.

(ii) If f_1 is a closed immersion and there is an isomorphism

$$\bigoplus_{n \geq 0} \mathfrak{m}_y^n / \mathfrak{m}_y^{n+1} \cong \bigoplus_{n \geq 0} \mathfrak{m}_x^n / \mathfrak{m}_x^{n+1}$$

of graded $\mathscr{O}_{\mathscr{X}_0}$-modules, then f is an isomorphism.

Proof. By Theorem 1.2.12, we may choose a minimal smooth presentations $V = \operatorname{Spec} B \to \mathscr{Y}$ such that $V \times_{\mathscr{Y}} \mathscr{Y}_0 \cong \operatorname{Spec} k$. Hence B is an artinian local ring, then so is $U = \operatorname{Spec} B \cong V \times_{\mathscr{Y}} \mathscr{X}$. Hence we can let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of local artinian rings.

For (i), this follows from [27] Lemma II.7.4. For (ii), this is trivial. □

Lemma 1.2.14. *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field with affine diagonal. Let $f : \mathscr{W} := [\operatorname{Spec} A/G] \to \mathscr{X}$ be a finite type morphism with G linearly reductive. If $w \in \operatorname{Spec} A$ has closed G-orbit and f induces an isomorphism of stabilizer groups at w, then there exists a G-invariant, affine, and open subscheme $U \subset \operatorname{Spec} A$ containing w such that $f|_{[U/G]}$ is affine.*

Proof. Let $\pi : \mathscr{W} \to \operatorname{Spec} A^G$. We may let $F : \mathscr{W} \to \mathscr{X}$ is quasi-finite as it is quasi-finite over some open set.

Choose a smooth presentation $V = \operatorname{Spec} B \to \mathscr{X}$, then

$$\begin{tikzcd}
\mathscr{W}_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & V = \operatorname{Spec} B \arrow[d] \\
\mathscr{W} \arrow[r, "f"] & \mathscr{X}
\end{tikzcd}$$

As $\mathscr{X}$ with affine diagonal, the map $V \to \mathscr{X}$ is affine. Hence $\mathscr{W}_V$ is cohomologically affine. By Proposition 6.3.28 in [4] we have:

- Suppose $\mathscr{Z}$ is a noetherian algebraic stack with affine diagonal and a good moduli space $\pi : \mathscr{Z} \to Z$. If the diagonal Δ_π is quasi-finite, then it is finite.

Hence $\mathscr{W}_V \to V$ is separated. From descent $\mathscr{W} \to \mathscr{X}$ is also separated and that the relative inertia $\mathscr{I}_{\mathscr{W}/\mathscr{X}} \to \mathscr{W}$ is finite. Since the fiber over w is trivial, there is an open neighborhood over which the relative inertia is trivial. Hence replace this we may let $\mathscr{I}_{\mathscr{W}/\mathscr{X}} \to \mathscr{W}$ is trivial. Hence it is representable. By Serre's criteria we get the result. □

Theorem 1.2.15 (Étale Local Structure of Algebraic Stacks)**.** *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal. For every point $x \in X(k)$ with linearly reductive stabilizer G_x there exists an affine étale morphism*

$$f : ([\operatorname{Spec} A/G_x], w) \to (\mathscr{X}, x)$$

which induces an isomorphism of stabilizer groups at w.

If $x \in \mathscr{X}$ is a smooth point, then there is also an étale morphism

$$f : ([\operatorname{Spec} A/G_x], w) \to ([T_{\mathscr{X},x}/G_x], 0).$$

Proof of the Smooth Case. Here we only give the proof of smooth case and tell you the difficulties of proof the general case in the remark.

Since x is locally closed, we may let it is closed. Hence $\mathscr{X}_0 := \mathbf{B}G_x \subset \mathscr{X}$ defined by $\mathscr{I}$. Let $\mathscr{X}_n$ to be the n-th nilpotent thickening of it. The Zariski tangent space $T_{\mathscr{X},x}$ can be identified with the normal space $(\mathscr{I}/\mathscr{I}^2)^\vee$, hence with a G_x-representation. Hence we can define $\mathscr{T} = [T_{\mathscr{X},x}/G_x]$ with $\mathscr{T}_0 := \mathbf{B}G_x$ and the n-th nilpotent thickening $\mathscr{T}_n$.

By Proposition 1.2.10 we get an affine $\mathscr{X}_n \to \mathbf{B}G_x$. By Proposition 1.2.9 inductively we have lifts:

$$\begin{tikzcd}
\mathscr{X}_n \arrow[r] \arrow[d, hook] & \mathscr{T} \arrow[d] \\
\mathscr{X}_{n+1} \arrow[r] \arrow[ur, dashed] & \mathbf{B}G_x
\end{tikzcd}$$

By some easy commutative algebra via smooth descent, we have $\mathscr{X}_1 \cong \mathscr{T}_1$. Hence by Lemma 1.2.13(ii) we have $\mathscr{X}_n \cong \mathscr{T}_n$.

Consider $\pi : \mathscr{T} \to T := T_{\mathscr{X},x} /\!\!/ G_x$ and $\widehat{\mathscr{T}} := \operatorname{Spec} \widehat{\mathscr{O}}_{T,\pi(0)} \times_T \mathscr{T} = [\operatorname{Spec} B/G]$ where B is of finite type over the noetherian complete local k-algebra $B^G = \widehat{\mathscr{O}}_{T,\pi(0)}$. By Remark 1.2.7 (ii) we know that $\widehat{\mathscr{T}}$ is coherently complete along $\mathscr{T}_0$ and by coherent Tannaka duality we get

$$\mathrm{MOR}(\widehat{\mathscr{T}}, \mathscr{X}) \to \varprojlim \mathrm{MOR}(\mathscr{T}_n, \mathscr{X}).$$

Hence we have

$$\begin{tikzcd}
 & X & \\
\mathscr{X}_n \cong \mathscr{T}_n \arrow[r] \arrow[ur] & \widehat{\mathscr{T}} \arrow[r] \arrow[d] \arrow[u, dashed] \arrow[dr, phantom, "\ulcorner"] & \mathscr{T} \arrow[d] \\
 & \operatorname{Spec} \mathscr{O}_{T,\pi(0)} \arrow[r] & T
\end{tikzcd}$$

Now by Artin Approximation, there exists an étale morphism $(U, u) \to (T, 0)$ where U is an affine scheme with a k-point $u \in U$ and a morphism $(U \times_T \mathscr{T}, (u, 0)) \to (\mathscr{X}, x)$ agreeing with $(\widehat{\mathscr{T}}, 0) \to (\mathscr{X}, x)$ in the first order. As $U \times_T \mathscr{T}$ is smooth at $(u, 0)$ and $\mathscr{X}$ is smooth at x, and as $U \times_T \mathscr{T} \to \mathscr{X}$ induces an isomorphism of tangent spaces and stabilizer groups at $(u, 0)$, hence the morphism $U \times_T \mathscr{T} \to \mathscr{X}$ is étale at $(u, 0)$. Finally, by Lemma 1.2.14 we get the result. $\square$

Remark 1.2.16. *We refer Section 6.5.5 in [4] for the proof of the general case. Now we point out that in the general case we also have $\mathscr{X}_1 \cong \mathscr{T}_1$. But we can only use the Lemma 1.2.13(i) to get a closed immersion $\mathscr{X}_n \to \mathscr{T}_n$. Also in the general case we can not deduce $U \times_T \mathscr{T} \to \mathscr{X}$ is étale from the isomorphism of tangent spaces! In order* *to solve this, we need a more general fact called* equivariant Artin algebraization theorem. *See Theorem 6.5.14 in [4] for the statement and the proof.*

Remark 1.2.17. *Actually the property in some more general setting we only have the following (which we will not use):*

- *Let S be a quasi-separated algebraic space. Let $\mathscr{X}$ be an algebraic stack locally of finite presentation and quasi-separated over S, with affine stabilizers. If $x \in |\mathscr{X}|$ is a point with image $s \in |S|$ such that the residue field extension $\kappa(x)/\kappa(s)$ is finite and the stabilizer of x is linearly reductive, then there exists $f : ([\operatorname{Spec} A/\mathrm{GL}_N], w) \to (\mathscr{X}, x)$ induces an isomorphism of stabilizer groups (such kind of maps called* quotient presentation*). If $\mathscr{X}$ has separated (resp. affine) diagonal, then there exists a such representable (resp. affine), étale quotient presentation.*

See [2] Theorem 1.1. In our case, this is proved in [3].

1.3 Existence of Good Moduli Space

Here we give a strategy for constructing good moduli spaces in 6.8.1 in [4].

Our main goal is to glue the étale local GIT quotient $[\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$ via the groupoid representations. Let $f : \mathscr{W} := [\operatorname{Spec} A/G_x] \to \mathscr{X}$ is affine étale with $W := \operatorname{Spec} A^{G_x}$. Let $\mathscr{R} := \mathscr{W} \times_{\mathscr{X}} \mathscr{W}$ which is of form $[\operatorname{Spec} B/G_x]$ as f is affine. Let $R = \operatorname{Spec} B^{G_x}$ and consider

$$\begin{tikzcd}
\mathscr{R} \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \mathscr{W} \arrow[r, "f"] \arrow[d] & \mathscr{X} \\
R \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & W &
\end{tikzcd}$$

Hence if q_1, q_2 defines an étale equivalence relation, the algebraic space quotient W/R is a good moduli space of $f(W)$. Then we have some chance to glue them.

By Luna's fundamental lemma 1.2.1 (its Corollary 1.2.2), in order to make q_1, q_2 as an étale equivalence relation, we need that for all closed points $r \in \mathscr{R}$ we have

(a) $p_1(r), p_2(r)$ are closed;

(b) p_1, p_2 induces isomorphisms of stabilizer groups at r.

As $f(w)$ is is closed and f induces an isomorphism of stabilizer groups. We just want to show that there is an open neighborhood $\mathscr{U}$ of w such that

(i) $f|_{\mathscr{U}}$ sends closed points map to closed points and stable under base change;

(ii) $f|_{\mathscr{U}}$ induces isomorphisms of stabilizer groups at closed points and stable under base change.

We will see that the Θ-completeness implies Θ-surjectivity which will implies (i); and S-completeness will implies (ii).

1.3.1 Basic Properties of Θ-Complete and S-Complete

Definition 1.3.1 (Θ-Completeness). *Define* $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$ *over* $\operatorname{Spec}\mathbb{Z}$ *and* $\Theta_R := \Theta \times \operatorname{Spec} R$ *for any DVR* R *with fracion field* K *and residue field* κ*. We can describe it as following cartesians:*

$$\begin{tikzcd}
 & \operatorname{Spec} R \arrow[dr, hook, "x\neq 0"] & & \mathbf{B}\mathbb{G}_{m,R} \arrow[dl, hook', "x=0"'] & \\
\operatorname{Spec} K \arrow[ur, hook] \arrow[dr] \arrow[r, phantom, "\lrcorner" rotate=-45] & & \Theta_R & & \mathbf{B}\mathbb{G}_{m,\kappa} \arrow[ul, hook'] \arrow[dl, hook'] \arrow[l, phantom, "\lrcorner" rotate=135] \\
 & \Theta_K \arrow[ur, hook, "\pi\neq 0"] & & \Theta_\kappa \arrow[ul, hook', "\pi=0"] &
\end{tikzcd}$$

Hence $\Theta_R\backslash 0 = \operatorname{Spec} R \cup_{\operatorname{Spec} K} \Theta_K$*. Hence* $\Theta_R\backslash 0 \to \mathscr{X}$ *is the data of morphisms* $\operatorname{Spec} R \to \mathscr{X}$ *and* $\Theta_K \to \mathscr{X}$ *together with an isomorphism of their restrictions to* $\operatorname{Spec} K$*.*

Then a locally noetherian algebraic stack $\mathscr{X}$ *is called* Θ-*complete if for any DVR* R*, every diagram*

$$\begin{tikzcd}
\Theta_R\backslash 0 \arrow[r] \arrow[d, hook] & \mathscr{X} \\
\Theta_R \arrow[ur, dashed] &
\end{tikzcd}$$

of solid arrows can be uniquely filled in.

Here is the figures of our stacks look like, see Remark 1.3.4:

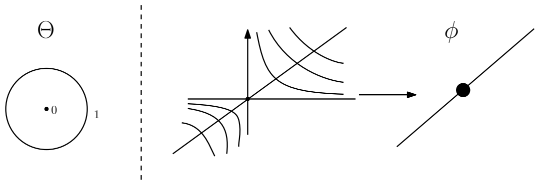


Figure 1.1: Θ and ϕ looks like

Definition 1.3.2 (S-Completeness). *For any DVR R with fracion field K and residue field κ, we define*

$$\phi_R := [\operatorname{Spec}(R[s,t]/(st-\pi))/\mathbb{G}_m]$$

where s and t have $\mathbb{G}_m$-weights 1 and -1, respectively. Now we have

$$\phi_R|_{s\neq 0} = [\operatorname{Spec}(R[s,t]_s/(t-\pi/s))/\mathbb{G}_m] = [\operatorname{Spec}(R[s]_s)/\mathbb{G}_m] \cong \operatorname{Spec} R$$

and similar for $t \neq 0$. Hence we can describe it as following cartesians:

$$\begin{tikzcd}
 & \operatorname{Spec} R \arrow[dr, hook, "s\neq 0"] & & \Theta_\kappa \arrow[dl, hook, "s=0"'] & \\
\operatorname{Spec} K \arrow[ur, hook] \arrow[dr, hook] \arrow[r, phantom, "<"] & & \phi_R & & \mathbf{B}\mathbb{G}_{m,\kappa} \arrow[ul, hook'] \arrow[dl, hook] \arrow[l, phantom, ">"] \\
 & \operatorname{Spec} R \arrow[ur, hook, "t\neq 0"] & & \Theta_\kappa \arrow[ul, hook, "t=0"] &
\end{tikzcd}$$

Hence $\phi_R \backslash 0 = \operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R \to \mathscr{X}$ is the data of two morphisms $\xi, \xi' : \operatorname{Spec} R \to \mathscr{X}$ together with an isomorphism $\xi_K \cong \xi'_K$ over $\operatorname{Spec} K$.

Then a locally noetherian algebraic stack $\mathscr{X}$ is called S-*complete* *if for any DVR R, every diagram*

$$\begin{tikzcd}
\phi_R \backslash 0 \arrow[r] \arrow[d, hook] & \mathscr{X} \\
\phi_R \arrow[ur, dashed] &
\end{tikzcd}$$

of solid arrows can be uniquely filled in.

Remark 1.3.3. *In the original paper [8], they introduce the Θ-completeness and* S-*completenes for morphisms of algebraic stacks, but we won't use them.*

Remark 1.3.4. *There is an interesting fact that the symbols Θ and ϕ is used because they look like the stacks they represent! See figure 1.1.*

There are many properties of Θ-completeness and S-completeness, here we introduce some of them.

Proposition 1.3.5. *We have the following properties:*

(i) A locally noetherian algebraic stack with affine diagonal is Θ-complete (resp. S-*complete), if and only if these diagrams, there exists a lift after an extension of DVRs $R \subset R'$. In particular, Θ-completeness and* S-*completeness can be verified on complete DVRs with algebraically closed residue fields.*

(ii) *Let* $f; \mathscr{X} \to \mathscr{Y}$ *be an affine morphism of locally noetherian algebraic stacks. If* $\mathscr{Y}$ *is* Θ*-complete (resp.* S*-complete), so is* $\mathscr{X}$*.*

(iii) *If* G *is a reductive group over an algebraically closed field* k*, then every quotient stack* $[\operatorname{Spec} A/G]$ *is* Θ*-complete and* S*-complete.*

(iv) *Let* $\mathscr{X}$ *be an algebraic stack of finite type over an algebraically closed field* k *with affine diagonal. If* $\pi : \mathscr{X} \to X$ *be a good moduli space, then* $\mathscr{X}$ *is* Θ*-complete. Moreover,* $\mathscr{X}$ *is* S*-complete if and only if* X *is separated.*

(v) *Let* $\mathscr{X}$ *be a noetherian algebraic stack with affine and quasi-finite diagonal. Then*

- *If* R *is a complete DVR, every map* $\Theta_R \to \mathscr{X}$ *(resp.* $\phi_R \to \mathscr{X}$*) factors through* $\Theta_R \to \operatorname{Spec} R$ *(resp.* $\phi_R \to \operatorname{Spec} R$*).*
- $\mathscr{X}$ *is* Θ*-complete. Moreover,* $\mathscr{X}$ *is* S*-complete if and only if it is separated.*

(vi) *If* $\mathscr{X}$ *be an algebraic stack locally of finite type over an algebraically closed field* k *with affine diagonal, then to verify that* $\mathscr{X}$ *is* Θ*-complete or* S*-complete, it suffices to check the lifting criterion for DVRs* R *essentially of finite type over* k*.*

Some Comments of Proofs. We will not give the whole proofs, but we will give some comments on it. The proofs we refer [8] or [4].

For (i), this follows from the fpqc descent.

For (ii), since Θ_R is regular and $0 \in \Theta_R$ is codimension 2, the pushforward of the structure sheaf along $\Theta_R \backslash 0 \to \Theta_R$ is the structure sheaf. Then by the definition we can get the result.

For (iii), first show the case of $\mathbf{B}\mathrm{GL}_n$. Indeed this follows from $0 \in \Theta_R$ is codimension 2 and Θ_R is regular, then vector bundles have unique extension. For general case, pick a faithful representation $G \subset \mathrm{GL}_n$ By the reductivity of G we get GL_n/G is affine by Matushima's result. As

$$\begin{tikzcd}
\mathrm{GL}_n/G \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \operatorname{Spec} k \arrow[d] \\
\mathbf{B}G \arrow[r] & \mathbf{B}\mathrm{GL}_n
\end{tikzcd}$$

and smooth descent we get $\mathbf{B}G \to \mathbf{B}\mathrm{GL}_n$ is affine. Hence the result follows from (ii).

For (iv), by the étale local structure of algebraic stacks and (i) we can show that the Θ-completeness follows from the local case (iii). For S-completeness, this from some arguments of valuative criterions.

For (v), the first one follows from deformation theory and coherent Tannaka duality. The second one follows from the first one and the valuative criterion.

For (vi), see Proposition 3.18 and Proposition 3.42 in [8]. □

Actually the S-completeness also have some relation to the reductivity of groups:

Theorem 1.3.6 (Cartan Decomposition and S-Completeness)**.** *Let G be a smooth affine algebraic group over an algebraically closed field k. Then the following are equivalent:*

(a) G is reducible.

(b) $\mathbf{B}G$ is S-complete.

(c) For any complete DVR R over k with residue field κ and fracion field K and for any $g \in G(K)$, there exists elements $h_1, h_2 \in G(R)$ and a 1-PS $\lambda : \mathbb{G}_m \to G$ such that $g = h_1 \lambda|_K h_2$

In particular, if $\mathscr{X}$ is an S-complete algebraic stack and $x \in \mathscr{X}$ is a closed point with smooth affine stabilizer G_x, then G_x is reductive.

Proof. We omitted the proof. Actually the equivalence of (a) and (c) is classical in [45]. For the complete proof we refer Proposition 6.8.45 in [4]. □

Proposition 1.3.7 (Stacky Destabilization Theorem)**.** *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal. Let $x \rightsquigarrow x_0$ be a specialization of k-points such that the stabilizer G_{x_0} is linearly reductive. Then there exists a morphism $[\mathbb{A}^1/\mathbb{G}_m] \to \mathscr{X}$ representing the specialization $x \rightsquigarrow x_0$.*

Proof. Here we will use a classical destabilization theorem, see Page 53 in [45] or Theorem 6.6.28 in [4]:

- Let G be a reductive algebraic group over an algebraically closed field k acting on an affine scheme X of finite type over k. Given $x \in X(k)$, there exists a 1-PS $\lambda : \mathbb{G}_m \to G$ such that $x_0 := \lim_{t \to 0} \lambda(t) \cdot x$ exists and has closed G-orbit.

Back to our proof. By the Theorem 1.2.15, we have étale morphism $f : ([\operatorname{Spec} A/G_{x_0}], w_0) \to (\mathscr{X}, x_0)$ which induces an isomorphism of stabilizer groups at w_0. After possibly replacing $\operatorname{Spec} A$ with a G_{x_0}-invariant affine subscheme, we can assume that w_0 is a closed point. The specialization $x \rightsquigarrow x_0$ lifts a specialization $w \rightsquigarrow w_0$ in $\operatorname{Spec} A$, and we can choose a representative $\tilde{w} \in \operatorname{Spec} A$ of the orbit corresponding to w. The Destabilization Theorem gives a 1-PS $\lambda : \mathbb{G}_m \to G_{x_0}$ such that $\tilde{w}_0 := \lim_{t \to 0} \lambda(t) \cdot \tilde{w}$ exists and has closed orbit. By the affine version of Theorem 1.1.3 we get there is a unique closed orbit in $\overline{G\tilde{w}}$, and thus $\tilde{w}_0 \in \operatorname{Spec} A$ maps to w_0. Hence the extension of λ induce $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1 \to \operatorname{Spec} A$. Hence we get $[\mathbb{A}^1/\mathbb{G}_m] \to [\operatorname{Spec} A/G_{x_0}] \to \mathscr{X}$ representing the specialization $x \rightsquigarrow x_0$. □

1.3.2 Θ-Surjectivity and Θ-Complete

Definition 1.3.8 (Θ-surjective)**.** *Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks and $\mathscr{X}(k)$ be a geometric point. We say f is Θ-surjective at x if every diagram:*

$$\begin{tikzcd}
\operatorname{Spec} k \arrow[r, "x"] \arrow[d, hook, "1"] & \mathscr{X} \arrow[d, "f"] \\
\Theta_k \arrow[r] \arrow[ur, dashed] & \mathscr{Y}
\end{tikzcd}$$

has a lift. We say that f is Θ-surjective if it is Θ-surjective at every geometric point.

Remark 1.3.9. *This condition is table under base change as it is equivalent to the surjectivity of*

$$\mathrm{ev}(f)_1 : \underline{\mathrm{MOR}}(\Theta, \mathscr{X}) \to \mathscr{X} \times_{\mathscr{Y}, \mathrm{ev}(f)_1} \underline{\mathrm{MOR}}(\Theta, \mathscr{Y}).$$

If $f : \mathscr{X} \to \mathscr{Y}$ of noetherian algebraic stacks where $\mathscr{Y}$ with affine and quasi-finite diagonal, then by Propostion 1.3.5(v) we know that f is Θ-surjective.

Lemma 1.3.10. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a separated, representable, and finite type morphism of noetherian algebraic stacks, then the lift in the definition of Θ-surjectivity is unique and the Θ-surjectivity is not depend on the fields represent the same point.*

Proof. The first one follows from descent and valuative criterion. The second one follows from some limit result, we omit it. ☐

Proposition 1.3.11. *Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks, each of finite type over an algebraically closed field k with affine diagonal. Let the closed points of $\mathscr{Y}$ have linearly reductive stabilizers. If f is Θ-surjective, then f sends closed points to closed points.*

Proof. Let $x \in |\mathscr{X}|$ closed and $f(x) \rightsquigarrow y_0$ be a specialization to a closed point. By Proposition 1.3.7, we have $\Theta \to \mathscr{Y}$ sends $1 \mapsto f(x), 0 \mapsto y_0$. Hence by $Theta$-surjectivity, we get a lift $g : \Theta \to \mathscr{X}$ sends $1 \mapsto x$. As x closed, this map g is trivial. So is $\mathscr{Y}$. Well done. ☐

Proposition 1.3.12. *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal such that the closed points of $\mathscr{X}$ have linearly reducetive stabilizers. Let $x \in \mathscr{X}$ be a closed point with affine étale morphism $f : ([\operatorname{Spec} A/G_x], w) \to (\mathscr{X}, x)$ inducing an isomorphism of stabilizers at w. Let $\pi : [\operatorname{Spec} A/G_x] \to \operatorname{Spec} A^{G_x}$. Then if $\mathscr{X}$ is Θ-complete, then there exists an open affine $U \subset \operatorname{Spec} A^{G_x}$ of $\pi(w)$ such that $f|_{\pi^{-1}(U)} : \pi^{-1}(U) \to \mathscr{X}$ is Θ-surjective.*

Proof. We omit the proof and refer Propisition 6.8.31 in [4] or Proposition 4.3(i) in [8] for more general case. ☐

Here we have a topology like GIT:

Proposition 1.3.13. *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal. Assume that $\mathscr{X}$ is Θ-complete and that the closed points of $\mathscr{X}$ have linearly reductive stabilizer. Then the closure of every k-point contains a unique closed point.*

Proof. If we have two of them, we then have two $\Theta \to \mathscr{X}$. Then we can glue them into $[\mathbb{A}^2/\mathbb{G}_m]\backslash 0 \to \mathscr{X}$. Consider the diagonal action and Θ-completeness, we get extension $\Psi : [\mathbb{A}^2/\mathbb{G}_m] \to \mathscr{X}$. Hence $\Psi(0,0)$ is a common specialization of $x = \Psi(1,0)$ and $x' = \Psi(0,1)$. Since x and x' are closed points, we have that $x = \Psi(0,0) = x'$. ☐

1.3.3 Unpunctured Inertia and S-Complete

We will only give a sketch of these because the proof of this main theorem is very complicated.

Definition 1.3.14 (Unpunctured Inertia)**.** *We say that a noetherian algebraic stack $\mathscr{X}$ has* unpunctured inertia *if for every closed point $x \in |\mathscr{X}|$ and every formally smooth morphism $p : (T,t) \to (\mathscr{X}, x)$ where T is the spectrum of a local ring with closed point t, every connected component of the inertia group scheme $\mathrm{Aut}_{\mathscr{X}}(p) \to T$ has non-empty intersection with the fiber over t.*

Proposition 1.3.15. *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal. Let $x \in |\mathscr{X}|$ be a closed point which have linearly reducetive stabilizers. Pick an affine étale morphism $f : ([\mathrm{Spec}\, A/G_x], w) \to (\mathscr{X}, x)$ inducing an isomorphism of stabilizers at w. and let $\pi : [\mathrm{Spec}\, A/G_x] \to \mathrm{Spec}\, A^{G_x}$. Then if $\mathscr{X}$ has unpunctured inertia, then there exists an open affine $U \subset \mathrm{Spec}\, A^{G_x}$ of $\pi(w)$ such that $f|_{\pi^{-1}(U)} : \pi^{-1}(U) \to \mathscr{X}$ induces isomorphisms of stabilizers at all points.*

Proof. Let $\mathscr{W} := [\mathrm{Spec}\, A/G_x]$. We just need to find an open $\mathscr{U} \subset \mathscr{W}$ of w such that $f|_{\mathscr{U}} : \mathscr{U} \to \mathscr{X}$ induce an isomorphism $\mathscr{I}_{\mathscr{U}} \cong \mathscr{U} \times_{\mathscr{X}} \mathscr{I}_{\mathscr{X}}$. Consider

$$\begin{tikzcd}
\mathscr{I}_{\mathscr{W}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{W} \times_{\mathscr{X}} \mathscr{I}_{\mathscr{X}} \arrow[d] \\
\mathscr{W} \arrow[r] & \mathscr{W} \times_{\mathscr{X}} \mathscr{W}
\end{tikzcd}$$

As f is affine étale, then $\mathscr{I}_{\mathscr{W}} \to \mathscr{W} \times_{\mathscr{X}} \mathscr{I}_{\mathscr{X}}$ is finite étale. Let $\mathscr{Z} \subset \mathscr{W} \times_{\mathscr{X}} \mathscr{I}_{\mathscr{X}}$ be the locus that is not an isomorphism. Then $\mathscr{Z}$ is closed and open substack. Let $p_1 : \mathscr{W} \times_{\mathscr{X}} \mathscr{I}_{\mathscr{X}} \to \mathscr{W}$ and then $w \notin p_1(\mathscr{Z})$.

Let a formally smooth morphism $p : (T, t) \to (\mathscr{X}, x)$ where T is the spectrum of a local ring with closed point t. Since $\mathscr{X}$ has unpunctured inertia, hence the preimage of $\mathscr{Z}$ in $\mathscr{W} \times_{\mathscr{X}} \mathscr{I}_{\mathscr{X}} \times_{\mathscr{X}} T$ is empty. Then $w \notin \overline{p_1(\mathscr{Z})}$, hence pick $\mathscr{U} := \mathscr{W} \backslash \overline{p_1(\mathscr{Z})}$ and well done. □

Theorem 1.3.16. *Let $\mathscr{X}$ be an algebraic stack of finite type over an algebraically closed field k with affine diagonal. Assume that the closed points have linearly reductive stabilizers. If $\mathscr{X}$ is S-complete, then $\mathscr{X}$ has unpunctured inertia.*

Proof. Omitted since this is very complicated. For our case we refer the proof of Theorem 6.8.40 in [4] and the general case we refer Theorem 5.3 in [8]. □

1.3.4 The Finally Statement and the Proof

Theorem 1.3.17. *Let $\mathscr{X}$ be an algebraic stack of finite tyoe over an algebraically closed field k of characteristic 0 with affine diagonal. There exists a good moduli space $\pi : \mathscr{X} \to X$ with X a separated algebraic space if and only if $\mathscr{X}$ is Θ-complete and S-complete.*

Moreover, X is proper if and only if $\mathscr{X}$ satisfies the existence part of the valuative criterion for properness.

Remark 1.3.18. *Here we follows the proof in [4] which I talked before. In paper [8] Theorem 5.4, we have a more general form which is characteristic independent:*

- *Let $\mathscr{X}$ be an algebraic stack of finite presentation over a quasi-separated and locally noetherian algebraic space S, with affine stabilizers and separated diagonal. Then $\mathscr{X}$ admits a good moduli space X separated over S if and only if we have*

  (1) every closed point of $\mathscr{X}$ has linearly reductive stabilizer;

  (2) $\mathscr{X} \to S$ is Θ-complete;

  (3) $\mathscr{X} \to S$ is S-complete.

  If $\mathscr{X}$ is locally reductive and has affine diagonal, then $\mathscr{X}$ admits an adequate moduli space X separated over S if and only if (2) and (3) hold. In both cases, if S is locally excellent and $\mathscr{X} \to S$ has affine diagonal, it suffices to check the filling conditions of Θ-completeness and S-completeness only for DVRs that are essentially finite type over S.

  Furthermore, in both cases $X \to S$ is proper if and only if $\mathscr{X} \to S$ satisfies the existence part of the valuative criterion for properness.

But we will not use this.

Proof of Theorem 1.3.17. If $\mathscr{X}$ has a good moduli space X which is a separated algebraic space, then $\mathscr{X}$ is Θ-complete and S-complete by Proposition 1.3.5(iv). Hence we just need to consider the converse.

By Theorem 1.3.6, as $\mathscr{X}$ is S-complete and over characteristic zero, then stabilizers of every closed points are linearly reductive. For any closed $x \in |\mathscr{X}|$ there is an affine étale morphism $([\operatorname{Spec} A/G_x], w) \to (\mathscr{X}, x)$ which is Θ-surjective and stabilizer preserving at all points since $\mathscr{X}$ is Θ-complete and S-complete by Proposition 1.3.12, Proposition 1.3.15 and Theorem 1.3.16. Since $\mathscr{X}$ is quasi-compact, we can choose finitely many closed points $x_i \in \mathscr{X}$ and morphisms $f_i : [\operatorname{Spec} A_i/G_{x_i}] \to \mathscr{X}$. Pick an embedding $G_{X_i} \hookrightarrow \mathrm{GL}_N$. Since $[\operatorname{Spec} A_i/G_{x_i}] \cong [\operatorname{Spec} A_i \times^{G_{x_i}} \mathrm{GL}_N/\mathrm{GL}_N]$, let $A = \prod_i (A_i \times^{G_{x_i}} \mathrm{GL}_N)$ and we get a surjective, affine, and étale morphism

$$f : \mathscr{X}_1 := [\operatorname{Spec} A/\mathrm{GL}_N] \to \mathscr{X}$$

which is Θ-surjective and stabilizer preserving at all points. As the characteristic of k is zero, then GL_N is linear reductive. Hence we have a good moduli space $\mathscr{X}_1 \to X_1 := \operatorname{Spec} A^{\mathrm{GL}_N}$.

Let $\mathscr{X}_2 := \mathscr{X}_1 \times_{\mathscr{X}} \mathscr{X}_1$ with two affine, étale, Θ-surjective and stabilizer preserving projections $p_1, p_2 : \mathscr{X}_2 \to \mathscr{X}_1$. As f affine, then $\mathscr{X}_2 \cong [\operatorname{Spec} B/\mathrm{GL}_N]$ with good moduli space $\mathscr{X}_2 \to X_2 := \operatorname{Spec} B^{\mathrm{GL}_N}$. Hence we have two cartesian diagrams by Luna's fundamental lemma 1.2.1:

$$\begin{tikzcd}
\mathscr{X}_2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \mathscr{X}_1 \arrow[d] \\
X_2 \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & X_1
\end{tikzcd}$$

By the universal property of good moduli space, $q_1, q_2 : X_2 \rightrightarrows X_1$ is an étale groupoid.

We claim that $q_1, q_2 : X_2 \rightrightarrows X_1$ is an étale equivalence relation. Pick any $x_1 \in X_1(k)$ and let $x_2, x_2' \in X_2$ are two points in the preimage of (x_1, x_1) in $(q_1, q_2) : X_2 \to X_1 \times X_1$. Let $\hat{x}_2, \hat{x}_2'$ be the unique closed points in their preimages by Proposition 1.3.13. As f is Θ-surjective, then $p_1(\hat{x}_2), p_2(\hat{x}_2), p_1(\hat{x}_2')$ and $p_2(\hat{x}_2')$ are closed over $x_1 \in X_1$. Hence they are all identified with the unique closed point $\hat{x}_1$ over x_1. On the other hand, since f is stabilizer preserving, the stabilizer groups of $\hat{x}_2$ and $\hat{x}_2'$ are the same as the stabilizer groups of $\hat{x}_1$ and of its image in $\mathscr{X}$. Let this stabilizer group by G. It follows that the fiber product of $(p_1, p_2) : \mathscr{X}_2 \to \mathscr{X}_1 \times \mathscr{X}_1$ along the inclusion of the residual gerbe $\mathcal{G}_{(\hat{x}_1, \hat{x}_1)} = \mathbf{B}G \times \mathbf{B}G \to \mathscr{X}_1 \times \mathscr{X}_1$ is isomorphic to $\mathbf{B}G$ and thus identified with the residual gerbe of a unique closed point. Therefore $x_2 = x_2'$. Hence we get the claim.

Now pick $X = X_1/X_2$ as an algebraic space. From étale descent, we have

$$\begin{tikzcd}
\mathscr{X}_2 \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] \arrow[d] & \mathscr{X}_1 \arrow[r, "f"] \arrow[d] & \mathscr{X} \arrow[d, dashed] \\
X_2 \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & X_1 \arrow[r, dashed] & X
\end{tikzcd}$$

By the descent of good moduli space we know $\mathscr{X} \to X$ is a good moduli space. As it is S-complete and Proposition 1.3.5(iv), we get X is separated. □

1.4 Semistable Reduction and Θ-Stability

1.4.1 Preliminaries: Θ-Stratification

Here we first define two stacks arising from the stack of coherent sheaves, see Proposition 3.1.2 for the similar argument.

Definition 1.4.1. *For an algebraic stack $\mathscr{X}$ over S, we define two notations:*

- *By definition for any stack $\mathscr{X}$ and point $\operatorname{Spec} k \to S$ a map $\mathbf{B}\mathbb{G}_{m,k} \to \mathscr{X}$ is a point $x \in \mathscr{X}(k)$ together with a cocharacter $\mathbb{G}_{m,k} \to \operatorname{Aut}_X(x)$. As the action of $\mathbb{G}_m$ on a vector space is the same as a grading on the vector space, we often think of a morphism $\mathbf{B}\mathbb{G}_m \to \mathscr{X}$ as a point of $\mathscr{X}$ equipped with a grading. Hence we define the* **stack of graded points** *in $\mathscr{X}$ to be $\operatorname{Grad}(\mathscr{X}) := \underline{\operatorname{Map}}_S(\mathbf{B}\mathbb{G}_m, \mathscr{X})$.*
- *By definition for any stack $\mathscr{X}$ and point $\operatorname{Spec} k \to S$ a map $f : \Theta_k \to \mathscr{X}$ is a point $x_1 \in \mathscr{X}(k)$ (as $f(1)$) together with a filtration of x_1 and as $x_0 = f(0)$ as the associated graded object. Hence we define the* **stack of filtrations** *in $\mathscr{X}$ to be $\operatorname{Filt}(\mathscr{X}) := \underline{\operatorname{Map}}_S(\Theta, \mathscr{X})$.*

Definition 1.4.2. *Let $\mathscr{X}$ be an algebraic stack locally of finite type over a noetherian scheme S.*

*(i) A Θ-**stratum** in $\mathscr{X}$ consists of a union of connected components $\mathscr{Z}^+ \subset \operatorname{Filt}(\mathscr{X})$ such that $\operatorname{ev}_1 : \mathscr{Z}^+ \to \mathscr{X}$ is a closed immersion.*

*(ii) A Θ-**stratification** of $\mathscr{X}$ indexed by a totally ordered set Γ is a cover of $\mathscr{X}$ by open substacks $\mathscr{X}_{\leq c}$ for $c \in \Gamma$ such that $\mathscr{X}_{\leq c} \subset \mathscr{X}_{\leq c'}$ for $c < c'$, along with a Θ-stratum $\mathscr{Z}_c^+ \subset \operatorname{Filt}(\mathscr{X}_{\leq c})$ in each $\mathscr{X}_{\leq c}$ whose complement is $\bigcup_{c' < c} \mathscr{X}_{\leq c'}$.*

We require that for any $x \in |\mathscr{X}|$ the subset $\{c \in \Gamma : x \in \mathscr{X}_{\leq c}\}$ has minimal element. We assume for convenience that Γ has a minimal element $0 \in \Gamma$.

(iii) We say that a Θ-stratification is well-ordered if for any point $x \in |\mathscr{X}|$, the totally ordered set $\{c \in \Gamma : \operatorname{ev}_1(\mathscr{Z}_c^+) \cap \overline{\{x\}} \neq \emptyset\}$ is well-ordered.

Definition 1.4.3. *Let $\mathscr{X}$ be an algebraic stack locally of finite type over a noetherian scheme S. Given a Θ-stratification, we refer to the open substack $\mathscr{X}^{\mathrm{ss}} := \mathscr{X}^{\leq 0}$ as the* semistable locus*. For any unstable point $x \in \mathscr{X}(k) \backslash \mathscr{X}^{\mathrm{ss}}(k)$, the Θ-stratification determines a canonical filtration $f : \Theta_k \to \mathscr{X}$ with $f(1) \cong x$, which we refer to as the* HN-filtration.

Remark 1.4.4. *The map $\mathbf{B}\mathbb{G}_m \hookrightarrow \Theta$ induce $\mathrm{gr} : \mathrm{Filt}(\mathscr{X}) \to \mathrm{Grad}(\mathscr{X})$ and projection $\Theta \to \mathbf{B}\mathbb{G}_m$ induce $\sigma : \mathrm{Grad}(\mathscr{X}) \to \mathrm{Filt}(\mathscr{X})$ as a section of gr. By Lemma 1.3.8 in [25], these maps define a canonical $\mathbb{A}^1$-deformation retract of $\mathrm{Filt}(\mathscr{X})$ to $\mathrm{Grad}(\mathscr{X})$.In particular induce bijections on connected components and we say they are the center $\mathscr{Z}$ of the Θ-stratum $\mathscr{Z}^+$.*

1.4.2 Semistable Reduction: Langton's Algorithm

Theorem 1.4.5 (Langton's Algorithm)**.** *Let $\mathscr{X}$ be an algebraic stack locally of finite type over an algebraically closed field k with affine diagonal. Assume that for any $x \in \mathscr{X}(k)$ the stabilizers G_x are smooth (for example, k is of characteristic zero).*

Let $\mathscr{Z}^+ \subset \mathscr{X}$ be a Θ-stratum. Let R be a DVR with fraction field K and residue field κ. Let $\xi_R : \operatorname{Spec} R \to \mathscr{X}$ such that the general point ξ_K is not mapped to $\mathscr{Z}^+$, but th special point ξ_κ is mapped to $\mathscr{Z}^+$:

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, hook] \arrow[d, "\xi_K"] & \operatorname{Spec} R \arrow[d, "\xi_R"] & \operatorname{Spec} \kappa \arrow[l, hook'] \arrow[d, "\xi_\kappa"] \\
\mathscr{X} \backslash \mathscr{Z}^+ \arrow[r, hook, "j"] & \mathscr{X} & \mathscr{Z}^+ \arrow[l, hook', "\iota"']
\end{tikzcd}$$

Then there exists an extension $R \to R'$ of DVRs with $K \to K' = \mathrm{Frac}(R')$ finite and an elementary modification *(that is, $h : \phi_R \to \mathscr{X}$ such that $\xi_{R'} \cong h|_{s \neq 0}$) $\xi'_{R'}$ of $\xi_{R'}$ such that $\xi'_{R'}$ lands in $\mathscr{X} \backslash \mathscr{Z}^+$.*

Remark 1.4.6. *This theorem holds for much general conditions (Theorem 6.3 in [8]): if $\mathscr{X}$ be an algebraic stack locally of finite type and quasi-separated, with affine stabilizers, over a noetherian algebraic space S. But we will not use that. In order to prove the general case, we will apply the non-local slice theorem 2.8 in [8]. But in our case we just need to use the Theorem 1.2.15.*

Sketch of Theorem 1.4.5. We have several steps:

- **Step 1.** Reduce to the case where $\mathscr{X}$ is quasi-compact.

  Proof of Step 1. Let $\sigma : \mathscr{Z} \to \mathscr{Z}^+$ be the center of the Θ-stratum $\mathrm{ev}_1 : \mathscr{Z}^+ \hookrightarrow \mathscr{X}$. Then for any point $x \in |\mathscr{Z}|$ and any open substack $\mathscr{U} \subset \mathscr{X}$ containing $\sigma(x)$, we just need to show that there is another open substack with $\sigma(x) \in \mathscr{V} \subset \mathscr{U}$ such that $\mathscr{X}^+ \cap \mathscr{V}$ is a Θ-stratum in $\mathscr{V}$.

Hence we only need to find a substack $\mathscr{V} \subset \mathscr{X}$ containing $\sigma(x)$ such that for any $f : \Theta_{k'} \to \mathscr{X}$, where k' is a field, with $f \in \mathscr{X}^+$ and $f(1) \in \mathscr{V}$, we have $f(0) \in \mathscr{V}$ as well. Indeed, let $\mathscr{U}' = (\mathrm{ev}_1 \circ \sigma)^{-1} \subset \mathscr{Z}$, and let $\mathscr{Z}' = \mathscr{Z} \backslash \mathscr{U}'$. Then the open substack

$$\mathscr{V} = \mathscr{U} \backslash (\mathscr{U} \cap \mathrm{ev}_1(\mathrm{gr}^{-1}(\mathscr{Z}'))) \subset \mathscr{X}$$

satisfies the condition. □

- **Step 2.** Consider a map $\xi : \operatorname{Spec} R \to \mathscr{X}$ as in the statement of the theorem. Assume that there is a smooth map $p : \mathscr{Y} \to \mathscr{X}$ such that $\mathscr{X}^+$ induces a Θ-stratum $p^{-1}(\mathscr{X}^+)$ in $\mathscr{Y}$ and the image of p contains the image of ξ. If we know the conclusion of the theorem holds for $\mathscr{Y}$, then show that the conclusion holds for $\mathscr{X}$ as well.

Proof of Step 2. As after an extension of R we may lift ξ to a map $\xi' : \operatorname{Spec} R' \to \mathscr{Y}$, this is trivial. □

- **Step 3.** Let $\mathscr{X}^+$ with center $\sigma : \mathscr{Z} \to \mathscr{X}^+$, and let $x_0 \in \mathscr{Z}(k)$ be a point and let $x := \sigma(x_0)$. Then there is a smooth representable morphism $p : [\operatorname{Spec} A/\mathbb{G}_m] \to \mathscr{X}$ whose image contains x and such that

$$p^{-1}(\mathscr{X}^+) = [\operatorname{Spec}(A/I_+)/\mathbb{G}_m] \hookrightarrow [\operatorname{Spec} A/\mathbb{G}_m]$$

where I_+ generated by the positive weight elements under $\mathbb{G}_m$.

Proof of Step 3. Since this need to analyse the properties of normal cone to Θ-stratum, we refer Lemma 6.9 and 6.10 in [8] and just give a sketch.

The point x_0 has $\mathbb{G}_m \to \mathrm{Aut}_{\mathscr{Z}}(x_0)$ which induce a 1-PS $\lambda : \mathbb{G}_m \to G_x$. WLOG let λ injective, then by Theorem 1.2.15 we get a smooth representable morphism

$$p : [\operatorname{Spec} A/\mathbb{G}_m] \to \mathscr{X}.$$

By some result (Lemma 6.9 in [8]) we can show that $\mathscr{X}_A^+ := [\operatorname{Spec}(A/I_+)/\mathbb{G}_m]$ satisfies $\mathscr{X}_A^+ \cong p^{-1}(\mathscr{X}^+)$ after shrink A. □

- **Step 4.** The theorem holds for $\mathscr{X} = [\operatorname{Spec} A/\mathbb{G}_m]$ and $\mathscr{X}^+ = [\operatorname{Spec}(A/I_+)/\mathbb{G}_m]$.

Proof of Step 4. This follow from an elementary calculation. See Lemma 6.7 in [8] for details. We will omit it for now. □

- **Step 5.** Finish the proof.

  Proof of Step 5. By the **Step 1**–**Step 4**, we conclude the theorem. ☐

Well done. ☐

Theorem 1.4.7 (Semistable Reduction)**.** *Let $\mathscr{X}$ be an algebraic stack locally of finite type over an algebraically closed field k with affine diagonal. Assume that for any $x \in \mathscr{X}(k)$ the stabilizers G_x are smooth (for example, k is of characteristic zero). Let $\mathscr{X}$ with a well-ordered Θ-stratification. Then for any morphism $\operatorname{Spec}(R) \to \mathscr{X}$, after an extension $R \to R'$ of DVRs with $K \to K' = \operatorname{Frac}(R')$ finite there is a modification (that is, another $\xi' : \operatorname{Spec} R \to \mathscr{X}$ such that $\xi|_K \cong \xi'|_K$) $\operatorname{Spec}(R') \to \mathscr{X}$, obtained by a finite sequence of elementary modifications, whose image lies in a single stratum of $\mathscr{X}$.*

Remark 1.4.8. *This theorem also holds for much general conditions (Theorem 6.5 in [8]): if $\mathscr{X}$ be an algebraic stack locally of finite type and quasi-separated, with affine stabilizers, over a noetherian algebraic space S. But we will not use that. The general version follows from the same proof induced by the general version of the Langton's Algorithm.*

Proof of Theorem 1.4.7. Actually this follows from Theorem 1.4.5 directly. Consider a map $\xi_R : \operatorname{Spec} R \to \mathscr{X}$ such that $\xi_\kappa \in \mathscr{X}_{c_0}^+$ and $\xi_K \in \mathscr{X}_c^+$ for $c_0 > c$, we may apply Theorem 1.4.5 iteratively to obtain a sequence of finite extensions of R and elementary modifications of ξ with special point in $\mathscr{X}_{c_i}^+$ for $c_0 > c_1 > \cdots$. Each $\mathscr{X}_{c_i}^+$ meets $\overline{\xi_K}$, so the well-orderedness condition guarantees that this procedure terminates, and it can only terminate when $c_i = c$. ☐

1.4.3 Comparison Between a Stack and Its Semistable Locus

Here is an easy consequence of the semistable reduction:

Proposition 1.4.9. *Let $\mathscr{X}$ be an algebraic stack locally of finite type over an algebraically closed field k with affine diagonal with smooth stabilizers for any $x \in \mathscr{X}(k)$. Let $\mathscr{X} = \bigcup_{c \in \Gamma} \mathscr{X}_{\leq c}$ be a well-ordered Θ-stratification. If $\mathscr{X} \to \operatorname{Spec} k$ satisfies the existence part of the valuative criterion for properness with respect to DVRs, then so does $\mathscr{X}_{\leq c}$ for every $c \in \Gamma$. In particular, if the semistable locus $\mathscr{X}^{\mathrm{ss}}$ is quasi-compact, then $\mathscr{X}^{\mathrm{ss}} \to \operatorname{Spec} k$ is universally closed.*

Proof. Using Theorem 1.4.7, we find the following process:

$$\begin{tikzcd}
\operatorname{Spec} K'' \arrow[r] \arrow[d, hook] & \operatorname{Spec} K' \arrow[r] \arrow[d, hook] & \operatorname{Spec} K \arrow[r] \arrow[d, hook] & \mathscr{X}_{\leq c} \arrow[r, hook] \arrow[d] & \mathscr{X} \arrow[dl] \\
\operatorname{Spec} R'' \arrow[r] \arrow[urrr, dashed] & \operatorname{Spec} R' \arrow[r] \arrow[urrr, dashed] & \operatorname{Spec} R \arrow[r] & \operatorname{Spec} k &
\end{tikzcd}$$

where $\operatorname{Spec} R'' \to \operatorname{Spec} R'$ is a sequence of elementary modifications. As $\phi_R \to \operatorname{Spec} R$ is a good moduli space and by the universal property of good moduli space, any elementary modification of a map $\operatorname{Spec} R \to \operatorname{Spec} k$ is trivial for some DVR R. It follows that our modified map $\operatorname{Spec} R' \to \mathscr{X}_{\leq c}$ is a lift of the original map $\operatorname{Spec}(R) \to \operatorname{Spec} k$. The final statement follows from that a finitely presented morphism of noetherian algebraic stacks $\mathscr{A} \to \mathscr{B}$ is universally closed by checking that $\mathscr{A} \times \mathbb{A}^n \to \mathscr{B} \times \mathbb{A}^n$ is closed for all n (see [26] Lemma 2.4.6). Well done. □

Now we will introduce the Θ-stability and its properties which is very important.

Definition 1.4.10. *Given a cohomology class $\ell \in H^2(\mathscr{X}, \mathbb{R})$, we say that a point $p \in |\mathscr{X}|$ is* unstable with respect to ℓ *if there is a filtration $f : \Theta_k \to \mathscr{X}$ with $f(1) = p$ and such that $f^*(\ell) \in H^2(\Theta_k, \mathbb{R}) \cong \mathbb{R}$ is positive. The* Θ-semistable locus $\mathscr{X}^{\mathrm{ss}}$ *is the set of points which are not unstable.*

Remark 1.4.11. *We don't care the maining of the cohomology here. If $\mathscr{X}$ is over $k \subset \mathbb{C}$, we consider the betti cohomology. If over other field we consider the Chow cohomology. If in general we consider the Neron-Severi group.*

Proposition 1.4.12. *Let $\mathscr{X}$ be an algebraic stack locally of finite type with affine diagonal over an algebraically closed k, and let $\mathscr{X}^{\mathrm{ss}}$ be the Θ-semistable points with respect to a class $\ell \in H^2(\mathscr{X}, \mathbb{R})$. Suppose that either*

(a) $\mathscr{X}^{\mathrm{ss}}$ is the open part of a Θ-stratification of $\mathscr{X}$, i.e., $\mathscr{X}^{\mathrm{ss}} = \mathscr{X}_{\leq 0}$, such that for each HN-filtration $g : \Theta_k \to \mathscr{X}$ of an unstable point one has $g^*(\ell) > 0$, or

(b) $\mathscr{X}^{\mathrm{ss}} \subset \mathscr{X}$ is open and $\mathscr{X}$ is Θ-complete.

Then

(i) if $\mathscr{X}$ is S*-complete, so is $\mathscr{X}^{\mathrm{ss}}$;*

(ii) if $\mathscr{X}$ is Θ-complete, so is $\mathscr{X}^{\mathrm{ss}}$.

Proof. We will use a result in Lemma 6.15 in [8]:

- **Lemma A.** Under the hypotheses of the propositon, given a filtration $f : \Theta_k \to \mathscr{X}$ such that $f(1)$ is semistable with repsect to ℓ, then $f^*(\ell) = 0$ if and only if $f(0)$ is semistable as well.

For S-completeness, consider a DVR R and

$$\begin{tikzcd}
\operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R \arrow[r] \arrow[d] & \mathscr{X}^{\mathrm{ss}} \arrow[r, hook] \arrow[d] & \mathscr{X} \\
\phi_R \arrow[r] \arrow[ru, dashed] \arrow[rru, dashed] & \operatorname{Spec} k &
\end{tikzcd}$$

Then we can have a lift $\phi_R \to \mathscr{X}$. As $\mathscr{X}^{\mathrm{ss}}$ is open in both cases, we just need to show the unique closed point maps into $\mathscr{X}^{\mathrm{ss}}$. As $(\pi, s, t) = (0,1,0), (0,1,1)$ maps to $\mathscr{X}^{\mathrm{ss}}$, restricting the map $\phi_R \to \mathscr{X}$ to the locus $\Theta_\kappa \cong \{s = 0\}$ and $\Theta_\kappa \cong \{t = 0\}$ give filtrations f_1 and f_2 in $\mathscr{X}$ with $f_i(1) \in \mathscr{X}^{\mathrm{ss}}$. If one has $f_i^*(\ell) < 0$ then the other has $f_j^*(\ell) > 0$ for $i \neq j$, which would contradict the fact that $f_i(1) \in \mathscr{X}^{\mathrm{ss}}$. Hence $f_i^*(\ell) = 0$ and by **Lemma A** we get $f(0) \in \mathscr{X}^{\mathrm{ss}}$.

For Θ-completeness, by the similar reason of the S-completeness we get $f : \Theta_R \backslash 0 \to \mathscr{X}^{\mathrm{ss}}$ with $f_K^*(\ell) = 0$. Let the extenion is $F : \Theta_R \to \mathscr{X}$. As the function $f \mapsto f^*(\ell) \in \mathbb{R}$, regarded as a function on $\mathrm{Filt}(\mathscr{X})$, is locally constant, then we get the result. ☐

As a summary of this section we have:

Theorem 1.4.13. *Let $\mathscr{X}$ be an algebraic stack locally of finite type with affine diagonal over a algebraically closed field k with smooth stabilizers for any $x \in \mathscr{X}(k)$, and let $\ell \in H^2(\mathscr{X}, \mathbb{R})$ be a class defining a semistable locus $\mathscr{X}^{\mathrm{ss}} \subset \mathscr{X}$ which is part of a well-ordered Θ-stratification of $\mathscr{X}$ compatible with ℓ. Then if $\mathscr{X}$ is either Θ-complete, S-complete, or satisfies the existence part of the valuative criterion for properness, then the same is true for $\mathscr{X}^{\mathrm{ss}}$.*

In particular, if in addition k is of characteristic 0, $\mathscr{X}$ is S-complete and Θ-complete, and $\mathscr{X}^{\mathrm{ss}}$ is quasi-compact, then there exists a separated good moduli space of $\mathscr{X}^{\mathrm{ss}}$ (and proper if $\mathscr{X} \to \operatorname{Spec} k$ satisfies the existence part of the valuative criterion for properness).

Remark 1.4.14. *Note that the Theorem 1.3.17, Theorem 1.4.5 (or Theorem 1.4.7) and Theorem 1.4.13 (or the propositions it represent) form the main results (Theorem A,B,C) of the paper [8].*

Again, the original results in [8] is much general in our case. But we will only use the case here so I omit these.

Chapter 2

Good Moduli Spaces for Objects in Abelian Categories

2.1 Moduli Problem for Objects in Abelian Categories

In this section we study the moduli functor for objects in a k-linear abelian category $\mathcal{A}$. The first paper about this in [10] due to Artin and Zhang, who explained that many of the results known for categories of quasi-coherent sheaves on a scheme can be carried out in an abstract setting.

This general setup is very useful as it include the case of moduli of coherent sheaves and moduli of complexes. This setup also leads to moduli problems in which the conditions of Θ-completeness, S-completeness, and unpunctured inertia can be checked rather easily.

Here we mainly follows the Section 7 in paper [8] and we assume k to be a algebraically closed field, althrough this is true for any commutative ring.

2.1.1 Special Objects in Abelian Categories

First we need to introduce some definitions in the abelian categories.

Definition 2.1.1. *Let $\mathcal{A}$ be a k-linear cocomplete abelian category.*

- *We say $E \in \mathcal{A}$ is finitely presentable (or compact) if the canonical map*

$$\varinjlim_{\alpha\in I} \operatorname{Hom}(E, F_\alpha) \to \operatorname{Hom}(E, \varinjlim_{\alpha\in I} F_\alpha)$$

  is an isomorphism for any small filtered system $\{F_\alpha\}_{\alpha\in I}$ in $\mathcal{A}$. Let $\mathcal{A}^{\mathrm{fp}}$ be the full subcategory consisting of finitely presentable objects.

- *We say $E \in \mathcal{A}$ is* finitely generated *if the same map is an isomorphism for any filtered system of monomorphisms, or equivalently, if $E = \bigcup_\alpha E_\alpha$ for a filtered system of subobjects, then $E = E_\alpha$ for some $\alpha \in I$.*
- *We say $E \in \mathcal{A}$ is* noetherian *if every ascending chain of subobjects of E terminates, or equivalently, if every subobject of E is finitely generated.*

Definition 2.1.2. *Hence we say a k-linear cocomplete abelian category $\mathcal{A}$:*

- *$\mathcal{A}$ is* locally of finite type *if every object in $\mathcal{A}$ is the union of its finitely generated subobjects.*
- *$\mathcal{A}$ is* locally finitely presented *if very object in $\mathcal{A}$ can be written as the filtered colimit of finitely presentable objects, and $\mathcal{A}^{\mathrm{fp}}$ is essentially small.*
- *$\mathcal{A}$ is* locally noetherian *if it has a set of noetherian generators.*

Remark 2.1.3. *If $\mathcal{A}$ is locally noetherian, then finitely generated, finitely presentable, and noetherian objects coincide, and the category $\mathcal{A}^{\mathrm{fp}}$ is closed under kernels and hence abelian. Our main results will assume that $\mathcal{A}$ is locally noetherian.*

2.1.2 Functors in Abelian Categories

Definition 2.1.4 (Tensor Product)**.** *For a k-linear cocomplete abelian category $\mathcal{A}$, there is a canonical k-bilinear* tensor functor

$$(-) \otimes_k (-) : \mathrm{Mod}_k \times \mathcal{A} \to \mathcal{A}$$

defined by the formula

$$\mathrm{Hom}_{\mathcal{A}}(M \otimes_k E, F) = \mathrm{Hom}_{\mathrm{Mod}_k}(M, \mathrm{Hom}_{\mathcal{A}}(E, F))$$

for objects $E, F \in \mathcal{A}$ and a k-module M.

Remark 2.1.5. *Actually if $M = \mathrm{coker}(k^I \to k^J)$, then $M \otimes_k E = \mathrm{coker}(E^I \to E^J)$ by the same matrix.*

This tensor functor commutes with filtered colimits and is right exact in each variable. If M is flat and $\mathcal{A}$ is locally noetherian then $M \otimes_k (-)$ is exact. See [10].

Definition 2.1.6. *We say $E \in \mathcal{A}$ is* flat *if $(-) \otimes_k E : \mathrm{Mod}_k \to \mathcal{A}$ is exact.*

Definition 2.1.7. *For a commutative k-algebra R, let $\mathcal{A}_R$ denote the category of R-module objects in $\mathcal{A}$, i.e., pairs (E, ξ_E) where $E \in \mathcal{A}$ and $\xi_E : R \to \mathrm{End}_{\mathcal{A}}(E)$ is a morphism of k-algebras, and a morphism $(E, \xi_E) \to (E', \xi_{E'})$ in $\mathcal{A}_R$ is a morphism $E \to E'$ in $\mathcal{A}$ compatible with the actions of ξ_E and $\xi_{E'}$.*

For a commutative k-algebra R, $\mathcal{A}_R$ is an R-linear abelian category and $\mathcal{A}_k = \mathcal{A}$. Given a homomorphism of commutative rings $\phi : R_1 \to R_2$, the forgetful functor $\phi_* : \mathcal{A}_{R_2} \to \mathcal{A}_{R_1}$ is faithfully exact, commutes with filtered colimits and faithful, and ϕ_* is fully faithful if ϕ is surjective. Moreover, ϕ_* admits a left adjoint $\phi^* : R_2 \otimes_{R_1} (-) : \mathcal{A}_{R_1} \to \mathcal{A}_{R_2}$.

Note that if $\mathcal{A}$ is locally noetherian and if $R \to S$ is a faithfully flat map of commutative k-algebras then $\mathcal{A}_R$ is equivalent to the category of objects in $\mathcal{A}_S$ equipped with a descent datum. Also we note that **we only consider the locally noetherian case**.

Definition 2.1.8. *Hence if $\mathcal{A}$ is locally noetherian, then we have a stack $\underline{\mathcal{A}}$ in the fppf topology on k-Alg.*

Hence we can define that for any algebraic stack $\mathscr{X}$ over k we can define

$$\mathcal{A}_{\mathscr{X}} := \mathrm{Map}_{\mathrm{Fibered-Cat}/k-\mathrm{alg}}(\mathscr{X}, \underline{\mathcal{A}}).$$

Remark 2.1.9. *If $\mathscr{X}$ is the quotient stack for a groupoid of affine schemes $\mathscr{X} = [X_1 \rightrightarrows X_0]$ with $X_i = \operatorname{Spec} R_i$, then descent implies that the category $\mathcal{A}_{\mathscr{X}}$ is naturally equivalent to the category of objects of $\mathcal{A}_{X_0}$ equipped with a descent datum. We will use this description for the stacks Θ and ϕ_R.*

Faithfully flat descent also allows one to extend the functor $R_2 \otimes_{R_1} (-) : \mathcal{A}_{R_1} \to \mathcal{A}_{R_2}$ above to a functor $f^* : \mathcal{A}_{\mathscr{Y}} \to \mathcal{A}_{\mathscr{X}}$ for any morphism of stacks $f : \mathscr{X} \to \mathscr{Y}$.

Lemma 2.1.10 (Pushforward)**.** *Suppose that $\mathcal{A}$ is locally noetherian. If $f : \mathscr{X} \to \mathscr{Y}$ is a quasi-compact morphism with affine diagonal of algebraic stacks then the functor $f^* : \mathcal{A}_{\mathscr{Y}} \to \mathcal{A}_{\mathscr{X}}$ admits a right adjoint f_* which commutes with filtered colimits and flat base change.*

Proof. Actually this is easy to see if we consider the groupoid in affine schemes. Then by faithfully flat descent we can get the result. See Lemma 7.6 in [8]. □

2.1.3 Moduli Functor of Abelian Categories

Definition 2.1.11. *Let k be an algebraically closed field and let $\mathcal{A}$ be a locally noetherian, cocomplete, and k-linear abelian category. Then we define the category $\mathscr{M}_{\mathcal{A}}$ fibered in groupoids over k-alg by assigning the groupoid*

$$\mathscr{M}_{\mathcal{A}}(R) := \langle \text{objects } E \in \mathcal{A}_R \text{ which are flat and finitely presented} \rangle .$$

Proposition 2.1.12. *The category fibered in groupoids $\mathscr{M}_{\mathcal{A}}$ is a stack in the big fppf topology on k-alg and extends naturally to a stack on the big fppf topology on schemes over k.*

Proof. This is just from some flat descent results and we will omit them, see [10] Theorem C8.6 and [8] Lemma 7.9. □

2.2 Valuative Criteria for the Stack $\mathscr{M}_{\mathcal{A}}$

2.2.1 Description of $\mathscr{M}_{\mathcal{A}}(\Theta_R)$ and $\mathscr{M}_{\mathcal{A}}(\phi_R)$

Here we will consider the Θ-completeness and S-completeness of $\mathscr{M}_{\mathcal{A}}$. First we need to describe $\mathscr{M}_{\mathcal{A}}([\operatorname{Spec} A/\mathbb{G}_m])$.

Definition 2.2.1. *Let k be an algebraically closed field and let $\mathcal{A}$ be a locally noetherian, cocomplete, and k-linear abelian category. Let $\mathcal{A}^{\mathbb{Z}} = \operatorname{Fun}(\mathbb{Z}, \mathcal{A})$ be a category of $\mathbb{Z}$-graded objects.*

Pick a $\mathbb{Z}$-graded k-algebra A, a $\mathbb{Z}$-graded A- module object is an object of $\mathcal{A}^{\mathbb{Z}}$ whose underlying object $E = \bigoplus_{n\in\mathbb{Z}} E_n \in \mathcal{A}$ is equipped with an A-module structure such that multiplication $A \otimes_k E \to E$ maps $A_n \otimes_k E_m$ into E_{n+m}. We denote $\mathcal{A}^{\mathbb{Z}}_A$ be the category of $\mathbb{Z}$-graded A-module objects.

Remark 2.2.2. *By [10] Proposition B7.5, we have the category $\mathcal{A}^{\mathbb{Z}}_A$ is abelian and locally noetherian if $\mathcal{A}_A$ is.*

We first need to describe $\mathcal{A}_{[\operatorname{Spec} A/\mathbb{G}_m]}$.

Now we encode the $\mathbb{Z}$-grading of a graded k-algebra A by a morphism of k- algebras $\sigma_A : A \to A[t^{\pm 1}]$ by $a = \bigoplus_n a_n \mapsto \sum_n a_n t^n$.

Now objects in $\mathcal{A}_{[\operatorname{Spec} A/\mathbb{G}_m]}$ are objects $E \in \mathcal{A}_A$ together with a cocycle, which can be encoded by a coaction morphism

$$\sigma : E \to E[t^{\pm 1}] := A[t^{\pm 1}] \otimes_A E.$$

We can write $\sigma = \sum_n \sigma_n t^n$ for $\sigma_n : E \to E$, morphisms in $\mathcal{A}$.

The cocycle condition on σ amounts to the condition that the following diagrams in $\mathcal{A}$ must commute:

$$\begin{tikzcd}
E \arrow[r, "\sum \sigma_n t^n"] \arrow[d, "\sum \sigma_n t^n"'] & E[t^{\pm 1}] \arrow[d, "t \mapsto tt'"'] \\
E[t^{\pm 1}] \arrow[r, "\sum \sigma_n (t')^n"] & E[t^{\pm 1}, (t')^{\pm 1}]
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\sum \sigma_n t^n"] \arrow[dr, "\mathrm{id}_E"'] & E[t^{\pm 1}] \arrow[d, "t \mapsto 1"] \\
 & E
\end{tikzcd}$$

Proposition 2.2.3. *Let k be an algebraically closed field and let $\mathcal{A}$ be a locally noetherian, cocomplete, and k-linear abelian category. Let A be a $\mathbb{Z}$-graded k-algebra. Then there is a natural equivalence $\mathcal{A}^{\mathbb{Z}}_A \to \mathcal{A}_{[\operatorname{Spec} A/\mathbb{G}_m]}$ that maps $E \in \mathcal{A}^{\mathbb{Z}}_A$ to the object of $\mathcal{A}_{[\operatorname{Spec} A/\mathbb{G}_m]}$ defined by the coaction morphism $\sigma = \sum_n \sigma_n t^n : E \to E[t^{\pm 1}]$, where $\sigma_n : E \to E$ is the k-linear.*

This restricts to an equivalence between $\mathscr{M}_{\mathcal{A}}([\operatorname{Spec} A/\mathbb{G}_m])$ and the groupoid of objects in $\mathcal{A}^{\mathbb{Z}}_A$ whose underlying non-graded A-module object is flat and finitely presented.

Proof. This is very trivial. Actually the cocycle diagram above is $\sum_{m,n} \sigma_m \sigma_n t^m (t')^n = \sum_n \sigma_n (tt')^n$ and $\sum_n \sigma_n = \mathrm{id}$. This implies that σ_n are a collection of mutually orthogonal idempotent endomorphisms of E that induce a direct sum decomposition $E = \bigoplus_n E_n$ in A, where E_n is the image of σ_n. Converse is trivial.

For the claim of $\mathscr{M}_{\mathcal{A}}([\operatorname{Spec} A/\mathbb{G}_m])$ follows from the fact that $\mathscr{M}_{\mathcal{A}}([\operatorname{Spec} A/\mathbb{G}_m])$ is a stack for fppf topology. □

Use this general fact, we can describe it for Θ_R and ϕ_R.

Corollary 2.2.4. *Let k be an algebraically closed field and let $\mathcal{A}$ be a locally noetherian, cocomplete, and k-linear abelian category. Let R be a k-algebra then the category $\mathcal{A}_{\Theta_R}$ is equivalent to the category of sequences of morphisms*

$$E: \quad \cdots \to E_{n+1} \xrightarrow{x} E_n \to \cdots$$

in $\mathcal{A}_R$ such that

- *along* $\operatorname{Spec} R \hookrightarrow \Theta_R$ *is* $\varinjlim_i E_i$, *and*
- *along* $\mathbf{B}\mathbb{G}_{m,R} \hookrightarrow \Theta_R$ *is* $\bigoplus_{n \in \mathbb{Z}} E_n / x E_{n+1}$.

This equivalence restricts to an equivalence between $\mathscr{M}_{\mathcal{A}}(\Theta_R)$ and the groupoid of $\mathbb{Z}$-weighted filtrations $\cdots \subset E_{n+1} \subset E_n \subset \cdots$ of an object E_∞ in $\mathcal{A}_R$ such that $E_n/E_{n+1} \in \mathcal{A}_R$ is flat and finitely presented, $E_n = E_\infty$ for $n \ll 0$ and $E_n = 0$ for $n \gg 0$.

Proof. The description of $\mathcal{A}_{\Theta_R}$ follows directly from Proposition 2.2.3 as E is just a $\mathbb{Z}$-graded $R[x]$-module. Along $\operatorname{Spec} R \hookrightarrow \Theta_R$, this is $E \otimes_{R[x]} R[x^{\pm 1}]$ and follows from the fact

$$R[x^{\pm 1}] = \varinjlim(\cdots \xrightarrow{x} R[x] \xrightarrow{x} R[x] \xrightarrow{x} \cdots).$$

Along $\mathbf{B}\mathbb{G}_{m,R} \hookrightarrow \Theta_R$, this is $E \otimes_{R[x]} R[x]/x = E/xE$. Well done. The flatness and finitly presented one omitted. See Corollary 7.13 in [8]. □

Corollary 2.2.5. *Let k be an algebraically closed field and let $\mathcal{A}$ be a locally noetherian, cocomplete, and k-linear abelian category. Let R be a DVR over k with uniformizing parameter π and residue field κ. The category $\mathcal{A}_{\phi_R}$ is equivalent to the category of diagrams in $\mathcal{A}_R$:*

$$\begin{tikzcd}
E\colon & \cdots \arrow[r, bend left, "s"] & E_{n-1} \arrow[l, bend left, "t"] \arrow[r, bend left, "s"] & E_n \arrow[l, bend left, "t"] \arrow[r, bend left, "s"] & E_{n+1} \arrow[l, bend left, "t"] \arrow[r, bend left, "s"] & \cdots \arrow[l, bend left, "t"] ,
\end{tikzcd}$$

satisfying $st = ts = \pi$. Under this equivalence the restriction of E

- *along* $\operatorname{Spec} R \overset{s\neq 0}{\hookrightarrow} \phi_R$ *is* $\varinjlim_n(\cdots \xrightarrow{s} E_{n-1} \xrightarrow{s} E_n \xrightarrow{s} \cdots)$,
- *along* $\operatorname{Spec} R \overset{t\neq 0}{\hookrightarrow} \phi_R$ *is* $\varinjlim_n(\cdots \xleftarrow{t} E_{n-1} \xleftarrow{t} E_n \xleftarrow{t} \cdots)$,
- *along* $\Theta_\kappa \overset{s=0}{\to} \phi_R$ *is the object corresponding to the sequence*
$$(\cdots \xleftarrow{t} E_n/sE_{n-1} \xleftarrow{t} E_{n+1}/sE_n \xleftarrow{t} \cdots),$$
- *along* $\Theta_\kappa \overset{t=0}{\to} \phi_R$ *is the object corresponding to the sequence*
$$(\cdots \xrightarrow{s} E_{n-1}/tE_n \xrightarrow{s} E_n/tE_{n+1} \xrightarrow{s} \cdots).$$

This equivalence restricts to an equivalence between $\mathscr{M}_{\mathcal{A}}(\phi_R)$ *and the groupoid consisting of objects* E *such that: (a)* s *and* t *are injective, (b)* $s : E_{n-1}/tE_n \to E_n/tE_{n+1}$ *is injective for all* n*, (c) each* E_n *is finitely presentable, (d)* $s : E_{n-1} \to E_n$ *is an isomorphism for* $n \gg 0$*, and (e)* $t : E_n \to E_{n-1}$ *is an isomorphism for* $n \ll 0$*.*

Proof. The description of $\mathcal{A}_{\phi_R}$ follows directly from Proposition 2.2.3. Moreover, we can show that flatness is characterized by conditions (a) and (b). And (c)–(e) to be flat and finitely presentable. These are boring and we refer Corollary 7.14 in [8]. ☐

2.2.2 Θ-Completeness and S-Completeness

Lemma 2.2.6. *Let* $j : U \hookrightarrow X$ *be an open subscheme of a regular noetherian scheme of dimension* 2 *whose complement is* 0*-dimensional. Then* $j_* : \mathcal{A}_U \to \mathcal{A}_X$ *maps flat objects to flat objects, and induces an equivalence between the full subcategory of flat objects over* X *and over* U*, with inverse given by* $j^* : \mathcal{A}_X \to \mathcal{A}_U$*.*

Proof. Just need to show that j_* preserves flat objects, and that both the unit and counit of the adjunction between j_* and j^* are equivalences on flat objects. By descent we may assume that $X = \operatorname{Spec} R$ is affine and U is the complement of a single closed point. Localizing further it suffices to consider the case of $X = \operatorname{Spec} R$ for a regular ring R of dimension 2 and U the complement of the closed point p whose maximal ideal is generated by a regular sequence x, y. In particular $U = \operatorname{Spec} R_x \cup \operatorname{Spec} R_y$.

Then in this case it is trivial that the unit and counit of the adjunction between j_* and j^* are equivalences on flat objects as $j_*E = \ker(E|_{R_x} \oplus E|_{R_y} \to E|_{R_{xy}})$.

Finally we must show that j_* preserves flat objects. We just need to show that $\operatorname{Tor}_1^R(R/\mathfrak{p}, j_*E) = 0$ for any $\mathfrak{p} \in \operatorname{Spec} R$. If $\mathfrak{p} \in U$ this follows from the flatness, so we just need to show $\operatorname{Tor}_1(\kappa, j_*E) = 0$ where $\kappa = R/(x,y)$. We need to show that tensoring j_*EE with the Koszul complex $0 \to R \to R \oplus R \to R \to \kappa \to 0$ gives an exact sequence
$$o \to j_*E \to j_*E \oplus j_*E \to j_*E.$$

This follows from j_* left exact. ☐

Proposition 2.2.7. *Let k be an algebraically closed field and let $\mathcal{A}$ be a locally noetherian, cocomplete, and k-linear abelian category. Then $\mathscr{M}_{\mathcal{A}}$ is S-complete with respect to any DVR R that is essentially of finite type over k.*

Proof. Let $j : \phi_R \backslash 0 \hookrightarrow \phi_R$ and take $E \in \mathscr{M}_{\mathcal{A}}(\phi_R \backslash 0)$, then by Lemma 2.2.6 $j_* E$ is flat. Hence we need to show that it is finitely presentable, i.e., we have to check conditions (c)–(e) of Corollary 2.2.5.

Let $j_s : \operatorname{Spec} R \hookrightarrow \phi_R$ and $j_t : \operatorname{Spec} R \hookrightarrow \phi_R$ and $j_{st} : \operatorname{Spec} K \hookrightarrow \phi_R$. As E is flat, it is defined by an object $F \in \mathcal{A}_K$ and two R-module subobjects $E_1, E_2 \subset F$ such that $K \otimes_R E_i \cong F$.

Now we have ???

☐

Proposition 2.2.8. *Let k be an algebraically closed field and let $\mathcal{A}$ be a locally noetherian, cocomplete, and k-linear abelian category. Then $\mathscr{M}_{\mathcal{A}}$ is Θ-complete with respect to any DVR R that is essentially of finite type over k.*

Proof. Let $j : \mathscr{U} := \Theta_R \backslash 0 \hookrightarrow \Theta_R$ and $E \in \mathscr{M}_{\mathcal{A}}(\mathscr{U})$. Again by Lemma 2.2.6 $j_* E$ is flat. Hence we need to show that it is finitely presentable.

Let the presentation $\mathbb{A}^1_R \to \Theta_R$, where the open subset $U \subset \mathbb{A}^1_R$ corresponding to $\mathscr{U}$ is covered by the two affine subschemes defined by $R[x] \subset K[x]$ and $R[x] \subset R[x^{\pm 1}]$. Now $E \in \mathcal{A}_{\mathscr{U}}$ corresponds to an object $F \in \mathcal{A}_K$, a R-submodule object $E_1 \subset F$ such that $K \otimes_R E_1 \cong F$, and a weighted descending filtration $\cdots F_{n+1} \subset F_n \subset \cdots \subset F$ satisfying the hypotheses of Corollary 2.2.4, then $j_* E$ corresponds to the graded $R[x]$-module object ??? ☐

2.3 Good Moduli Space of Semistable Objects

2.3.1 Some Basic Properties

Lemma 2.3.1. *Let $f : X \to Y$ be a quasi-compact morphism of algebraic spaces locally of finite type over a field. Then f satisfies the valuative criterion for properness for DVRs if and only if it satisfies the lifting criterion for DVRs essentially of finite type over k.*

Proof. We refer Lemma A.11 in [8] for the proof. ☐

Proposition 2.3.2. *Let k be an algebraically closed field and let $\mathcal{A}$ be a locally noetherian, cocomplete, and k-linear abelian category. Then the stack $\mathscr{M}_{\mathcal{A}}$ satisfies the valuative criterion for universal closedness with respect to DVRs which are essentially of finite type over k.*

Proof. If R is a DVR, as in commutative algebra, an object $E \in \mathcal{A}_R$ is flat if and only if it is torsion free follows from [10] Lemma C1.12 as the condition is equivalent to the vanishing of Tor_1. Let $j : \mathrm{Spec}(K) \to \mathrm{Spec}(R)$, then for any $E \in \mathcal{A}_K$, we can write $j_*E = \bigcup_\alpha F_\alpha$ as a directed union of finitely generated (hence finitely presentable) subobjects which must be torsion free. If E is finitely generated then $E = \bigcup_\alpha F_\alpha \otimes_R K$ must stabilize, so there is some flat and finitely presentable object F_α extending E. $\square$

Lemma 2.3.3. *Let k be an algebraically closed field and let $\mathcal{A}$ be a locally noetherian, cocomplete, and k-linear abelian category. If $\mathscr{M}_\mathcal{A}$ is an algebraic stack with affine stabilizers, κ is a field over k, and $E \in \mathscr{M}_\mathcal{A}(\kappa)$ represents a closed point, then E is a semisimple object in $\mathcal{A}_\kappa$.*

Proof. As E is finitely presented, it can not be expressed as an infinite sum of non-zero objects. Therefore, we only have to show that every finite filtration of E splits. Now by Corollary 2.2.4 any finite filtration of E corresponds to a map $\Theta_\kappa \to \mathscr{M}_\mathcal{A}$ mapping $1 \mapsto E$. Since E is a closed point, the resulting map must factor through a map $\Theta_\kappa \to \mathbf{B}_\kappa \mathrm{Aut}_{\mathscr{M}_\mathcal{A}}(E)$. We know from the classification of torsors ([25] Proposition A.0.1) on Θ_κ that any such map factors through the projection $\Theta_\kappa \to \mathbf{B}_\kappa \mathbb{G}_m$, and thus the corresponding filtration of E is split. $\square$

Proposition 2.3.4. *Let k be an algebraically closed field and let $\mathcal{A}$ be a locally noetherian, cocomplete, and k-linear abelian category. If $\mathscr{M}_\mathcal{A}$ is an algebraic stack locally of finite presentation over k, then $\mathscr{M}_\mathcal{A}$ has affine diagonal.*

Proof. If R is a valuation ring over k with fraction field K and $E, F \in \mathscr{M}_\mathcal{A}(R)$, then $F \to F \otimes_R K$ is injective and hence so is the restriction map

$$\mathrm{Hom}_R(E,F) \to \mathrm{Hom}_R(E, K \otimes_R F) \cong \mathrm{Hom}_K(K \otimes_R E, K \otimes_R F).$$

Hence by the valuative criterion we get the diagonal of $\mathscr{M}_\mathcal{A}$ is separated.

Next we claim that for any ring R over k and $E, F \in \mathscr{M}_\mathcal{A}(R)$, the functor $R'/R \mapsto \mathrm{Hom}_{R'}(R' \otimes_R E, R' \otimes_R F)$ is a separated algebraic space $\underline{\mathrm{Hom}}_R(E,F)$ locally of finite presentation over R. Indeed, observe that the subfunctor $P \subset \underline{\mathrm{Aut}}_R(E \oplus F)$ classifying automorphisms of the form $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ is representable by a closed subspace, because it is the preimage of the closed identity section under the map of separated R-spaces

$$\underline{\mathrm{Aut}}_R(E \oplus F) \to \underline{\mathrm{Aut}}_R(E \oplus F), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}.$$

Next observe that we have a group homomorphism $P \to \underline{\mathrm{Aut}}_R(E) \times \underline{\mathrm{Aut}}_R(F)$ over R given by

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & D \end{pmatrix}.$$

Hence the preimage of closed identity section is the subgroup classifying automorphisms of the form $\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$, which is $\underline{\mathrm{Hom}}_R(E,F)$. Hence we get the claim.

Now let $X := \underline{\mathrm{Hom}}_R(E,F)$ be that separated algebraic space locally finitely presented over R. Then the natural action of $\mathbb{G}_m$ on X with unique extension to $\mathbb{A}^1$. Hence $X = X^+ \overset{\mathrm{ev}_0}{\to} X^{\mathbb{G}_m}$ is affine (even is an affine fibration) by Theorem 6.6.7 in [4]. Moreover we can see that $X^{\mathbb{G}_m} \cong \operatorname{Spec} R \hookrightarrow X$ as a zero section. Hence X is affine.

Finally, the algebraic R-space $\underline{\mathrm{Isom}}_R(E,F)$ is the closed subspace of $\underline{\mathrm{Hom}}_R(E,F) \times \underline{\mathrm{Hom}}_R(F,E)$ which is also affine as well. Hence the $\mathscr{M}_{\mathcal{A}}$ has affine diagonal. □

Theorem 2.3.5. *Let k be an algebraically closed field of characteristic 0 and let $\mathcal{A}$ be a locally noetherian, cocomplete, and k-linear abelian category. Assume that $\mathscr{M}_{\mathcal{A}}$ is an algebraic stack locally of finite type over k. Then any quasi-compact closed substack $\mathscr{X} \subset \mathscr{M}_{\mathcal{A}}$ admits a proper good moduli space, and in this case points of $\mathscr{X}$ must parameterize objects of $\mathcal{A}$ of finite length.*

Proof. Except the final assertion, this Theorem follows directly from Proposition 1.3.5(vi), Theorem 1.3.17, Proposition 2.2.7, Proposition 2.2.8, Proposition 2.3.2 and a easy result in Proposition 3.48 in [8] for the properness.

For the final fact, by Lemma 2.3.3 the closed points of $\mathscr{X}$ are represented by semisimple objects in $\mathcal{A}_\kappa$ for fields κ of finite type over k. □

2.3.2 Stability Condition and Good Moduli Spaces

Definition 2.3.6. *We denote by $\pi_0(\mathscr{M}_{\mathcal{A}})$ the set of connected components of the stack $\mathscr{M}_{\mathcal{A}}$. For any $v \in \pi_0(\mathscr{M}_{\mathcal{A}})$, we let $\mathscr{M}_{\mathcal{A}}^v \subset \mathscr{M}_{\mathcal{A}}$ be the corresponding open and closed substack.*

Now let a locally constant function on $|\mathscr{M}_{\mathcal{A}}|$ as

$$p_v : |\mathscr{M}_{\mathcal{A}}| \to \pi_0(\mathscr{M}_{\mathcal{A}}) \to V$$

where V is a totally ordered abelian group, such that $p_v(E) = 0$ for any $E \in \mathscr{M}_{\mathcal{A}}^v$, and p_v is additive in the sense that $p_v(E \oplus F) = p_v(E) + p_v(F)$.

Definition 2.3.7. *We will say that a point of $\mathscr{M}_{\mathcal{A}}^v$ represented by $E \in \mathcal{A}_\kappa$ for some algebraically closed field κ over k, is p_v-semistable if for any subobject $F \subset E$, $p_v(F) \leq 0$ and p_v-unstable otherwise.*

Remark 2.3.8. *Here are two points we will consider.*

- *This definition is unaffected by embedding V in a larger totally ordered group, so we may assume that V is a totally ordered vector space over $\mathbb{R}$ by the Hahn embedding theorem.*

- *As* $\underline{\mathrm{Map}}(\Theta, \mathscr{M}_{\mathcal{A}}^v) \times_{\mathrm{ev}_1, \mathscr{M}_{\mathcal{A}}^v, [E]} \operatorname{Spec}\kappa$ *is an algebraic space locally of finite type over* κ*, if there is a destabilizing subobject of* E *after base change to an arbitrary field extension* κ'/κ*, then there is a destabilizing subobject for* E *over* κ*, so this definition does not depend on the choice of representative.*

Definition 2.3.9. *Using Corollary 2.2.4 to identify maps* $f : \Theta_\kappa \to \mathscr{M}_{\mathcal{A}}$ *with* $\mathbb{Z}$*-weighted descending filtrations* $\cdots \subset E_{w+1} \subset E_w \subset \cdots$ *in* $\mathcal{A}_\kappa$*, we define a locally constant function* $\ell : |\underline{\mathrm{Map}}_k(\Theta, \mathscr{M}_{\mathcal{A}}^v)| \to V$ *as*

$$\ell(\cdots \subset E_{w+1} \subset E_w \subset \cdots) := \sum_w w p_v(E_w/E_{w+1}).$$

Lemma 2.3.10. *A point* $x \in |\mathscr{M}_{\mathcal{A}}^v|$ *is* p_v*-unstable if and only if there is some* $f \in |\underline{\mathrm{Map}}_k(\Theta, \mathscr{M}_{\mathcal{A}}^v)|$ *such that* $f(1) = x$ *and* $\ell(f) > 0$*.*

Proof. If $F \subset E$ is a destabilizing subobject, then we consider the filtration $F : \cdots \subset E_2 = 0 \subset E_1 = F \subset E_0 = E = \cdots$. This filtration has $\ell(F) = p_v(F) > 0$.

Conversely, given a filtration such that $\ell(E_i) := \sum_w w p_v(E_w/E_{w+1}) > 0$ and $p_v(E) = \sum_w p_v(E_w/E_{w+1}) = 0$ it follows that for some index i we have

$$p_v(E_i) = \sum_{w \geq i} p_v(E_w/E_{w+1}) > 0$$

so one of the filtration steps will be destabilizing. □

Remark 2.3.11. *We know that in Definition 1.4.10 we have another stability condition. The stability condition here is some kind of generalization as we see in Lemma 2.3.10. Actually Proposition 1.4.12 is hold in our case as the proof of* ***Lemma A*** *in it applies verbatim. Remark 6.16 in [8] gives the general condition over this and we omitted.*

Theorem 2.3.12. *Let* k *be an algebraically closed field of characteristic* 0 *and let* $\mathcal{A}$ *be a locally noetherian, cocomplete, and* k*-linear abelian category. Assume that* $\mathscr{M}_{\mathcal{A}}$ *is an algebraic stack locally of finite type over* k*. Let* $v \in \pi_0(\mathscr{M}_{\mathcal{A}})$ *be a connected component, and let* $p_v : \pi_0(\mathscr{M}_{\mathcal{A}}) \to V$ *be an additive function defining a notion of* p_v*-semistability on* $\mathscr{M}_{\mathcal{A}}^v$*, as above.*

If the substack of p_v*-semistable points* $\mathscr{M}_{\mathcal{A}}^{v,\mathrm{ss}} \subset \mathscr{M}_{\mathcal{A}}^v$ *is open and quasi-compact, then* $\mathscr{M}_{\mathcal{A}}^{v,\mathrm{ss}}$ *admits a separated good moduli space. If in addition* $\mathscr{M}_{\mathcal{A}}^{v,\mathrm{ss}}$ *is the open piece of a* Θ*-stratification of* $\mathscr{M}_{\mathcal{A}}^v$*, then* $\mathscr{M}_{\mathcal{A}}^{v,\mathrm{ss}}$ *admits a proper good moduli space.*

Proof. We have seen that $\mathscr{M}_{\mathcal{A}}^v$ has affine diagonal, and with respect to essentially finite type DVRs $\mathscr{M}_{\mathcal{A}}^v$ is Θ-reductive, S-complete and satisfies the existence part of the valuative criterion for properness.

By Remark 2.3.11 and Proposition 1.3.5(vi) we know that $\mathscr{M}_{\mathcal{A}}^{v,\mathrm{ss}}$ is Θ-reductive and S-complete. As $\mathscr{M}_{\mathcal{A}}^{v,\mathrm{ss}}$ is quasi-compact, we find that by Theorem 1.3.17 there is a separated good moduli space $\mathscr{M}_{\mathcal{A}}^{v,\mathrm{ss}} \to M$. Then by Proposition 1.4.9 and Proposition 2.3.2 applied to the Θ-stratification of $\mathscr{M}_{\mathcal{A}}^{v}$ imply that $\mathscr{M}_{\mathcal{A}}^{v,\mathrm{ss}}$ satisfies the existence part of the valuative criterion for properness with respect to essentially finite type DVRs and hence M is proper over $\operatorname{Spec} k$ by Lemma 2.3.1. □

Chapter 3

Good Moduli Space of Semistable Sheaves

3.1 Moduli Stack of Coherent Sheaves

3.1.1 Construction of the Moduli Stack of Coherent Sheaves

Now we consider the moduli space of coherent sheaves over some smooth projective variety X over $\mathbb{C}$. Then we have the Chern character map

$$\gamma : K(X) \xrightarrow{\mathrm{ch}} \mathrm{CH}^*(X)_{\mathbb{Q}} \xrightarrow{\mathrm{cl}} H^{2*}(X, \mathbb{Q}).$$

(or we can use ℓ-adic cohomology) Let Γ be the image of this map.

By Grothendieck-Riemann-Roch theorem (see Chapter 15 in [22]),

$$P(\mathscr{F}, m) = \chi(\mathscr{F}(m)) = \int_X \mathrm{ch}(\mathscr{F}(m))\mathrm{td}(\mathcal{T}_X),$$

then we find that the information of $v \in \Gamma$ is equivalent to the information of the Hilbert polynomial χ. So we can use both of them when X is smooth. If X is just a projective scheme, then we will only to use the Hilbert polynomial.

Theorem 3.1.1. *Let X be a connected projective k-scheme for some field k, we let $\underline{\mathrm{Coh}}_P(X)$ the category fibred in groupoid over* $\mathrm{Sch}/\mathbb{C}$ *sending a k-scheme T to the groupoid of T-flat families $\mathscr{E} \in \mathrm{Coh}(X \times T)$ such that any restriction $\mathscr{E}_t \in \mathrm{Coh}(X)$ has the Hilbert polynomial P, the morphisms in the above groupoid are given by isomorphisms of $\mathscr{E}$.*

Then $\underline{\mathrm{Coh}}_P(X)$ is an algebraic stack locally of finite type over k of affine diagonal. Also, we have the algebraic stack $\underline{\mathrm{Coh}}(X) = \coprod_P \underline{\mathrm{Coh}}_P(X)$.

Proof. Easy to see that $\underline{\mathrm{Coh}}_P(X)$ is actually a stack, we first claim that it is an algebraic stack in a natural way.

For each integer N, we claim there is an open substack $\mathscr{U}_N \subset \underline{\mathrm{Coh}}_P(X)$ parameterizing coherent sheaves $\mathscr{E}$ such that $\mathscr{E}(N)$ generated by global sections and $H^i(X, \mathscr{E}(N)) = 0$ for any $i > 0$. Actually this is trivial by some application of cohomology and base change. As $\underline{\mathrm{Coh}}_P(X) = \bigcup_N \mathscr{U}_N$, we just need to show $\mathscr{U}_N$ is an algebraic stack locally of finite type over k.

For each N, we consider the quotient scheme

$$Q_N := \underline{\mathrm{Quot}}_X^P(\mathscr{O}_X(-N)^{P(N)}).$$

Again by some application of cohomology and base change, we find that there is an open subscheme $Q'_N \subset Q_N$ parameterizing quotients $q : \mathscr{O}_X(-N)^{P(N)} \twoheadrightarrow \mathscr{F}$ such that $H^0(q(N))$ is surjective and $H^i(X, \mathscr{F}(n)) = 0$ for all $i > 0$.

We have a natural map $Q'_N \to \mathscr{U}_N$ maps $[\mathscr{O}_X(-N)^{P(N)}]$ to $\mathscr{F}$. We observe that Q'_N is also $\mathrm{GL}_{P(N)}$-invariant, then this map descends to

$$\Psi^{\mathrm{pre}} : [Q'_N/\mathrm{GL}_{P(N)}]^{\mathrm{pre}} \to \mathscr{U}_N$$

which is fully faithful since every automorphism of a coherent sheaf $\mathscr{E}$ on $X \times S$ induces an automorphism of $p_{2,*}\mathscr{E}(N) = \mathscr{O}_S^{P(N)}$ i.e. an element of $\mathrm{GL}_{P(N)}(S)$, and this element acts on $\mathscr{O}_X(-N)^{P(N)}$ preserving the quotient $\mathscr{E}$.

After stackification, we have another fully faithful map $\Psi : [Q'_N/\mathrm{GL}_{P(N)}] \to \mathscr{U}_N$ which is also essentially surjective by the constructions. Hence we have

$$\mathscr{U}_N \cong [Q'_N/\mathrm{GL}_{P(N)}], \quad \underline{\mathrm{Coh}}_P(X) = \bigcup_N [Q'_N/\mathrm{GL}_{P(N)}].$$

Hence $\underline{\mathrm{Coh}}_P(X)$ is an algebraic stack locally of finite type over k. □

3.1.2 Basic Facts of the Moduli Stack of Coherent Sheaves

Proposition 3.1.2. *Let X be a projective scheme over an algebraically closed field k. For a noetherian k-algebra R, $\mathrm{MOR}_k(\Theta_R, \underline{\mathrm{Coh}}(X))$ is equivalent to the groupoid of pairs $(\mathscr{E}, \mathscr{E}_*)$ where $\mathscr{E}$ is a coherent sheaf on X_R flat over R and*

$$\mathscr{E}_* : 0 \subset \cdots \subset \mathscr{E}_{i-1} \subset \mathscr{E}_i \subset \cdots \subset \mathscr{E}$$

is a filtration such that $\mathscr{E}i = 0$ for $i \ll 0$, $\mathscr{E}_i = E$ for $i \gg 0$, and each factor $\mathscr{E}_i/\mathscr{E}_{i-1}$ is flat over R. A morphism is an isomorphism $\mathscr{E} \to \mathscr{E}'$ of coherent sheaves compatible with the filtration.

Under this correspondence, the morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ sends 1 to E and 0 to the associated graded $\mathrm{gr}\, \mathscr{E}_* = \bigoplus_i \mathscr{E}_i/\mathscr{E}_{i-1}$.

Proof. A morphism $\Theta_R \to \underline{\mathrm{Coh}}(X)$ correspond to a coherent sheaf $\mathscr{F}$ on $X \times \Theta_R$ flat over Θ_R. By smooth descent, this corresponds to a coherent sheaf on $X \times \mathbb{A}^1_R$ flat over $\mathbb{A}^1_R$ together with a $\mathbb{G}_m$-action. Pushing forward $\mathscr{F}$ along the affine morphism $X \times \Theta_R \to X \times \mathbf{B}\mathbb{G}_{m,R}$, we see that $\mathscr{F}$ also corresponds to a graded $\mathscr{O}_{X_R}[x]$-module flat over $R[x]$. Then $\mathscr{F} = \bigoplus_i \mathscr{E}_i$ with each $\mathscr{E}_i$ a coherent sheaf on X_R, then multiplication by x induces maps $x : \mathscr{E}_i \to \mathscr{E}_{i+1}$ which are necessarily injective as $\mathscr{F}$ is flat over $R[x]$, hence torsion free. Since $\mathscr{F}$ is finitely generated as a graded $R[x]$-module, there exists finitely many homogeneous generators with bounded degree. Thus $\mathscr{E}_i = \mathscr{E}$ for $i \gg 0$. On the other hand, considering the $\mathscr{O}_{X_R}[x]$-module $\mathscr{E}_{\geq d} := \bigoplus_{i \geq d} \mathscr{E}_i \subset \mathscr{F}$, the ascending chain

$$\cdots \subset \mathscr{E}_{\geq d} \subset \mathscr{E}_{\geq d-1} \subset \cdots \subset \mathscr{F}$$

must terminate as $\mathscr{F}$ is noetherian. It follows that $\mathscr{E}_i = 0$ for $i \ll 0$. Since $\mathscr{F}$ is flat as an $R[x]$-module, the quotient $\mathscr{F}/x\mathscr{F} = \bigoplus_i \mathscr{E}_i/\mathscr{E}_{i-1}$ is flat as an R-module and thus each factor $\mathscr{E}_i/\mathscr{E}_{i-1}$ is flat over R. The converse is similar and we omit it. □

Theorem 3.1.3. *For every projective scheme X over an algebraically closed field k, the algebraic stack $\underline{\mathrm{Coh}}(X)$ (and hence $\underline{\mathrm{Coh}}_P(X)$) is Θ-complete and S-complete.*

Remark 3.1.4. *We remark that a map $\phi_R \to \underline{\mathrm{Coh}}(X)$ is the same data as two opposite filtration $\mathscr{E}_*$ and $\mathscr{F}^*$ (that is, $\mathscr{E}_i/\mathscr{E}_{i-1} \cong \mathscr{F}^i/\mathscr{F}^{i+1}$) such that $\mathscr{E}_i = 0$ and $\mathscr{F}_i = \mathscr{F}$ for $i \ll 0$, and $\mathscr{E}_i = \mathscr{E}$ and $\mathscr{F}_i = 0$ for $i \gg 0$. In this case, under this map $(1,0) \mapsto \mathscr{E}$, $(0,1) \mapsto \mathscr{F}$ and $(0,0) \mapsto \operatorname{gr}\mathscr{E}_*$.*

Proof. Here we just give an idea. For the entire proof we refer Proposition 6.8.23 in [4].

For Θ-completeness, by Proposition 3.1.2 we know that a map $\Theta_R\backslash 0 \to \underline{\mathrm{Coh}}(X)$ corresponds to a coherent sheaf $\mathscr{E}$ on X_R flat over R and a $\mathbb{Z}$-graded filtration $F_* : \cdots F_{i-1} \subset F_i \subset \cdots \subset \mathscr{E}_K$ such that $F_i = \mathscr{E}_K$ for $i \gg 0$ and $F_i = 0$ for $i \ll 0$, and F_i/F_{i-1} is flat over R. Viewing $\mathscr{E}$ is a subsheaf of $\mathscr{E}_K$, we define $\mathscr{E}_i := F_i \cap \mathscr{E}$. Then $\mathscr{E}_i/\mathscr{E}_{i-1}$ is torsion-free, hence flat over R. This defines $\Theta_R \to \underline{\mathrm{Coh}}(X)$.

For S-completeness, given a map $\phi_R\backslash 0 \to \underline{\mathrm{Coh}}(X)$ corresponding to coherent sheaves $\mathscr{E}$ and $\mathscr{F}$ flat over R and an isomorphism $\alpha : \mathscr{E}_K \cong \mathscr{F}_K$. Let $j : \phi_R\backslash 0 \subset \phi_R$, $j_s, j_t : \operatorname{Spec} R \to \phi_R$ (with $s \neq 0$ and $t \neq 0$), and $j_{st} : \operatorname{Spec} K \to \phi_R$ (with $st \neq 0$). We compute the pushforward as the equalizer

$$0 \to (\mathrm{id} \times j)_*\mathscr{M} \to (\mathrm{id} \times j_s)_*\mathscr{E} \oplus (\mathrm{id} \times j_t)_*\mathscr{F} \to (\mathrm{id} \times j_{st})_*\mathscr{F}_K$$

where the last map is $(a, b) \mapsto a - \alpha(b)$. We can compute the last two sheaves and show that $j_*\mathscr{M}$ is coherent and flat over ϕ_R like Proposition 2.2.7. Hence we get the result. □

Theorem 3.1.5. *For every projective scheme X over an algebraically closed field k, let $\mathscr{U} \subset \underline{\mathrm{Coh}}(X)$ be an open substack.*

(i) *The substack $\mathscr{U}$ is Θ-complete if and only if for every DVR R (with fraction field K and residue field κ), coherent sheaf $\mathscr{E}$ on X_R flat over R, and $\mathbb{Z}$-graded filtration $\mathscr{E}_*$ with $\mathscr{E}_i = 0$ for $i \ll 0$, $\mathscr{E}_i = \mathscr{E}$ for $i \gg 0$ and with each $\mathscr{E}_i/\mathscr{E}_{i-1}$ flat over R, then if $\mathscr{E}$ and $\mathrm{gr}(\mathscr{E}_*|_K)$ are in $\mathscr{U}$, so is $\mathrm{gr}(\mathscr{E}_*|_\kappa)$.*

(ii) *If for every pair of opposite filtrations $\mathscr{E}_*$ and $\mathscr{F}^*$ of $\mathscr{E}, \mathscr{F} \in \mathscr{U}(k)$, we have the associated graded $\mathrm{gr}\,\mathscr{E}_* \in \mathscr{U}(k)$, then the substack $\mathscr{U}$ is S-complete.*

Proof. These are easy. As by Theorem 3.1.3, $\underline{\mathrm{Coh}}(X)$ is Θ-complete and S-complete, the valuative criteria for $\mathscr{U}$ are equivalent to the existence of lifts for all commutative diagrams:

$$\begin{tikzcd}
\Theta_R\backslash 0 \arrow[r] \arrow[d, hook] & \mathscr{U} \arrow[d, hook] \\
\Theta_R \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Coh}}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\phi_R\backslash 0 \arrow[r] \arrow[d, hook] & \mathscr{U} \arrow[d, hook] \\
\phi_R \arrow[r] \arrow[ur, dashed] & \underline{\mathrm{Coh}}(X)
\end{tikzcd}$$

Hence we need to show that the images of 0 under the unique fillings $\Theta_R \to \underline{\mathrm{Coh}}(X)$ and $\phi_R \to \underline{\mathrm{Coh}}(X)$ are contained in $\mathscr{U}$. Hence theese two results are follows from this and the description as above. ☐

3.2 Basic Theory of Semistable Sheaves

Our aim is to find a moduli space of sheaves which is of finite type! Actually $\underline{\mathrm{Coh}}_P(X)$ is never of finite type and one can show that even on the smooth projective curves, $\underline{\mathrm{Coh}}_P(X)$ has no good moduli space. Consider $\{\mathscr{O}(n) \oplus \mathscr{O}(-n)\}$ on $\mathbb{P}^1$, then this can not parametrized by a scheme of finite type. Hence we need some more conditions.

3.2.1 Basic Properties

Fix X be a projective scheme over a field k with $H = \mathscr{O}(1)$. Now if $\mathscr{F}$ be a coherent sheaf of dimension $d = \dim X$ with Hilbert polynomial $P(\mathscr{F}, m) = \sum_{i=0}^{d} \alpha_i(\mathscr{F}) \frac{m^i}{i!}$, then we can define $\mathrm{rank}(\mathscr{F}) := \frac{\alpha_d(\mathscr{F})}{\alpha_d(\mathscr{O}_X)}$. If X is integral, this is the usual definition.

For polynomials $f_i \in \mathbb{Q}[m]$ for $i = 1, 2$, we define $f_1 < (\leq) f_2$ if $f_1(m) < (\leq) f_2(m)$ for $m \gg 0$.

Definition 3.2.1. *Fix (X, H) as above and $\mathscr{F}$ be a coherent sheaf of dimension d.*

(i) *We define the* slope *$\mu_H(\mathscr{F}) := \frac{c_1(\mathscr{F}) \cdot H^{d-1}}{\mathrm{rank}(\mathscr{F})}$;*

(ii) *we call $\mathscr{F}$ is μ_H-*(semi)stable* if for any $0 \subset \mathscr{E} \subset \mathscr{F}$ with $0 < \mathrm{rank}\,\mathscr{E} < \mathrm{rank}\,\mathscr{F}$ we have $T_{d-2}(\mathscr{F}) = T_{d-1}(\mathscr{F})$ and $\mu_H(\mathscr{E}) < (\leq) \mu_H(\mathscr{F})$;*

(iii) *we consider the Hilbert polynomial* $P(\mathscr{F},m) = \sum_{i=0}^{d}\alpha_i(\mathscr{F})\frac{m^i}{i!}$, *then we have* $\alpha_d(\mathscr{F}) = \operatorname{rank}(\mathscr{F})\cdot H^d$ *and* $\alpha_{d-1}(\mathscr{F}) = \frac{1}{2}\operatorname{rank}(\mathscr{F})\deg T_X + \deg\mathscr{F}$. *We define the* **reduced Hilbert polynomial** *is*

$$p(\mathscr{F},m) = \frac{P(\mathscr{F},m)}{\alpha_d(\mathscr{F})} = \frac{m^d}{d!} + \frac{1}{H^d}\left(\frac{1}{2}\deg\mathscr{F} + \mu_H(\mathscr{F})\right)\frac{m^{d-1}}{(d-1)!} + \textit{lower terms.}$$

(iv) *Define* $\mathscr{F}$ *is* H-**(semi)stable** *if it is pure and for any* $0 \subsetneq \mathscr{E} \subsetneq \mathscr{F}$, *we have* $p(\mathscr{E},m) < (\leq) p(\mathscr{F},m)$.

(v) *Define* $\mathscr{F}$ *is* **geometrically** H-**stable** *if for any base field extension* $X_K = X \times_k \operatorname{Spec}(K)$ *the pull-back* $\mathscr{F}_K$ *is stable.*

Remark 3.2.2. *Here we have some remarks.*

- *As the Harder-Narasimhan filtration is unique (Theorem 3.2.9) and stable under field extension (Proposition 3.2.10), we don't need the* **geometrically** H-**ss**.
- *We can define* $\mathscr{F}$ *is* μ_H-*(semi)stable if for any* $0 \subsetneq \mathscr{E} \subsetneq \mathscr{F}$ *with* $0 < \operatorname{rank}\mathscr{E} < \operatorname{rank}\mathscr{F}$, *we have* $\operatorname{rank}(\mathscr{F})\deg(\mathscr{E}) < (\leq)\operatorname{rank}(\mathscr{E})\deg(\mathscr{F})$. *This is obviously the same definition except that it does not require explicitly that* $T_{d-2}(\mathscr{F}) = T_{d-1}(\mathscr{F})$. *But this can be easy to be deduced.*
- *Similarly, we can define* $\mathscr{F}$ *is* H-*(semi)stable if for any* $0 \subsetneq \mathscr{E} \subsetneq \mathscr{F}$, *we have* $\alpha_d(\mathscr{F})P(\mathscr{E},m) < (\leq)\alpha_d(\mathscr{E})p(\mathscr{F},m)$. *This is obviously the same definition except that it does not require explicitly that* $\mathscr{F}$ *is pure. But applying the inequality to* $\mathscr{E} = T_{d-1}(\mathscr{F})$ *(maximal subsheaf of dimension* $\leq d-1$*), this implies* $T_{d-1}(\mathscr{F}) = 0$, *i.e. it is pure.*
- *If* $\mathscr{F}$ *is pure of dimension* d, *then we also can use saturated subsheaves, proper quotient sheaves with* $\alpha_d > 0$ *and even proper purely* d*-dimensional quotient sheaves to define the* H-*(semi)stable!*

  The proof is trivial by using the trivial exact sequence. See Proposition 1.2.6 in [29] for the proof.

Remark 3.2.3.

- *Easy to see that when it is pure, then*

$$\mu_H\text{-stable} \Rightarrow H\text{-stable} \Rightarrow H\text{-ss} \Rightarrow \mu_H\text{-ss};$$

- *if* $\dim X = 1$, *then* μ_H-*(semi)stable iff* H-*(semi)stable.*

Lemma 3.2.4. *Let* $\mathscr{F},\mathscr{G}$ *are* H-*ss of dimension* d. *Then*

(i) *if* $p(\mathscr{F}) > p(\mathscr{G})$, *then* $\operatorname{Hom}(\mathscr{F},\mathscr{G}) = 0$;

(ii) *let* $p(\mathscr{F}) = p(\mathscr{G})$*. If* $\mathscr{F}$ *is moreover* H*-stable, then any* $\phi : \mathscr{F} \to \mathscr{G}$ *either zero or injection. Similarly if* $\mathscr{G}$ *is moreover* H*-stable, then any* $\phi : \mathscr{F} \to \mathscr{G}$ *either zero or surjection.*

(iii) *If* $p(\mathscr{F}) = p(\mathscr{G})$ *and* $\alpha_d(\mathscr{F}) = \alpha_d(\mathscr{G})$*, then any non-trivial homomorphism* $f : \mathscr{F} \to \mathscr{G}$ *is an isomorphism provided* $\mathscr{F}$ *or* $\mathscr{G}$ *is* H*-stable.*

Proof. For (i), let nontrivial f with image $\mathscr{E}$, then $p(\mathscr{F}) \leq p(\mathscr{E}) \leq p(\mathscr{G})$ which is impossible. Hence $\mathrm{Hom}(\mathscr{F}, \mathscr{G}) = 0$.

For (ii), this is the similar reason in the proof of (i).

For (iii), this is the similar reason in the proof of (i). □

Corollary 3.2.5. *If* $\mathscr{E}$ *is a* H*-stable sheaf, then* $\mathrm{End}(\mathscr{E})$ *is a finite dimensional division algebra over* k*. In particular, if* k *is algebraically closed, then* $k \cong \mathrm{End}(\mathscr{E})$*, i.e.* $\mathscr{E}$ *is a simple sheaf.*

Example 3.2.1. *(i) Any line bundles over smooth projective curves are* H*-stable. See Example 1.2.10 in [29].*

(ii) For an algebraically closed field k *of zero characteristic, the bundle* $\Omega_{\mathbb{P}^n}$ *is* H*-stable. See Section 1.4 in [29].*

3.2.2 The Harder-Narasimhan Filtration

We consider a classical result due to Grothendieck as a motivation of the Harder-Narasimhan filtration.

Theorem 3.2.6 (Grothendieck)**.** *Let* $\mathscr{E}$ *be a vector bundle of rank* r *on* $\mathbb{P}^1$*, then there is a uniquely determined decreasing sequence of integers* $a_1 \geq \cdots \geq a_r$ *such that* $E \cong \mathscr{O}(a_1) \oplus \cdots \oplus \mathscr{O}(a_r)$*.*

Proof. For $r = 1$ this is trivial. Let the theorem holds for all vector bundles of rank $< r$ and that $\mathscr{E}$ is a vector bundle of rank r.

Take any saturation of any rank 1 subsheaf of $\mathscr{E}$. As $\mathbb{P}^1$ is a smooth curve, then it is a line bundle of form $\mathscr{O}(a)$. Let a_1 be the maximal number with this property. Hence $\mathscr{E}/\mathscr{O}(a_1) \cong \bigoplus_{i=2}^r \mathscr{O}(a_i)$ with $a_2 \geq \cdots \geq a_r$. We claim that $a_1 \geq a_2$. Indeed, consider

$$0 \to \mathscr{O}(-1) \to \mathscr{E}(-1 - a_1) \to \bigoplus_{i=2}^{r} \mathscr{O}(a_i - a_1 - 1) \to 0.$$

Since $\Gamma(\mathscr{E}(-1 - a_1)) = \mathrm{Hom}(\mathscr{O}(1 + a_1), \mathscr{E})$ and a_1 be the maximal number with non-trivial $\mathrm{Hom}(\mathscr{O}(a), \mathscr{E})$, then $\Gamma(\mathscr{E}(-1 - a_1)) = 0$. By the long exact sequence we get $H^0(\mathscr{O}(a_i - 1 - a_1)) = 0$ for all i. Hence $a_i < a_1 + 1$. Hence we get the claim.

Next we claim the sequence $0 \to \mathscr{O}(a_1) \to \mathscr{E} \to \bigoplus_{i=2}^r \mathscr{O}(a_i) \to 0$ split. This follows from the Serre duality

$$\operatorname{Ext}^1\left(\bigoplus_{i=2}^r \mathscr{O}(a_i), \mathscr{O}(a_1)\right)^\vee \cong \bigoplus_{i=2}^r \operatorname{Hom}(\mathscr{O}(a_1), \mathscr{O}(a_i - 2)) = 0.$$

Finally, the uniqueness is not hard to prove. We omit it. □

Again we let X be a projective scheme over some field k with a fixed ample line bundle H.

Definition 3.2.7. *Fix $\mathscr{E} \in \operatorname{Coh}(X)$ is pure of dimension d. A Harder-Narasimhan filtration (or HN-filtration) of $\mathscr{E}$ is*

$$0 = \mathrm{HN}_0(\mathscr{E}) \subset \mathrm{HN}_1(\mathscr{E}) \subset \cdots \subset \mathrm{HN}_l(\mathscr{E}) = \mathscr{E}$$

such that $\mathrm{gr}_i^{\mathrm{HN}}(\mathscr{E}) := \mathrm{HN}_i(\mathscr{E})/\mathrm{HN}_{i-1}(\mathscr{E})$ which are H-ss of dimension d and $p(\mathrm{gr}_i^{\mathrm{HN}}(\mathscr{E})) > p(\mathrm{gr}_{i+1}^{\mathrm{HN}}(\mathscr{E}))$ for all i. We define $p_{\max}(\mathscr{E}) := p(\mathrm{gr}_1^{\mathrm{HN}}(\mathscr{E}))$ and $p_{\min}(\mathscr{E}) := p(\mathrm{gr}_l^{\mathrm{HN}}(\mathscr{E}))$;

Lemma 3.2.8. *If $\mathscr{F}, \mathscr{G}$ is pure of dimension d with $p_{\min}(\mathscr{F}) > p_{\max}(\mathscr{G})$, then $\operatorname{Hom}(\mathscr{F}, \mathscr{G}) = 0$.*

Proof. If $f : \mathscr{F} \to \mathscr{G}$ is non-trivial. Let $i > 0$ be the minimal with $f(\mathrm{HN}_i(\mathscr{F})) \neq 0$ and $j > 0$ the minimal with $f(\mathrm{HN}_i(\mathscr{F})) \subset \mathrm{HN}_j(\mathscr{G})$. Hence we get a non-trivial $\bar{f} : \mathrm{gr}_i^{\mathrm{HN}}(\mathscr{F}) \to \mathrm{gr}_j^{\mathrm{HN}}(\mathscr{G})$. But this is impossible by $p_{\min}(\mathscr{F}) > p_{\max}(\mathscr{G})$ and Lemma 3.2.4(i). □

Theorem 3.2.9. *Let $\mathscr{E}$ be a pure coherent sheaf of dimension d. Then there always exists a unique Harder-Narasimhan filtration.*

Proof. Here we will use a result (see Lemma 1.3.5 in [29]):

- Let $\mathscr{E}$ be a purely d-dimensional sheaf. Then there is a subsheaf $\mathscr{F} \subset \mathscr{E}$ such that for all subsheaves $\mathscr{G} \subset \mathscr{E}$ one has $p(\mathscr{F}) \geq p(\mathscr{G})$, and in case of equality $\mathscr{F} \supset \mathscr{G}$. Moreover, $\mathscr{F}$ is uniquely determined and semistable. It is called the maximal destabilizing subsheaf of $\mathscr{E}$.

Let $\mathscr{E}_1$ be its maximal destabilizing subsheaf. By induction we may assume $\mathscr{E}/\mathscr{E}_1$ has a Harder-Narasimhan filtration

$$0 \subset \mathscr{G}_0 \subset \mathscr{G}_1 \subset \cdots \subset \mathscr{G}_{l-1} = \mathscr{E}/\mathscr{E}_1.$$

Let $\mathscr{E}_{i+1} \subset \mathscr{E}$ be the preimage of $\mathscr{G}_i$. Just need to show that $p(\mathscr{E}_1) > p(\mathscr{E}_2/\mathscr{E}_1)$. If this were false, we would have $p(\mathscr{E}_2) \geq p(\mathscr{E}_1)$ contradicting the maximality of $\mathscr{E}_1$.

For the uniqueness, consider two Harder-Narasimhan filtrations $\mathscr{E}_*, \mathscr{E}'_*$. Let $p(\mathscr{E}'_1) \geq p(\mathscr{E}_1)$. Let j be minimal with $\mathscr{E}'_1 \subset \mathscr{E}_j$. Then we have

$$p(\mathscr{E}_j/\mathscr{E}_{j-1}) \geq p(\mathscr{E}'_1) \geq p(\mathscr{E}_1) \geq p(\mathscr{E}_j/\mathscr{E}_{j-1}).$$

Hence $p(\mathscr{E}'_1) = p(\mathscr{E}_1)$ and $j = 1$ and $\mathscr{E}'_1 \subset \mathscr{E}_1$. Similarly we get $\mathscr{E}'_1 \supset \mathscr{E}_1$, hence $\mathscr{E}'_1 = \mathscr{E}_1$. Using induction again we get the result. □

Proposition 3.2.10. *Let $\mathscr{E}$ be a pure sheaf of dimension d and let K/k be a field extension. Then*

$$\mathrm{HN}_*(E \otimes_k K) = \mathrm{HN}_*(E) \otimes_k K.$$

In particular, the H-ss sheaves stable under base field extension.

Proof. We do not care about this. We refer the proof of Theorem 1.3.7 in [29]. □

3.2.3 The Jordan-Hölder Filtration

As we all know, the Harder-Narasimhan filtration shows that the H-ss sheaves form the building blocks for all the coherent sheaves. But the Jordan-Hölder filtration shows that the H-stable sheaves form the building blocks for all H-ss sheaves.

Again we let X be a projective scheme over some field k with a fixed ample line bundle H.

Definition 3.2.11. *Fix $\mathscr{E} \in \mathrm{Coh}(X)$. Let $\mathscr{E}$ is H-ss, a Jordan-Hölder filtration (or JH-filtration) of $\mathscr{E}$ is*

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_l = \mathscr{E}$$

such that $\mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E}) := \mathscr{E}_i/\mathscr{E}_{i-1}$ are H-stable and $p(\mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E})) = p(\mathscr{E})$ for all i. We define $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}) := \bigoplus_{i=1}^{l} \mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E})$.

Remark 3.2.12. *Unlike the Harder-Narasimhan filtration, the Jordan-Hölder filtration is NOT unique. For example we let the direct sum of two line bundles of the same degree one.*

Theorem 3.2.13. *Jordan-Hölder filtrations always exist. Up to isomorphism, the sheaf $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}) = \bigoplus_{i=1}^{l} \mathrm{gr}_i^{\mathrm{JH}}(\mathscr{E})$ does not depend on the choice of the Jordan-Hölder filtration.*

Proof. Any filtration of $\mathscr{E}$ by semistable sheaves with reduced Hilbert polynomial $p(\mathscr{E})$ has a maximal refinement, whose factors are necessarily stable. The uniqueness of $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E})$ is not hard to show. We refer 1.5.2 in [29]. □

Definition 3.2.14. *Two H-ss sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ with the same reduced Hilbert polynomial are called S-equivalent if $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}_1) \cong \mathrm{gr}^{\mathrm{JH}}(\mathscr{E}_2)$.*

Definition 3.2.15. *If $\mathscr{E}$ is H-ss, we call $\mathscr{E}$ is H-polystable if it is the direct sum of stable sheaves. In this case $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E}) = \mathscr{E}$.*

Remark 3.2.16. *We will show that the good moduli space of moduli stack of H-ss sheaves actually parametrizes only S-equivalence classes of H-ss sheaves! As we saw above, every S-equivalence class of H-ss sheaves contains exactly one polystable sheaf up to isomorphism. Thus, the good moduli space of H-ss sheaves in fact parametrizes polystable sheaves. See Theorem 3.4.3.*

Actually the S stands for Seshadri as S-completeness is a geometric property reminiscent of how the S-equivalence relation on sheaves implies separatedness of the moduli space.

Remark 3.2.17. *(i) By the similar arguments of Jordan-Hölder filtrations, on can show that every semistable sheaf $\mathscr{E}$ contains a unique non-trivial maximal H-polystable subsheaf of the same reduced Hilbert polynomial. This sheaf is called the socle of $\mathscr{E}$.*

(ii) One can use some basic properties of socles to find that if $\mathscr{E}$ is a simple sheaf, then it is H-stable if and only if it is geometrically H-stable. Hence in particular if k is algebraically closed and $\mathscr{E}$ is a H-stable sheaf, then $\mathscr{E}$ is also gemetrically H-stable. See 1.5.10 and 1.5.11 in [29].

(iii) For μ_H-ss, they define $\mathrm{Coh}_{d,d'}(X) = \mathrm{Coh}_d(X)/\mathrm{Coh}_{d'-1}(X)$ and consider the μ-ss on it using $\hat{\mu}(\mathscr{E}) = \frac{\alpha_{d-1}(\mathscr{E})}{\alpha_d(\mathscr{E})}$. And when $d' = d-1$, this is just the definition before. In this space there also have the Harder-Narasimhan filtrations and Jordan-Hölder filtrations. For the general arguments we refer Section 1.6 in [29].

(iv) For μ, there are several properties for torsion-free sheaves $\mathscr{F}, \mathscr{G}$ on the normal variety ([29] Page 29):

- $\mu(\mathscr{E}(a)) = \mu(\mathscr{E}) + a \deg X$, *similar for* $\mu_{\min}, \mu_{\max}$;
- $\mu_{\min}(\mathscr{E} \oplus \mathscr{F}) = \min(\mu_{\min}(\mathscr{E}), \mu_{\min}(\mathscr{F}))$, *similar for* $\mu_{\max}$;
- $\mu_{\min}(\mathscr{F}) \geq \mu_{\min}(\mathscr{E})$ *for* $\mathscr{E} \twoheadrightarrow \mathscr{F}$;
- $\mu_{\max}(\mathscr{E}) \leq \mu_{\max}(\mathscr{F})$ *for* $\mathscr{E} \hookrightarrow \mathscr{F}$.

3.3 Moduli Stack of Semistable Sheaves

3.3.1 The Mumford-Castelnuovo Regularity and Boundedness

In this section we will give some useful criterion about boundedness of families of sheaves.

Let X be a projective scheme over k with very ample $H = \mathscr{O}_X(1)$.

Definition 3.3.1. *Let m be an integer. A coherent sheaf $\mathscr{F}$ is said to be m-***regular** *, if for all $i > 0$ we have $H^i(X, \mathscr{F}(m-i)) = 0$.*

The **Mumford-Castelnuovo regularity** *of a coherent sheaf $\mathscr{F}$ is the number*

$$\operatorname{reg}(\mathscr{F}) = \inf\{m \in \mathbb{Z} : \mathscr{F} \text{ is } m\text{-regular}\}.$$

Lemma 3.3.2. *There are universal polynomials $P_i \in \mathbb{Q}[T_0, ..., T_i]$ such that the following holds: Let $\mathscr{F}$ be a coherent sheaf of dimension $\leq d$ and let $H_1, ..., H_d$ be an $\mathscr{F}$-regular sequence of hyperplane sections. If $\chi(\mathscr{F}|_{\bigcap_{j \leq i} H_j}) = a_i$ and $h^0(\mathscr{F}|_{\bigcap_{j \leq i} H_j}) \leq b_i$, then*

$$\operatorname{reg}(\mathscr{F}) \leq P_d(a_0 - b_0, ..., a_d - b_d).$$

Proof. See [30] for the original proof. □

Lemma 3.3.3. *The following properties of a flat family of sheaves $\mathscr{F}$ on $X \to S$ are equivalent:*

- *(i) The family is bounded.*
- *(ii) There is a uniform bound $\operatorname{reg}(\mathscr{F}_s) \leq \rho$ for all $s \in S$.*

Proof. See [23] for the original proof. □

Then we have two nice criterion about boundedness of sheaves.

Theorem 3.3.4 (Kleiman Criterion)**.** *Let flat family of sheaves $\mathscr{F}$ on $X \to S$ with the same Hilbert polynomial P. Then this family is bounded if and only if there are constants $C_i, i = 0, ..., d = \deg P$ such that for every $\mathscr{F}_s$ there exists an $\mathscr{F}_s$-regular sequence of hyperplane sections $H_1, ..., H_d$, such that*

$$h^0(\mathscr{F}_s|_{\bigcap_{j \leq i} H_j}) \leq C_i.$$

Proof. Follows from Lemma 3.3.2 and Lemma 3.3.3. □

Theorem 3.3.5 (Grothendieck)**.** *Let P be a polynomial and ρ an integer. Then there is a constant C depending only on P and ρ such that the following holds:*

- *If X be a projective scheme on k with very ample divisor H and if $\mathscr{E} \in \operatorname{Coh}(X)$ is a d-dimensional sheaf with Hilbert polynomial P and Mumford-Castelnuovo regularity $\operatorname{reg}(\mathscr{E}) \leq \rho$ and if $\mathscr{F} \in \operatorname{Coh}(X)$ is a purely d-dimensional quotient sheaf of $\mathscr{E}$ then $\hat{\mu}(\mathscr{F}) \geq C$.*

Moreover, the family of purely d-dimensional quotients $\mathscr{F}$ with $\hat{\mu}(\mathscr{F})$ bounded from above is bounded. In particular the set of Hilbert polynomials of pure quotients with fixed $\hat{\mu}(\mathscr{F})$ is finite.

Proof. After embedding them into the projective space $\mathbb{P}^d$, we may consider $X = \mathbb{P}^d$. Hence we have $\mathscr{G} := V \otimes \mathscr{O}(-\rho) \twoheadrightarrow \mathscr{E}$ where $\operatorname{rank} V = P(\rho)$, so we just need to consider $\mathscr{G}$. Pick a quotient $q : \mathscr{G} \to \mathscr{F}$ of rank s, then

$$\bigwedge^s q : \bigwedge^s V \otimes \mathscr{O}(-s\rho) \to \det \mathscr{F} = \mathscr{O}(\deg \mathscr{F})$$

gives $\deg \mathscr{F} \geq -s\rho$. Hence

$$\hat{\mu}(\mathscr{F}) = \frac{\deg \mathscr{F} + \operatorname{rank} \mathscr{F} \alpha_{d-1}(\mathscr{O}_X)}{\alpha_d(\mathscr{F})} \geq -\rho + \alpha_{d-1}(\mathscr{O}_X).$$

For the final part, we let $\hat{\mu} \leq C'$. It is enough to show that the family of pure quotient sheaves $\mathscr{F}$ of rank $0 < s \leq \operatorname{rank}(\mathscr{G}) = P(\rho)$ and with $l = \deg \mathscr{F} = s(C' - \alpha_{d-1}(\mathscr{O}_X))$ is bounded. Consider $\psi : \mathscr{G} \otimes \bigwedge^{s-1} \mathscr{G} \overset{\wedge}{\to} \bigwedge^s \mathscr{G} \overset{\det q}{\to} \mathscr{O}(l)$ and $\psi^\vee : G \to \mathscr{O}(l) \otimes \bigwedge^{s-1} \mathscr{G}^\vee$. Let U denote the dense open subscheme where $\mathscr{F}$ is locally free. Then $\ker(\psi^\vee)|_U = \ker(q)|_U$. Since the quotients of $\mathscr{G}$ corresponding to these two subsheaves of $\mathscr{G}$ are torsion free and since they coincide on a dense open subscheme of $\mathbb{P}^d$, we must have $\ker(\psi^\vee) = \ker(q)$ everywhere, i.e. $\mathscr{F} \cong \operatorname{Im} \psi^\vee$. Now, the family of such image sheaves certainly is bounded. □

3.3.2 Basic Construction and Openness of Semistable Sheaves

Definition 3.3.6. *We define the stack* $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$ *send a scheme* T *to a families of* H*-ss sheaves on* $X \times T \to T$*. Similarly we define* $\underline{\mathrm{Coh}}_P^{\mathrm{H-s}}(X)$ *send a scheme* T *to a families of geometrically* H*-stable sheaves on* $X \times T \to T$*.*

Proposition 3.3.7. *The following properties of coherent sheaves are open in flat families: being simple, of pure dimension,* H*-ss, or geometrically* H*-stable.*

Proof. Let $f : X \to S$ be a projective morphism of Noetherian schemes (as the property is local) and let $\mathscr{O}_X(1)$ be an f-very ample invertible sheaf on X. Let $\mathscr{F}$ be a flat family of d-dimensional sheaves with Hilbert polynomial P on the fibres of f. For each $s \in S$, a sheaf $\mathscr{F}_s$ is simple iff $\hom_{\kappa(s)}(\mathscr{F}_s, \mathscr{F}_s) = 1$. Thus openness here is an immediate consequence of the semicontinuity properties for relative Ext-sheaves.

Next we consider pure dimension (P1), H-ss (P2), and geometrically H-stable (P3) which can be characteristics by the Hilbert polynomials of quotient sheaves. Consider the following several sets:

$$\begin{aligned} A = \big\{ P'' : \deg(P'') = d, \hat{\mu}(P'') \leq \hat{\mu}(P) & \text{ and there is a geometric point } s \in S \\ \text{and a surjection } \mathscr{F}_s \to \mathscr{F}'' & \text{ onto a pure sheaf with } P(\mathscr{F}'') = P'' \big\}; \end{aligned}$$

$$A_1 = \{P'' \in A : \deg(P - P'') \leq d - 1\}; \quad A_2 = \{P'' \in A : p'' < p\};$$

$$A_3 = \{P'' \in A : p'' \leq p \text{ and } P'' < P\}.$$

By Theorem 3.3.5 we get the set A is finite. For each polynomial $P'' \in A$ we consider $\pi : Q(P'') = \underline{\mathrm{Quot}}_{X/S}(\mathscr{F}, P'') \to S$ be the projective morphism. Hence $\pi(Q(P''))$ is closed. As $\mathscr{F}_s$ has (Pi) if and only if $s \notin \bigcup_{P'' \in A_i} \pi(Q(P'')) \subset S$. Well done. □

Corollary 3.3.8. *We have open substacks*

$$\underline{\mathrm{Coh}}_P^{\mathrm{H-s}}(X) \subset \underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \subset \underline{\mathrm{Coh}}_P(X)$$

which parameterizing H-ss sheaves and geometrically H-stable sheaves, are all algebraic stacks locally of finite type.

Proof. Follows from the Theorem 3.3.7. □

3.3.3 Boundedness I: The Grauert-Mülich Theorem

In this sections we will assume the base field k is an algebraically closed field of characteristic zero!

In 2004, Langer in [34] and [33] proved the positive and mixed characteristic of the boundedness of semistable sheaves and also gives a generalized Le Potier-Simpson type bound for the number of global sections. See also [24] for a modern proof.

Since I don't care about the fields either not algebraically closed or not of characteristic zero, so we just introduce the characteristic zero case which is more easier.

We may use the Theorem 3.3.4 to show the boundedness. Hence we need to investigate the behavior of sheaves resticted to the intersections of hyperplanes. Actually the Grauert-Mülich theorem and the Le Potier-Simpson estimate are what we want.

Before we discuss the notations and main results, we will introduce a family-version of the Harder-Narasimhan filtration:

Theorem 3.3.9 (The Relative Harder-Narasimhan Filtration)**.** *Let S be an integral k-scheme of finite type, let $f : X \to S$ be a projective morphism and let H be an f-ample invertible sheaf on X. Let $\mathscr{F}$ be a flat family of d-dimensional coherent sheaves on the fibres of f. There is a projective birational morphism $g : T \to S$ of integral k-schemes and a filtration*

$$0 = \mathrm{HN}_0(\mathscr{F}) \subset \mathrm{HN}_1(\mathscr{F}) \subset \cdots \mathrm{HN}_l(\mathscr{F}) = \mathscr{F}_T$$

such that

- *(a)* $\mathrm{HN}_i(\mathscr{F})/\mathrm{HN}_{i-1}(\mathscr{F})$ *are T-flat;*
- *(b) there is a dense open subscheme $U \subset T$ such that* $\mathrm{HN}_*(\mathscr{F})_t = g_X^* \mathrm{HN}_*(\mathscr{F}_{g(t)})$ *for all $t \in U$.*

Moreover, $(g, \mathrm{HN}_*(\mathscr{F}))$ is universal in the sense that if $g' : T' \to S$ is any dominant morphism of integral schemes and if $\mathcal{F}'_*$ is a filtration of $\mathscr{F}_{T'}$ satisfying these two properties, then there is an S-morphism $h : T' \to T$ with $\mathcal{F}'_* = h_X^* \mathrm{HN}_*(\mathscr{F})$.

Proof. See [44] for the original proof. Also in Theorem 2.3.2 in [29]. ☐

Now let X be a normal projective variety over k of $\dim n \geq 2$ with very ample $H = \mathscr{O}_X(1)$. Let $V_a := H^0(X, \mathscr{O}_X(a))$ and $\Pi_a := \mathbb{P}(V_a^\vee) = |\mathscr{O}_X(a)|$. Let

$$\begin{tikzcd} Z_a = \{(D,x) \in \Pi_a \times X : x \in D\} \arrow[r, "q"] \arrow[d, "p"'] & X \\ \Pi_a & \end{tikzcd}$$

The scheme-structure of Z_a is easy: consider $\mathscr{K}$ be the kernel of $V_a \otimes \mathscr{O}_X \to \mathscr{O}_X(a)$, then $Z_a = \mathbb{P}(\mathscr{K}^\vee)$.

Let $(a_1, ..., a_l)$ be a fixed finite sequence of positive integers, $0 < l < n$. Let $\Pi = \prod_i \Pi_{a_i}$ with $p_i : \Pi \to \Pi_{a_i}$ and $Z = Z_{a_1} \times_X \cdots \times_X Z_{a_l}$ and

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & X \\ \Pi & \end{tikzcd}$$

with $q_i : Z \to Z_{a_i}$.

Lemma 3.3.10. *Let $\mathscr{E}$ be a torsion free coherent sheaf on X and $\mathscr{F} := q^*\mathscr{E}$.*

(i) There is a nonempty open subset $S' \subset \Pi$ such that the morphism $p_{S'} : Z_{S'} \to S'$ is flat and such that for all $s \in S'$ the fibre Z_s is a normal irreducible complete intersection of codimension l in X;

(ii) There is a nonempty open subset $S \subset S'$ such that the family $\mathscr{F}_S = q^*\mathscr{E}|_{Z_s}$ is flat over S and such that for all $s \in S$ the fibre $\mathscr{F}_s \cong \mathscr{E}|_{Z_s}$ is torsion free.

Proof. Lemma 3.3.1 in [29]. Just an easy Bertini-type lemma. ☐

By the relative Harder-Narasimhan filtration 3.3.9, we have

$$0 = \mathscr{F}_0 \subset \cdots \mathscr{F}_j = \mathscr{F}_S$$

such that $\mathscr{F}_i/\mathscr{F}_{i-1}$ are S-flat and there is a dense open subscheme $S_0 \subset S$ such that for all $s \in S_0$ the fibres $(\mathscr{F}_*)_s$ form the Harder-Narasimhan filtration of $\mathscr{F}_s = \mathscr{E}|_{Z_s}$.

WLOG we let $S_0 = S$. Now S connected, we let $\mu_i = \mu((\mathscr{F}_i/\mathscr{F}_{i-1})_s)$ with $\mu_i > \mu_{i+1}$. Define the number

$$\delta\mu = \max\{\mu_i - \mu_{i+1} : i = 1, ..., j-1\}.$$

Remark 3.3.11. *Then $\delta\mu = \delta\mu(\mathscr{E}|_{Z_s})$ for a general point $s \in \Pi$, and $\delta\mu$ vanishes if and only if $\mathscr{E}|_{Z_s}$ is μ_H-ss for general s.*

Theorem 3.3.12 (Generalized Grauert-Mülich Theorem)**.** *Let $\mathscr{E}$ be a μ_H-ss torsion free sheaf. Then there is a nonempty open subset $S \subset \Pi$ such that for all $s \in S$ the following inequality holds:*

$$0 \leq \delta\mu(\mathscr{E}|_{Z_s}) \leq \max\{a_i\} \deg X \cdot \prod_i a_i.$$

Proof. WLOG we let $\delta\mu > 0$. Let i such that $\delta\mu = \mu_i - \mu_{i+1}$. Let $\mathscr{F}' = \mathscr{F}_i, \mathscr{F}'' = \mathscr{F}/\mathscr{F}'$ and for all $s \in S$ the sheaves $\mathscr{F}'_s, \mathscr{F}''_s$ are torsion-free. And $\mu_{\min}(\mathscr{F}'_s) = \mu_i$ and $\mu_{\max}(\mathscr{F}''_s) = \mu_{i+1}$. Pick Z_0 be a maximal open set of Z_S such that $\mathscr{F}|_{Z_0}$ and $\mathscr{F}''|_{Z_0}$ are locally free of rank r, r''. Then $\mathscr{F}|_{Z_0} \to \mathscr{F}''|_{Z_0}$ defines $\phi : Z_0 \to \underline{\mathrm{Grass}}_X(\mathscr{E}, r'')$.

Consider $d\phi : \mathcal{T}_{Z/X}|_{Z_0} \to \phi^*\mathcal{T}_{\underline{\mathrm{Grass}}_X(\mathscr{E},r'')/X}$. As

$$\phi^*\mathcal{T}_{\underline{\mathrm{Grass}}_X(\mathscr{E},r'')/X} = \mathscr{H}om(\mathscr{F}', \mathscr{F}'')|_{Z_0},$$

we get $d\phi$ correspond to $\Phi : (\mathscr{F}' \otimes \mathcal{T}_{Z/X})|_{Z_0} \to \mathscr{F}''|_{Z_0}$.

We claim that Φ_s were not zero for a general point $s \in S$. If it is, making S smaller if necessary, this supposition would imply that Φ is zero. As $q : Z \to X$ is a bundle, we have $X_0 := q(Z_0)$ is open and $\mathrm{codim}(X \backslash X_0, X) \geq 2$ and $\mathscr{E}|_{X_0}$ is locally free. Hence we have

$$\begin{tikzcd}
 & & \underline{\mathrm{Grass}}(\mathscr{E}|_{X_0}, r'') \arrow[d, shift right] \\
Z_0 \arrow[urr, "\phi"] \arrow[rr, "q_0"'] \arrow[d, hook] & & X_0 \arrow[u, dashed, shift right, "\rho"'] \arrow[d, hook] \\
Z_S \arrow[r, hook] \arrow[d, "p_S"'] \arrow[dr, phantom, "\ulcorner", very near start] & Z \arrow[r, "q"] \arrow[d, "p"] & X \\
S \arrow[r, hook] & \Pi &
\end{tikzcd}$$

Now q_0 is smooth of connected fibers and ϕ is constant on the fibres of q_0 and hence factors through a morphism ρ (here we need char$k = 0$). But such ρ corresponds to a locally free quotient $\mathscr{E}|_{X_0} \to \mathscr{E}''$ of rank r'' with the property that $\mathscr{E}''|_{Z_s \cap X_0}$ is isomorphic to $\mathscr{F}''|_{Z_s \cap Z_0}$ for general s. Since by assumption $\mathscr{F}''_s$ is a destabilizing quotient of $\mathscr{F}_s$, any extension of $\mathscr{E}''$ as a quotient of $\mathscr{E}$ is destabilizing. This contradicts the assumption that $\mathscr{E}$ is μ_H-ss.

Hence Φ_s is nonzero for general $s \in S$, that is, Φ_s is a non-trivial element in $\mathrm{Hom}_{\mathcal{C}}(\mathscr{F}'_s \otimes \mathcal{T}_{Z/X}|_{Z_s}, \mathscr{F}''_s)$ where $\mathcal{C} := \mathrm{Coh}_{n-l,n-l-1}(Z_s)$. By the similar result of Lemma 3.2.8, we have

$$\mu_{\min}(\mathscr{F}'_s \otimes \mathcal{T}_{Z/X}|_{Z_s}) \leq \mu_{\max}(\mathscr{F}''_s).$$

The Koszul complex associated to the evaluation map $e : V_a \otimes \mathscr{O}_X \to \mathscr{O}_X(a)$ provides us a surjection $\bigwedge^2 V_a \otimes \mathscr{O}_X(-a) \to \ker e \cong \mathscr{K}$ and hence a surjection

$$\bigoplus_i \bigwedge^2 V_{a_i} \otimes_k q^*\mathscr{O}_X(-a_i) \otimes p^*\mathscr{O}(1) \to \bigoplus_i q^*\mathscr{K}_{a_i} \otimes p^*\mathscr{O}(1) \to \mathcal{T}_{Z/X}.$$

Hence a surjection

$$\left(\bigoplus_i \bigwedge^2 V_{a_i} \otimes_k q^*\mathscr{O}_X(-a_i) \right)\Bigg|_{Z_s} \to \mathcal{T}_{Z/X}|_{Z_s}.$$

Hence we get

$$\begin{aligned} \mu_{\min}(\mathcal{T}_{Z/X}|_{Z_s} \otimes \mathscr{F}'_s) &\geq \mu_{\min}\left(\bigoplus_i \bigwedge^2 V_{a_i} \otimes_k q^*\mathscr{O}_X(-a_i) \otimes \mathscr{F}'|_{Z_s} \right) \\ &= \min_i\{\mu_{\min}(\mathscr{O}_{Z_s}(-a_i) \otimes \mathscr{F}'_s)\} \\ &= \mu_{\min}(\mathscr{F}'_s) - \max\{a_i\} \cdot \deg Z_s. \end{aligned}$$

Hence combining these two inequality, we have

$$\begin{aligned} \delta\mu &= \mu_{\min}(\mathscr{F}'_s) - \mu_{\max}(\mathscr{F}''_s) \\ &\leq \max\{a_i\} \cdot \deg Z_s = \max\{a_i\} \deg X \cdot \prod_i a_i. \end{aligned}$$

Hence we get the result. □

Theorem 3.3.13. *Let X be a normal projective variety over an algebraically closed field of characteristic zero. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are μ_H-ss sheaves, then $\mathscr{F}_1 \otimes \mathscr{F}_2$ is μ_H-ss too.*

Proof. Omitted, see Section 3.2 in [29]. □

Remark 3.3.14. *As a corollary of this theorem, we have $\mu_{\min}(\mathscr{F}_1 \otimes \mathscr{F}_2) = \mu_{\min}(\mathscr{F}_1) + \mu_{\min}(\mathscr{F}_2)$ by tensoring their HN-filtrations, similar for $\mu_{\max}$ and μ.*

Corollary 3.3.15. *Let X be a normal projective variety of dimension n and let $H = \mathscr{O}_X(1)$ be a very ample line bundle. Let $\mathscr{F}$ be a μ_H-ss coherent $\mathscr{O}_X$-module of rank r. Let Y be the intersection of $s < n$ general hyperplanes in the linear system $|\mathscr{O}_X(1)|$. Then*

$$\mu_{\min}(\mathscr{F}|_Y) \geq \mu(\mathscr{F}) - \frac{r-1}{2} \deg(X)$$

and

$$\mu_{\max}(\mathscr{F}|_Y) \leq \mu(\mathscr{F}) + \frac{r-1}{2} \deg(X).$$

Proof. WLOG we let $\mathscr{F}$ is a torsion free sheaf. Pick $\mu_1, ..., \mu_j$ and $r_1, ..., r_j$ be the slopes and ranks of μ_H-HN filtration of $\mathscr{F}|_Y$. By Theorem 3.3.12 we have $0 \leq \mu_i - \mu_{i+1} \leq \deg X$. Hence $\mu_i \geq \mu_1 - (i-1)\deg X$. Hence we have

$$\begin{aligned}\mu(\mathscr{F}) = \sum_{i=1}^{j} \frac{r_i \mu_i}{r} &\geq \mu_1 - \sum_{i=1}^{j} (i-1)\frac{r_i}{r} \deg X \\ &\geq \mu_1 - \frac{\deg X}{r} \sum_{i=1}^{r} (i-1) = \mu_{\max}(\mathscr{F}|_Y) - \deg X \frac{r-1}{2}.\end{aligned}$$

Similar for $\mu_{\min}(\mathscr{F}|_Y)$. ☐

3.3.4 Boundedness II: The Le Potier-Simpson Estimate

In this sections we will assume the base field k is an algebraically closed field of characteristic zero!

Lemma 3.3.16. *Suppose that X is a normal projective variety of dimension d and that $\mathscr{F}$ is a torsion free sheaf of* $\operatorname{rank}(\mathscr{F})$. *Then for any $\mathscr{F}$-regular sequence of hyperplane sections $H_1, ..., H_d$ and $X_v = H_1 \cap \cdots \cap H_{d-v}$ the following estimate holds for all $v = 1, ..., d$:*

$$\frac{h^0(X_v, \mathscr{F}|_{X_v})}{\deg X \cdot \operatorname{rank} \mathscr{F}} \leq \frac{1}{v!} \left[\frac{\mu_{\max}(\mathscr{F}|_{X_1})}{\deg X} + v \right]_+^v$$

where for any $x \in \mathbb{R}$ we define $[x]_+ := \max\{0, x\}$.

Proof. Let $\mathscr{F}_v := \mathscr{F}|_{X_v}$. Using induction on v.

Let $v = 1$. Since we have $h^0(X_1, \mathscr{F}_1) \leq \sum_i h^0(X_1, \operatorname{gr}_i^{\mathrm{HN}}(\mathscr{F}_1))$ and the right hand side of the estimate in the lemma is monotonously increasing with μ, we may assume WLOG that $\mu(\mathscr{F}_1) = \mu_{\max}(\mathscr{F}_1)$, i.e. that $\mathscr{F}_1$ is μ_H-semistable. Hence

$$h^0(x_1, \mathscr{F}_1) \leq h^0(x_1, \mathscr{F}_1(-l)) + \operatorname{rank}(\mathscr{F}) \cdot l \deg X.$$

By Lemma 3.2.4(i), we find that $h^0(x_1, \mathscr{F}_1(-l)) = \hom(\mathscr{O}_{X_1}(l), \mathscr{F}_1) = 0$ if $l > \mu(\mathscr{F}_1)/\deg X$. Now pick $l = \lfloor \mu(\mathscr{F}_1)/\deg X \rfloor + 1$ and well done.

Now let this is right for $v-1$. Consider

$$0 \to \mathscr{F}_v(-k-1) \to \mathscr{F}_v(-k) \to \mathscr{F}_{v-1}(-k) \to 0, \quad k = 0, 1, \cdots.$$

Hence inductively derives estimates

$$\begin{aligned}h^0(X_v, \mathscr{F}_v) &\leq h^0(X_v, \mathscr{F}_v(-l)) + \sum_{i=0}^{l-1} h^0(X_{v-1}, \mathscr{F}_{v-1}(-i)) \\ &\leq \sum_{i=0}^{\infty} h^0(X_{v-1}, \mathscr{F}_{v-1}(-i)).\end{aligned}$$

By induction hypothesis one has

$$\frac{h^0(X_v,\mathscr{F}_v)}{\operatorname{rank}(\mathscr{F})\deg X} \leq \frac{1}{(v-1)!}\int_{-1}^{C}\left[\frac{\mu_{\max}(\mathscr{F}_1)}{\deg X}+(v-1)-t\right]_+^{v-1} dt$$

where C is the maximum of -1 and the smallest zero of the integrand. Evaluating the integral yields the bound of the lemma. □

Theorem 3.3.17 (The Le Potier-Simpson Estimate)**.** *Suppose that X is a projective variety over an algebraically closed k of characteristic zero. For any purely d-dimensional coherent sheaf $\mathscr{F}$ of multiplicity $\alpha_d(\mathscr{F}) = r(\mathscr{F})$ there is an $\mathscr{F}$-regular sequence of hyperplane sections $H_1, ..., H_d$ and $X_v = H_1 \cap \cdots \cap H_{d-v}$ the following estimate holds for all $v = 1, ..., d$:*

$$\frac{h^0(X_v,\mathscr{F}|_{X_v})}{r(\mathscr{F})} \leq \frac{1}{v!}\left[\hat{\mu}_{\max}(\mathscr{F})+r(\mathscr{F})^2+\frac{1}{2}(r(\mathscr{F})+d)-1\right]_+^{v}.$$

Proof. First we claim that when W is a normal projective variety of dimension d and that $\mathscr{K}$ is a torsion free sheaf of $\operatorname{rank}(\mathscr{K})$, there is an $\mathscr{K}$-regular sequence $H_1, ..., H_d$ such that the following estimate holds for all $v = 1, ..., d$:

$$\frac{h^0(W_v,\mathscr{K}|_{W_v})}{\deg W\cdot\operatorname{rank}(\mathscr{K})} \leq \frac{1}{v!}\left[\frac{\mu_{\max}(\mathscr{K})}{\deg W}+\frac{\operatorname{rank}(\mathscr{K})-1}{2}+v\right]_+^{v}.$$

Indeed,

Now we can use this claim to reduce to the general case. Let $i : X \hookrightarrow \mathbb{P}^N$ be the closed embedding induced by $H = \mathscr{O}_X(1)$. Let $\mathscr{F}$ as $i_*\mathscr{F}$ on $\mathbb{P}^N$, let $Z = \operatorname{supp}(\mathscr{F})$ and choose a linear subspace L of dimension $N-d-1$ which does not intersect Z (right for infinite field). Consider projection $\pi : Z \hookrightarrow \mathbb{P}^N\backslash L \to Y \cong \mathbb{P}^d$ which is a finite map with $\mathscr{O}_Z(1) = \pi^*\mathscr{O}_Y(1)$. As $\mathscr{F}$ is pure, we know that $\pi_*\mathscr{F}$ is torsion-free and $\operatorname{rank}(\mathscr{F}) = \operatorname{rank}(\pi_*\mathscr{F})$. Hence

$$\hat{\mu}(\mathscr{F}) = \hat{\mu}(\pi_*\mathscr{F}) = \mu(\pi_*\mathscr{F}) + \frac{d+1}{2}.$$

A $\pi_*\mathscr{F}$-regular sequence of hyperplanes H_i' in Y induces an $\mathscr{F}$-regular sequence of hyperplane sections H_i on X. Let $Y_v = H_1' \cap \cdots \cap H_{d-v}'$, then $\pi_*\mathscr{F}|_{\mathscr{X}} = \pi_*(\mathscr{F})|_{X_v}$ and hence $h^0(F|_{X_v}) = h^0(\pi_*(\mathscr{F})|_{X_v})$.

- **Lemma 3.3.17.A.** The sheaf $\mathscr{A} := \pi_*\mathscr{O}_Z$ is a torsion free sheaf with

$$\mu_{\min}(\mathscr{A}) \geq -\operatorname{rank}(\mathscr{A}) \geq -\operatorname{rank}(\pi_*\mathscr{F})^2 = -r(\mathscr{F})^2.$$

Proof of Lemma 3.3.17.A. As $\pi_*\mathscr{F}$ is an $\mathscr{A}$-module,we have algebra homomorphism $\mathscr{A} \to \mathscr{E}nd(\pi_*\mathscr{F})$ which is injective since Z is the support of $\mathscr{F}$. Hence $\mathscr{A}$ is torsion free with rank less or equal to $\operatorname{rank}(\pi_*\mathscr{F})^2 = r(\mathscr{F})^2$.

Actually we have $\mathbb{P}^N \backslash L \cong \underline{\operatorname{Spec}}_Y \operatorname{Sym} \mathscr{O}_Y(-1)^{\oplus(N-d)}$, let $\mathscr{W} := \mathscr{O}_Y(-1)^{\oplus(N-d)}$. Then this induce a surjection $\phi : \operatorname{Sym}\mathscr{W} \to \mathscr{A}$. Consider the filtration $F_p\mathscr{A} := \phi\left(\bigoplus_{i=0}^{p} \operatorname{Sym}^i\mathscr{W}\right)$. As $\mathscr{A}$ is coherent the filtration is bounded. Moreover, since the multiplication $\mathscr{W} \otimes \operatorname{gr}_p^F \mathscr{A} \to \operatorname{gr}_{p+1}^F \mathscr{A}$ is surjective, hence if $\operatorname{gr}_p^F \mathscr{A}$ is torsion, the same is true for all $\operatorname{gr}_{p+i}^F \mathscr{A}$, $i \geq 0$. In particular, if $\operatorname{gr}_p^F \mathscr{A}$ is not torsion then $p \leq \operatorname{rank}(\mathscr{A})$. Hence the cokernel of $\phi : \bigoplus_{i=0}^{\operatorname{rank}(\mathscr{A})} \operatorname{Sym}^i\mathscr{W} \to \mathscr{A}$ is torsion. Hence $\mu_{\min}(\mathscr{A}) \geq \mu_{\min}(\operatorname{Sym}^{\operatorname{rank}(\mathscr{A})}\mathscr{W}) = -\operatorname{rank}(\mathscr{A})$. □

- **Lemma 3.3.17.B.** We have

$$\mu_{\max}(\pi_*\mathscr{F}) \leq \hat{\mu}_{\max}(\mathscr{F}) + r(\mathscr{F})^2 - \frac{d+1}{2}.$$

Proof of Lemma 3.3.17.B. Let $\mathscr{G}$ be the maximal destabilizing submodule of $\pi_*\mathscr{F}$, and let $\mathscr{G}'$ be the image of the multiplication map $\mathscr{A} \otimes \mathscr{G} \to \pi_*\mathscr{F}$. Then $\mathscr{G}' = \pi_*\mathscr{G}''$ for some $\mathscr{O}_Z$-submodule $\mathscr{G}'' \subset \mathscr{F}$. It follows that

$$\begin{aligned}
\hat{\mu}_{\max}(\mathscr{F}) &\geq \hat{\mu}(\mathscr{G}'') = \hat{\mu}(\mathscr{G}') = \mu(\mathscr{G}') + \hat{\mu}(\mathscr{O}_Y) \\
&\geq \mu_{\min}(\mathscr{A} \otimes \mathscr{G}) + \hat{\mu}(\mathscr{O}_Y) \\
&= \mu(\mathscr{G}) + \mu_{\min}(\mathscr{A}) + \hat{\mu}(\mathscr{O}_Y) \\
&\geq \mu_{\max}(\pi_*\mathscr{F}) - r(\mathscr{F})^2 + \frac{d+1}{2}
\end{aligned}$$

using Theorem 3.3.13 and Lemma 3.3.17.A. □

Now by Lemma 3.3.17.B, we have

$$\mu_{\max}(\pi_*\mathscr{F}) + v + \frac{\operatorname{rank}(\pi_*\mathscr{F}) - 1}{2} \leq \hat{\mu}_{\max}(\mathscr{F}) + r(\mathscr{F})^2 + \frac{r(\mathscr{F}) - 1}{2} + \frac{d-1}{2}.$$

By the claim for $\pi_*\mathscr{F}$ and this inequality, we get the result. □

3.3.5 Boundedness III: The Main Results

In this sections we will assume the base field k is an algebraically closed field of characteristic zero!

Theorem 3.3.18. *Let $f : X \to S$ be a projective morphism of schemes of finite type over k and let $\mathscr{O}_X(1)$ be an f-ample line bundle. Let P be a polynomial of degree d, and let μ_0 be a rational number. Then the family of purely d-dimensional sheaves on the fibres of f with Hilbert polynomial P and maximal slope $\hat{\mu}_{\max} \leq \mu_0$ is bounded. In particular, the family of H-ss sheaves with Hilbert polynomial P is bounded.*

Proof. Covering S by finitely many open subschemes and replacing H by an appropriate high tensor power, if necessary, we may assume that f factors through an embedding $X \hookrightarrow S \times \mathbb{P}^N$. Thus we may reduce to the case $S = \operatorname{Spec}(k), X = \mathbb{P}^N$. By Theorem 3.3.17, we can find for each purely d-dimensional coherent sheaf $\mathscr{F}$ a regular sequence of hyperplanes $H_1, ..., H_d$ such that $h^0(F|_{H_1 \cap \cdots \cap H_i}) \leq C$ for all $i = 0, ..., d$, where C is a constant depending only on the dimension and degree of X and the multiplicity and maximal slope of $\mathscr{F}$. Since these are given or bounded by P and μ_0, respectively, the bound is uniform for the family in question. Hence the result follows from this and the Kleiman Criterion 3.3.4. □

Corollary 3.3.19. *The open moduli substack $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \subset \underline{\mathrm{Coh}}_P(X)$ of H-ss sheaves is an algebraic stack of finite type.*

3.3.6 Harder-Narasimhan Stratification

In order to use Theorem 2.3.12, we need to find a Θ-stratification on $\underline{\mathrm{Coh}}(X)$. We will follows [46] and give some idea.

Definition 3.3.20. *Consider $\mathbb{Q}[t]$ be the polynomial ring in the variable t. An element $f \in \mathbb{Q}[t]$ is called a numerical polynomial if $f(\mathbb{Z}) \subset \mathbb{Z}$. Let the set of all Harder-Narasimhan types, denoted by* **HNT**, *be the set consisting of all finite sequences $(f_1, \ldots, f_p)$ of numerical polynomials in $\mathbb{Q}[t]$, where p is allowed to vary over all integers ≥ 1, such that the following three conditions are satisfied:*

(a) We have $0 < f_1 < \cdots < f_p$ in $\mathbb{Q}[t]$.

(b) The polynomials f_i are all of the same degree, say d.

(c) The following inequalities are satisfied

$$\frac{f_1}{r_d(f_1)} > \frac{f_2 - f_1}{r_d(f_2) - r_d(f_1)} > \cdots > \frac{f_p - f_{p-1}}{r_d(f_p) - r_d(f_{p-1})}.$$

Remark 3.3.21. *By Theorem 3.2.9, for any coherent sheaf $\mathscr{E}$ on X, a projective scheme over a field k, we have the unique Harder-Narasimhan filtration of $\mathscr{E}$. That is, we have*

$$0 = \mathrm{HN}_0(\mathscr{E}) \subset \mathrm{HN}_1(\mathscr{E}) \subset \cdots \subset \mathrm{HN}_l(\mathscr{E}) = \mathscr{E}$$

such that $\mathrm{gr}_i^{\mathrm{HN}}(\mathscr{E})$ *are* H*-ss of dimension* d *and for all* i *we have* $p(\mathrm{gr}_i^{\mathrm{HN}}(\mathscr{E})) > p(\mathrm{gr}_{i+1}^{\mathrm{HN}}(\mathscr{E}))$. *Hence the ordered* l*-tuple*

$$\mathrm{HN}(\mathscr{E}) := (P(\mathrm{HN}_1(\mathscr{E})), \ldots, P(\mathrm{HN}_l(\mathscr{E}))) \in \mathbf{HNT}$$

is called the Harder-Narasimhan type *of* $\mathscr{E}$.

Now we will use the relative version of Harder-Narasimhan filtration again. Note that in Theorem 3.3.9 we know after a modification we have such thing.

Now consider such $f : X \to S$ and $\mathscr{E}$ flat over S, then we can consider the Harder-Narasimhan function of $\mathscr{E}$ is the function

$$|S| \to \mathbf{HNT}, \quad s \mapsto \mathrm{HN}(\mathscr{E}_s).$$

One can show (see [52]) that it is upper-semicontinuous w.r.t. the partial order in our usual meaning.

Remark 3.3.22. *In this case, for any* $\tau \in \mathbf{HNT}$, *the corresponding level set*

$$|S|^{\tau}(\mathscr{E}) = \{s \in |S| \text{ such that } \mathrm{HN}(\mathscr{E}_s) = \tau\}$$

is locally closed in $|S|$, *the subset* $|S|^{\leq\tau}(\mathscr{E}) = \bigcup_{\alpha\leq\tau} |S|^{\alpha}(E) \subset |S|$ *is open in* $|S|$, *and* $|S|^{\tau}(\mathscr{E})$ *is closed in* $|S|^{\leq\tau}(\mathscr{E})$.

Here is our main theorem in this section and also the main theorem of [46]:

Theorem 3.3.23. *Let* $X \to S$ *be a projective morphism over a locally noetherian scheme* S, *with an* f*-ample line bundle* H. *Let* $\mathscr{E}$ *be a coherent sheaf on* X *which is flat over* S, *such that the restriction* $\mathscr{E}_s$ *is a pure-dimensional sheaf on* X_s *for each* $s \in S$. *Let* $\tau = (f_1, ..., f_l) \in \mathbf{HNT}$. *Then we have the following.*

- *(i) Each* Harder-Narasimhan stratum $|S|^{\tau}(\mathscr{E})$ *of* $\mathscr{E}$ *has a unique structure of a locally closed subscheme* $S^{\tau}(\mathscr{E})$ *of* S, *with the following universal property: a morphism* $T \to S$ *factors via* $S^{\tau}(\mathscr{E})$ *if and only if the pullback* $\mathscr{E}_T$ *on* $X \times_S T$ *admits a relative Harder-Narasimhan filtration of type* τ.
- *(ii) A relative Harder-Narasimhan filtration on* $\mathscr{E}$, *if it exists, is unique.*
- *(iii) For any morphism* $f : T \to S$ *of locally noetherian schemes, the schematic stratum* $T^{\tau}(\mathscr{E}_T)$ *equals the schematic inverse image of* $S^{\tau}(\mathscr{E})$ *under* f.

Proof. See Theorem 5 in [46] for the proof. □

Corollary 3.3.24 (Harder-Narasimhan Stratification). *Let* X *be a projective scheme over a field* k. *The stack of all flat families of pure-dimensional coherent sheaves on* X *with fixed Harder-Narasimhan type* τ *form an algebraic stack* $\underline{\mathrm{Coh}}^{\tau}(X)$ *over* k, *which is a locally closed substack of the algebraic stack* $\underline{\mathrm{Coh}}(X)$. *Similarly, we have the open substack* $\underline{\mathrm{Coh}}^{\leq\tau}(X) \subset \underline{\mathrm{Coh}}(X)$. *These data form a* Θ*-stratification.*

Proof. See Theorem 8 in [46] for the proof. These form a Θ-stratification by the fact Remark 3.3.22 and Theorem 3.3.23(i). □

3.4 Good Moduli Space of Semistable Sheaves

3.4.1 Existence of Good Moduli Space of Semistable Sheaves

Theorem 3.4.1. *If X is a projective scheme over an algebraically closed field k of characteristic zero, then the stack $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$ is Θ-complete and S-complete and has a proper good moduli space*

$$\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \to \mathrm{Coh}_P^{\mathrm{H-ss}}(X).$$

Proof. We define a map $p_P : \underline{\mathrm{Coh}}(X) \to V_d$ where V_d be the vector space of polynomials of degree $\leq d$ with the totally order we have used and P is a fixed (Hilbert-)polynomial. Let $P = \sum_{i=0}^{d} \alpha_i(P)\frac{t^i}{i!}$ as before, then we let $p_P(\mathscr{G}) := \alpha_d(P)P(\mathscr{G}) - \alpha_d(\mathscr{G})P$ where $P(\mathscr{G})$ be the Hilbert polynomial. As p_P is just

$$p_P : |\underline{\mathrm{Coh}}(X)| = \pi_0(\underline{\mathrm{Coh}}(X)) \to V_d$$

which satisfies the condition in Theorem 2.3.12. As be definition p_P-semistability is just H-semistability, by Theorem 2.3.12, Corollary 3.3.19 and Corollary 3.3.24 we get the proper good moduli space

$$\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \to \mathrm{Coh}_P^{\mathrm{H-ss}}(X).$$

Well done. □

3.4.2 Points In the $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$

Here we follows the idea in [7] and Lemma 4.1.2 in [29]. The paper [7] consider the moduli stack of μ-ss vector bundles over projective curves instead of our case, but the proofs are similar.

Proposition 3.4.2. *Let X is a projective scheme over an algebraically closed field k.*

(i) If $\mathscr{E}$ is a H-ss sheaf, then the corresponding k-point $[\mathscr{E}] \in \underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)(k)$ contains the point $[\mathrm{gr}^{\mathrm{JH}}\,\mathscr{E}]$ in its closure.

(ii) A point $[\mathscr{E}] \in \underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)(k)$ is closed if and only if $\mathscr{E}$ is polystable.

Proof. For (i), if $\mathscr{E}$ is a H-ss but not H-stable, there exists a non-split extension $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ of H-ss sheaves of the same reduced Hilbert polynomial follows from a Jordan-Hölder filtration. Let $\mathscr{G}$ be the universal family over the affine line in $\mathrm{Ext}^1(\mathscr{E}'', \mathscr{E}')$ spanned by this extension, then $\mathscr{G}$ is a family of H-ss sheaves on X parameterized by $\mathbb{A}^1$ such that $\mathscr{G}_t \cong \mathscr{E}$ if $t \neq 0$ and $\mathscr{G}_0 \cong \mathscr{E}' \oplus \mathscr{E}''$. Hence we get

$$[\mathscr{G}] : \mathbb{A}^1 \to \underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X), \quad 0 \mapsto [\mathscr{E}' \oplus \mathscr{E}''], t \mapsto [\mathscr{E}] \text{ for } t \neq 0.$$

It follows that $[\mathscr{E}' \oplus \mathscr{E}'']$ is contained in the closure of $[\mathscr{E}]$. Iterating this construction for $\mathscr{E}'$ and $\mathscr{E}''$ shows that $[\mathrm{gr}^{\mathrm{JH}}\,\mathscr{E}]$ is in the closure of $[\mathscr{E}]$.

For (ii), if $[\mathscr{E}] \in \underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)(k)$ is closed, then $\mathscr{E}$ is H-polystable directly by (i). Conversely, if $\mathscr{E}$ is polystable which is not closed, then take another $[\mathscr{F}]$ in its closure. By (i) we know that $[\mathrm{gr}^{\mathrm{JH}}\,\mathscr{F}]$ is in the closure of $[\mathscr{F}]$ and since no two points can be in the closure of each other, we must have $\mathscr{E} \ncong \mathrm{gr}^{\mathrm{JH}}\,\mathscr{F}$. On the other hand, if $\mathscr{E}_i$ is stable with the same reduced Hilbert polynomial as $\mathscr{E}$, then $\mathscr{E}_i$ appears as a direct summand of $\mathscr{E}$ with multiplicity $\hom_X(\mathscr{E}_i, \mathscr{E})$ and similarly for $\mathrm{gr}^{\mathrm{JH}}(\mathscr{F})$. For any $\mathscr{E}_i$, the function $\hom_X(\mathscr{E}_i, -)$ is upper semicontinuous in the second variable, since $[\mathrm{gr}^{\mathrm{JH}}(\mathscr{F})]$ is in the closure of $[\mathscr{E}]$, so we have $\hom_X(\mathscr{E}_i, \mathscr{E}) \leq \hom_X(\mathscr{E}_i, \mathrm{gr}^{\mathrm{JH}}(\mathscr{F}))$. This means that any stable summand of $\mathscr{E}$ appears in $\mathrm{gr}^{\mathrm{JH}}(\mathscr{F})$ with at least the same multiplicity. But $\mathscr{E}$ and $\mathscr{F}$ have the same rank, so we must have $\mathscr{E} \cong \mathrm{gr}^{\mathrm{JH}}(\mathscr{F})$, a contradiction. Thus, $[\mathscr{E}]$ is closed. □

Theorem 3.4.3. *Let X is a projective scheme over an algebraically closed field k of characteristic zero. Then the good moduli space $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \to \mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ induce bijections between the k-points of the good moduli space and S-equivalence classes of H-ss sheaves.*

In particular the good moduli space of H-ss sheaves $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ parameterizing the H-polystable sheaves.

Proof. By Theorem 1.1.3(ii), two k-points $[\mathscr{E}], [\mathscr{E}'] \in \underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$ map to the same point in $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ if and only if the closures of $\{[\mathscr{E}]\}$ and $\{[\mathscr{E}']\}$ in $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$ intersect.

On the one hand, if $\mathscr{E}$ is any H-ss vector bundle, then by Proposition 3.4.2(i) $\mathscr{E}$ contains $\mathrm{gr}^{\mathrm{JH}}(\mathscr{E})$ in its closure, so both points map to the same point in $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$. On the other hand, if $\mathscr{E}$ and $\mathscr{E}'$ are H-polystable and nonisomorphic, then by Proposition 3.4.2(ii), the corresponding points in $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$ are closed and distinct, hence map to distinct points in $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$. □

3.4.3 Projectivity

Now we have the good moduli space $\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \to \mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ and $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ is a proper algebraic space. Hence by Tag 0D36 to show $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ is a projective scheme we just need to show there is an ample line bundle on it.

Let X is a projective scheme over an algebraically closed field k of characteristic 0.

Construction 1 (draft) –Modern Method

May we use the similar method of section 4,5 in [7]?

Pick universal coherent sheaf $\mathscr{E}_{\mathrm{univ}}$ over $X \times \underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)$.

$$\begin{tikzcd}
 & X \times \underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \arrow[dl, "q"'] \arrow[dr, "p"] & \\
X & & \underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X)
\end{tikzcd}$$

For any $\mathscr{G}$ on X define

$$\mathscr{L}_{\mathscr{G}} := \left(\det(\mathbf{R}p_*(q^*\mathscr{G} \otimes \mathscr{E}_{\mathrm{univ}}))\right)^\vee .$$

Proposition 3.4.4. *Descend this into* $L_{\mathscr{G}}$ *over* $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$*?*

Theorem 3.4.5. *In this case* $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ *is a projective scheme over* k*.*

Proof. Some vanishing theorem to get that $L_{\mathscr{G}}$ semiample which induce a quasi-finite (hence finite) $f : \mathrm{Coh}_P^{\mathrm{H-ss}}(X) \to \mathbb{P}^M$. Hence $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ will be a proper scheme. Let $H = f^*\mathscr{O}_{\mathbb{P}^M}(1)$ which is ample. Hence $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ is a projective scheme. ☐

Construction 2–GIT Method

Here we follows chapter 4 in [29] and section 6.7 in [4].

Actually from Theorem 3.3.18, there is an integer N such that for any H-ss sheaf $\mathscr{F}$ with Hilbert polynomial P, $\mathscr{F}$ is N-regular. Hence $\mathscr{F}(N)$ is globally generated and $h^0(\mathscr{F}(N)) = P(N)$. Hence by the proofs of Proposition 3.3.7 and above, there is an open subscheme U of the Quot scheme $\underline{\mathrm{Quot}}_{X,P}(\mathscr{O}_X(-N)^{P(N)})$ parameterizing H-ss sheaves and inducing an isomorphism $k^{\oplus P(N)} = H^0(\mathscr{O}_X^{P(N)}) \cong H^0(\mathscr{F}(N))$ which is invariant under the natural action of $\mathrm{GL}_{P(N)}$ on $\underline{\mathrm{Quot}}_{X,P}(\mathscr{O}_X(-N)^{P(N)})$. Hence

$$\underline{\mathrm{Coh}}_P^{\mathrm{H-ss}}(X) \cong [U/\mathrm{GL}_{P(N)}].$$

As for any such $\mathscr{F}$ in U, $\mathrm{Aut}(\mathscr{F}) \hookrightarrow \mathrm{GL}_{P(N)}$ is just the stabilizer at $[\mathscr{O}_X(-N)^{P(N)} \twoheadrightarrow \mathscr{F}]$. Hence the the scalar matrixes are contained in the stabilizer of any point in $\underline{\mathrm{Quot}}_{X,P}(\mathscr{O}_X(-N)^{P(N)})$. Instead of the action of $\mathrm{GL}_{P(N)}$ we will therefore consider the actions of $\mathrm{PGL}_{P(N)}$ and $\mathrm{SL}_{P(N)}$. We will use the $\mathrm{SL}_{P(N)}$ as it is easier to find an $\mathrm{SL}_{P(N)}$-linearization ample line bundle as below.

Consider

$$\begin{tikzcd}
 & \underline{\mathrm{Quot}}_{X,P}(\mathscr{O}_X(-N)^{P(N)}) \times X \arrow[dl, "q"'] \arrow[dr, "p"] & \\
X & & \underline{\mathrm{Quot}}_{X,P}(\mathscr{O}_X(-N)^{P(N)})
\end{tikzcd}$$

Pick $\mathscr{F}_{\mathrm{univ}}$ be the universal quotient sheaf on $\underline{\mathrm{Quot}}_{X,P}(\mathscr{O}_X(-N)^{P(N)}) \times X$. For $l \gg 0$ we define

$$\mathscr{L}_l := \det(p_*(q^*\mathscr{O}(l) \otimes \mathscr{F}_{\mathrm{univ}})).$$

By the construction of quotient scheme, $\mathscr{L}_l$ is very ample for $l \gg 0$ which also have a natural $\mathrm{GL}_{P(N)}$-linearization. We fix such $l \gg 0$.

Consider the closure $\overline{U} \subset \underline{\mathrm{Quot}}_{X,P}(\mathscr{O}_X(-N)^{P(N)})$ of U and fix $\mathscr{L}_l$ on it for $l \gg 0$ with $\mathrm{SL}_{P(N)}$-linearization.

Theorem 3.4.6. *In this case we have* $U = \overline{U}^{\mathrm{GIT-ss}}(\mathscr{L}_l)$ *and* $U^s = \overline{U}^{\mathrm{GIT-s}}(\mathscr{L}_l)$ *where* U^s *are the locus of geometrically* H*-stable sheaves.*

Proof. See Theorem 4.3.3 in [29]. ☐

Hence by the basic theory of GIT (one can see Theorem 6.7.6 in [4] or [45]), we get:

Corollary 3.4.7. *The good moduli space* $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ *is a projective scheme.*

3.4.4 Dimension of Moduli Space of Semistable Sheaves

Let X be a projective scheme over an algebraically closed field of characteristic zero. Here we follows [29] section 4.5 and Appendix D in [4].

Proposition 3.4.8. *The first order deformations of a coherent sheaf* $\mathscr{E}$ *on* X *up to isomorphism are bijective to the group* $\mathrm{Ext}^1(\mathscr{E},\mathscr{E})$.

Proof. Pick a first order deformation $\mathscr{E}_1$ of a coherent sheaf $\mathscr{E}$. Then $\mathscr{E}_1$ is flat over $k[\varepsilon]$. Hence tensor $0 \to k \to k[\varepsilon] \to k \to 0$ we get

$$0 \to \mathscr{E} \to \mathscr{E}_1 \to \mathscr{E} \to 0$$

which gives an element of $\mathrm{Ext}^1(\mathscr{E},\mathscr{E})$. Conversely, given a such extension, then we have a such flat family over $k[\varepsilon]$ by Remark A.2.7 in [4]. ☐

Definition 3.4.9. *Consider a functor* $\mathcal{D} : \mathrm{Art}_k^{\mathrm{op}} \to (\mathrm{Sets})$ *such that* $\mathcal{D}(k)$ *is a single emelent. The functor* $\mathcal{D}$ *is said to have an* obstruction theory *with values in a finite dimensional* k*-vector space* U*, if the following hold:*

- *(a) For each small extension* $A' \to A$ *with kernel* J*, there is a map of sets* $\mathrm{ob} : \mathcal{D}(A) \to U \otimes J$ *such that the sequence* $\mathcal{D}(A') \to \mathcal{D}(A) \overset{\mathrm{ob}}{\to} U \otimes J$ *is exact.*
- *(b) Different small extensions is natural with respective to* ob.

Proposition 3.4.10. *Consider the deformation functor* $\mathcal{D}_{\mathscr{F}} : \mathrm{Art}_k^{\mathrm{op}} \to (\mathrm{Sets})$ *by*

$$A \mapsto \{\mathscr{F}_A \in \mathrm{Coh}(X_A) : \mathscr{F}_A \otimes_A k \cong \mathscr{F} \text{ and flat over } A\}/\cong.$$

Then $\mathcal{D}_{\mathscr{F}}$ *have an obstruction theory with values in* $\mathrm{Ext}^2(\mathscr{F},\mathscr{F})$.

Proof. This is easy by the injective resolutions and the definitions. For details we refer [29] 2.A.6. ☐

Theorem 3.4.11. *Let $\mathscr{F}$ is H-stable over X as a point $[\mathscr{F}] \in \mathrm{Coh}_P^{\mathrm{H-ss}}(X)$, then $\widehat{\mathscr{O}}_{\mathrm{Coh}_P^{\mathrm{H-ss}}(X),[\mathscr{F}]}$ pro-represents the deformation functor $\mathcal{D}_{\mathscr{F}} : \mathrm{Art}_k^{\mathrm{op}} \to (\mathrm{Sets})$ by*

$$A \mapsto \{\mathscr{F}_A \in \mathrm{Coh}(X_A) : \mathscr{F}_A \otimes_A k \cong \mathscr{F} \text{ and flat over } A\}/\cong .$$

Proof. There is a natural map of functors $\mathcal{D}_{\mathscr{F}} \to \widehat{\mathscr{O}}_{\mathrm{Coh}_P^{\mathrm{H-ss}}(X),[\mathscr{F}]}$ by the openess of H-stability. In this locus we have the geometric quotient $U^{\mathrm{s}} \to \mathrm{Coh}_P^{\mathrm{H-s}}(X)$. By the Luna's étale slice theorem 1.2.4, let $q \in U^{\mathrm{s}}$ be a point in the fibre over $[\mathscr{F}]$, then there exists $q \in S \subset U^{\mathrm{s}}$ open such that $\widehat{\mathscr{O}}_{S,[q]} \cong \widehat{\mathscr{O}}_{\mathrm{Coh}_P^{\mathrm{H-ss}}(X),[\mathscr{F}]}$. The universal family on $U^{\mathrm{s}} \times X$, restricted to $S \times X$, induces a map $\widehat{\mathscr{O}}_{S,[q]} \to \mathcal{D}_{\mathscr{F}}$ which yields the required inverse. ☐

Corollary 3.4.12. *Let $\mathscr{F}$ be a H-stable point. Then the Zariski tangent space of $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ at $[\mathscr{F}]$ is canonically given by*

$$T_{[\mathscr{F}]}\mathrm{Coh}_P^{\mathrm{H-ss}}(X) \cong \mathrm{Ext}^1(\mathscr{F},\mathscr{F}).$$

If $\mathrm{Ext}^2(\mathscr{F},\mathscr{F}) = 0$, then $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ is smooth at $[\mathscr{F}]$. In general, there are bounds

$$\mathrm{ext}^1(\mathscr{F},\mathscr{F}) \geq \dim_{[\mathscr{F}]} \mathrm{Coh}_P^{\mathrm{H-ss}}(X) \geq \mathrm{ext}^1(\mathscr{F},\mathscr{F}) - \mathrm{ext}^2(\mathscr{F},\mathscr{F}).$$

Proof. Here we will use a conclusion of pure commutative algebra (see the detailed proof at Proposition 2.A.11 in [29]):

- Suppose that such functor $\mathcal{D}$ is pro-represented by a couple (R, ξ) and has an obstruction theory with values in an r-dimensional vector space U. Let $d = \dim(\mathfrak{m}_R/\mathfrak{m}_R^2)$, then
$$d \geq \dim R \geq d - r.$$
Moreover if $r = 0$, then R is isomorphic to a ring of formal power series in d variables.

Now by this result, Proposition 3.4.8, Proposition 3.4.10 and Theorem 3.4.11, we get the result. ☐

Remark 3.4.13. *When X is a smooth projective variety, using* $\det : \mathrm{Coh}_P^{\mathrm{H-ss}}(X) \to \underline{\mathrm{Pic}}(X)$ *and its differential at a H-stable sheaf $\mathscr{F}$, which is*

$$\mathrm{Tr} : T_{[\mathscr{F}]}\mathrm{Coh}_P^{\mathrm{H-ss}}(X) \cong \mathrm{Ext}^1(\mathscr{F},\mathscr{F}) \to H^1(\mathscr{O}_X) \cong T_{[\det(\mathscr{F})]}\underline{\mathrm{Pic}}(X),$$

then one can show that:

- *Let* $M(\mathscr{Q})$ *be the fibre of the morphism* $\det : \mathrm{Coh}_P^{\mathrm{H-ss}}(X) \to \underline{\mathrm{Pic}}(X)$ *over the point* $[\mathscr{Q}]$. *Then for any* H*-stable* $\mathscr{F}$ *with* $\det \mathscr{F} = \mathscr{Q}$ *we have*

  $$T_{[\mathscr{F}]}M(\mathscr{Q}) \cong \mathrm{Ext}^1(\mathscr{F}, \mathscr{F})_0.$$

  If $\mathrm{Ext}^2(\mathscr{F}, \mathscr{F})_0 = 0$, *then* $\mathrm{Coh}_P^{\mathrm{H-ss}}(X)$ *and* $M(\mathscr{Q})$ *are all smooth at* $[\mathscr{F}]$. *Moreover we have*

  $$\mathrm{ext}^1(\mathscr{F}, \mathscr{F})_0 \geq \dim_{[\mathscr{F}]} M(\mathscr{Q}) \geq \mathrm{ext}^1(\mathscr{F}, \mathscr{F})_0 - \mathrm{ext}^2(\mathscr{F}, \mathscr{F})_0.$$

  Here $\mathrm{Ext}^i(\mathscr{F}, \mathscr{F})_0$ *means the kernel of trace.*

We refer the books [21] and Theorem 4.5.4 in [29].

Chapter 4

Bridgeland Stability and Its Good Moduli Space

4.1 Moduli Stack of Universally Gluable Complexes

Here we will follows papers [25] and [38] (or Tag 0DLB and Tag 0DPV) Here many results hold for algebraic spaces. But we only care about the schemes.

Definition 4.1.1. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{E}^*$ be a complex of $\mathscr{O}_X$-modules. We say $\mathscr{E}^*$ is* strictly perfect *if $\mathscr{E}^i$ is zero for all but finitely many i and $\mathscr{E}^i$ is a direct summand of a finite free $\mathscr{O}_X$-module for all i.*

Definition 4.1.2. *Let X be a scheme. An object $E \in \mathbf{D}(\mathscr{O}_X)$ is* pseudo-coherent *if it is represented by $\mathscr{E}^*$ such that there exists an open covering $X = \bigcup_i U_i$ and for each i a morphism of complexes $\alpha_i : \mathscr{E}_i^* \to \mathscr{E}^*|_{U_i}$ where $\mathscr{E}_i^*$ is strictly perfect on U_i and $H^j(\alpha_i)$ is an isomorphism for all j.*

Definition 4.1.3. *Let $f : X \to S$ be a morphism of schemes which is flat and locally of finite presentation. An object $E \in \mathbf{D}(\mathscr{O}_X)$ is* perfect relative to S *or S-perfect if E is pseudo-coherent and E locally has finite tor dimension as an object of $\mathbf{D}(f^{-1}\mathscr{O}_S)$.*

Definition 4.1.4. *Let $f : X \to S$ be a flat morphism of schemes. An S-perfect complex $E \in \mathbf{D}(\mathscr{O}_X)$ is* gluable *if $\mathbf{R}f_*\mathbf{R}\mathscr{H}om(E, E) \in \mathbf{D}(\mathscr{O}_X)^{\geq 0}$. It is* universally gluable *if this remains true upon arbitrary base change $T \to S$.*

Definition 4.1.5. *Let S be a scheme. Let $f : X \to B$ be a proper, flat, and of finite presentation morphism of schemes over S. Let $\mathscr{D}^b_{\mathrm{pug}}(X/B)$ be the fibred category over (Sch/S) of bounded universally gluable complex with coherent cohomology.*

Theorem 4.1.6. *Let S be a scheme. Let $f : X \to B$ be a proper, flat, and of finite presentation morphism of schemes over S. Then $\mathscr{D}^b_{\mathrm{pug}}(X/B)$ is an algebraic stack locally of finite presentation over S which has affine diagonal.*

Proof. This require some more advanced stack theory and deformation theory. Actually we don't care this proof. We refer Tag 0DLN or [38] Theorem 4.2.1.

We only show that $\mathscr{D}^b_{\mathrm{pug}}(X/B)$ has affine diagonal. We just need to show: given a scheme T over B and objects $E, E' \in \mathbf{D}(\mathscr{O}_{X_T})$ such that (T, E) and (T, E') are objects of the fibre category of $\mathscr{D}^b_{\mathrm{pug}}(X/B)$ over T, then $\underline{\mathrm{Isom}}(E, E') \to T$ is affine. Here we need use a part of the proof of the algebraicity (Tag 0DLC):

- In this case the functor $H = \underline{\mathrm{Hom}}(E, E')$ is an algebraic space affine over T.

Take functors $H' = \underline{\mathrm{Hom}}(E', E), I = \underline{\mathrm{Hom}}(E, E)$ and $I' = \underline{\mathrm{Hom}}(E', E')$. Then these are all algebraic spaces affine over T. We find that we have the cartesian

$$\begin{tikzcd}
\underline{\mathrm{Isom}}(E, E') \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & T \arrow[d, "\sigma"] \\
H' \times_T H \arrow[r, "c"] & I \times_T I'
\end{tikzcd}$$

where $c(\varphi', \varphi) = (\varphi \circ \varphi', \varphi' \circ \varphi)$ and $\sigma = (\mathrm{id}, \mathrm{id})$. Hence $\underline{\mathrm{Isom}}(E, E') \to T$ is affine. □

4.2 Basic Facts of t-Structures

4.2.1 Basic Definitions of t-Structures

Here we give a basic introduction of t-structures. We follows the lecture notes [17]. Given a triangulated category $\mathscr{D}$.

Definition 4.2.1. *Pick two full subcategories $\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0}$ of $\mathscr{D}$, We call the pair $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ to be a t-structure over $\mathscr{D}$ if*

(a) Let $\mathscr{D}^{\leq n} := \mathscr{D}^{\leq 0}[-n], \mathscr{D}^{\geq n} := \mathscr{D}^{\geq 0}[-n]$, then $\mathscr{D}^{\leq -1} \subset \mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 1} \subset \mathscr{D}^{\geq 0}$.

(b) We have $\mathrm{Hom}(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 1}) = 0$.

(c) For any $X \in \mathscr{D}$, there exists $Y \in \mathscr{D}^{\geq 0}$ and $Z \in \mathscr{D}^{\geq 1}$ filled the following distinguished triangle:

$$Y \to X \to Z \to Y[1].$$

Define $\mathscr{D}^{\heartsuit} = \mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq 0}$ to be the heart of this t-structure. If $\bigcap_n \mathscr{D}^{\geq n} = \bigcap_n \mathscr{D}^{\leq n} = 0$, then we call this t-structure non-degenerate.

Example 4.2.1. *(i) Given any triangulated category $\mathscr{D}$. we have the trivial t-structure $\mathscr{D}^{\leq 0} = \mathscr{D}, \mathscr{D}^{\geq 0} = 0$.*

(ii) *Given an abelian category $\mathscr{A}$ and its derived category $\mathscr{D} := \mathbf{D}(\mathscr{A})$. Let*

$$\mathscr{D}^{\leq 0} = \{K \in \mathscr{D} : H^i(K) = 0, i > 0\};$$
$$\mathscr{D}^{\geq 0} = \{K \in \mathscr{D} : H^i(K) = 0, i < 0\}.$$

One can see the first condition of t-structure holds trivially. The third condition we just need to consider:

$$\tau^{\leq 0}(X) \to X \to \tau^{\geq 1}(X) \to \tau^{\leq 0}(X)[1].$$

For the second, consider $K \in \mathscr{D}^{\leq 0}, L \in \mathscr{D}^{\geq 1}$, pick $f : K \to L$ and its representation:

$$\begin{tikzcd}
 & \tau^{\leq 0}K' \arrow[dl, "{\alpha,\mathrm{qis}}"'] \arrow[d, dashed, "{\beta,\mathrm{qis}}"] \arrow[dr, "g"] & \\
K & K' \arrow[l, "\mathrm{qis}"] \arrow[r] & L
\end{tikzcd}$$

where we replace K' to be $\tau^{\leq 0}K'$. As $g = 0$, we have $f = 0$. Hence we get a t-structure.

Hence the t-structure is some kind of generalization of derived categories.

4.2.2 Canonical Functors about t-structures

Lemma 4.2.2. *Let $\mathscr{D}$ be a triangulated category, for $i \in \{1, 2\}$ we consider two distinguished triangle:*

$$X \to Y \to Z \xrightarrow{d_i} X[1],$$

Then if $\operatorname{Hom}(X[1], Z) = 0$, we have $d_1 = d_2$.

Proof. Consider:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\mathrm{id}_X"] & Y \arrow[r] \arrow[d, "\mathrm{id}_Y"] & Z \arrow[r, "d_1"] \arrow[d, dashed, "c"] & X[1] \arrow[d, "\mathrm{id}_X"] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r, "d_2"] & X[1]
\end{tikzcd}$$

where c follows from the definition of triangulated category. By some diagram chase we get $\mathrm{id}_Z = c$, well done. ☐

Proposition 4.2.3. *Let $(\mathscr{D}^{\leq 0}, \mathscr{D}^{\geq 0})$ be a t-structure of $\mathscr{D}$.*

(i) *The inclusion $\mathscr{D}^{\leq n} \to \mathscr{D}$ has a right adjoint $\tau^{\leq n} : \mathscr{D} \to \mathscr{D}^{\leq n}$ given by a canonical $\tau^{\leq n}(X) \to X$.*

(ii) *The inclusion* $\mathscr{D}^{\geq n} \to \mathscr{D}$ *has a left adjoint* $\tau^{\geq n} : \mathscr{D} \to \mathscr{D}^{\geq n}$ *given by a canonical* $\tau^{\geq n}(X) \to X$.

(iii) *For any* $X \in \mathscr{D}$, *there exists a unique* $\delta : \tau^{\geq n+1}(X) \to \tau^{\leq n}(X)[1]$ *such that we have the distinguished triangle:*

$$\tau^{\leq n}(X) \to X \to \tau^{\geq n+1}(X) \xrightarrow{\delta} \tau^{\leq n}(X)[1]$$

(standard triangle) and δ *is functorial.*

Proof. By shifting n times we can let $n = 0$. By the definition (c) of t-structure we can get (i)(ii). Again (iii) follows from definition (c) of t-structure and the uniqueness follows from Lemma 4.2.2 and definition (b) of t-structure. □

Corollary 4.2.4. *About* $\tau^{\leq n}, \tau^{\geq n}$, *we have the following:*

(i) *We have* $\tau^{\leq n}(X[m]) = (\tau^{\leq n+m}(X))[m]$ *and* $\tau^{\geq n}(X[m]) = (\tau^{\geq n+m}(X))[m]$.

(ii) $X \in \mathscr{D}^{\leq n}$ *if and only if* $\tau^{\leq n}(X) \cong X$ *if and only if* $\tau^{>n}(X) = 0$; *and* $X \in \mathscr{D}^{\geq n}$ *if and only if* $\tau^{\geq n}(X) \cong X$ *if and only if* $\tau^{<n}(X) = 0$.

(iii) *For distinguished triangle* $X \to Y \to Z \to X[1]$, *if* $X, Z \in \mathscr{D}^{\leq n}$, *then* $Y \in \mathscr{D}^{\leq n}$; *if* $X, Z \in \mathscr{D}^{\geq n}$, *then* $Y \in \mathscr{D}^{\geq n}$.

(iv) *If* $a < b$, *then* $\tau^{\leq a} \circ \tau^{\leq b} = \tau^{\leq a} = \tau^{\leq b} \circ \tau^{\leq a}$ *and* $\tau^{\leq a} \circ \tau^{\geq b} = 0 = \tau^{\geq b} \circ \tau^{\leq a}$, *and* $\tau^{\geq a} \circ \tau^{\geq b} = \tau^{\geq b} = \tau^{\geq b} \circ \tau^{\geq a}$.

(v) *Fir any* $a, b \in \mathbb{Z}$ *we have canonical* $\tau^{\leq a} \circ \tau^{\geq b} \cong \tau^{\geq b} \circ \tau^{\leq a}$.

Proof. (i)(ii)(iv) follows from the definitions and adjointness. (iii) is easy to verity. (v) is complicated and we refer [28] Proposition 8.1.8. □

4.2.3 The Properties of Heart $\mathscr{D}^{\heartsuit}$

Fix a triangulated category $\mathscr{D}$.

Definition 4.2.5. *Define* $H^0 : \mathscr{D} \to \mathscr{D}^{\heartsuit}$ *as* $X \mapsto (\tau^{\leq 0} \circ \tau^{\geq 0})(X)$ *and* $H^n(-) := H^0((-)[n])$.

Lemma 4.2.6. *We have:*

(i) *For any* $X \in \mathscr{D}$, *we have*

$$H^n(X)[-n] \to \tau^{\geq n}(X) \to \tau^{\geq n+1}(X) \to H^n(X)[-n+1],$$

In particular, if $X \in \mathscr{D}^{\geq a}$ *then* $X \in \mathscr{D}^{\geq n}$ *if and only if* $H^i(X) = 0$ *for any* $i < n$.

(ii) *For distinguished triangle* $X \to Y \to Z \to X[1]$, *if* $X, Z \in \mathscr{D}^{\heartsuit}$, *then* $Y \in \mathscr{D}^{\heartsuit}$.

Proof. (i) using standard triangle of $\tau^{\geq n}(X)$. (ii) use Corollary 4.2.4(iii) twice. ☐

Remark 4.2.7. *For (ii), if $X, Y \in \mathscr{D}^{\heartsuit}$, then Z may not in heart. Consider:*

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{i} \mathbb{Z} \oplus \mathbb{Z}[1] \xrightarrow{\mathrm{pr}_2} \mathbb{Z}[1].$$

Theorem 4.2.8. *$\mathscr{D}^{\heartsuit}$ is an abelian category.*

Proof. Additive follows from Corollary 4.2.4(iii). Now let $X, Y \in \mathscr{D}^{\heartsuit}$ and $f : X \to Y$. Pick a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to X[1].$$

Easy to see that $Z \in \mathscr{D}^{\leq 0} \cap \mathscr{D}^{\geq -1}$.
•**Claim 1.** $Y \to Z \to H^0(Z)$is the cokernel of f and $H^{-1}(Z) \to Z[-1] \to X$ is the kernel of f.

Fix $W \in \mathscr{D}^{\heartsuit}$, taking $\mathrm{Hom}(-, W)$ we get the exact sequence

$$0 \to \mathrm{Hom}(H^0(Z), W) \to \mathrm{Hom}(Y, W) \to \mathrm{Hom}(Z, W).$$

By definition we get the claim 1.
•**Claim 2.** We have $\mathrm{coim}(f) \cong \mathrm{Im}(f)$.

For canonical map $\alpha : Y \to \mathrm{coker}(f), \beta : \ker f \to X$ we have $\mathrm{coim}(f) = \mathrm{coker}\beta = \mathrm{cone}(\beta)$. Similarly $\mathrm{Im}(f) = \ker \alpha = \mathrm{cone}(\alpha)[-1]$. So we just need to show $\mathrm{cone}(\beta) \cong \mathrm{cone}(\alpha)[-1]$. By octahedral axiom we have the exact diagram:

$$\begin{tikzcd}
H^{-1}(Z) \arrow[r, "\beta"] \arrow[d] & X \arrow[r] \arrow[d, "\mathrm{id}_X"] & \mathrm{cone}(\beta) \arrow[d] \\
Z[-1] \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Y \arrow[d] \\
H^0(Z)[-1] \arrow[r] & 0 \arrow[r] & Q
\end{tikzcd}$$

Then all the elements in $\mathscr{D}^{\heartsuit}$. By the final row we have $Q \cong H^0(Z)$. It's not hard to see that $Y \to Q$ and $Y \to H^0(Z)$ are the same. Hence by the right column we get the claim. ☐

Remark 4.2.9. *Actually the heart of t-structure determined the t-structure itself: since the heart determined H^0, then we get the t-structure after shifting.*

Corollary 4.2.10. *Let $X, Y, Z \in \mathscr{D}^{\heartsuit}$, then $0 \to X \xrightarrow{a} Y \xrightarrow{b} Z \to 0$ exact if and only if there is a distinguished triangle $X \xrightarrow{a} Y \xrightarrow{b} Z \to X[1]$.*

Proof. $\Leftarrow$ from $\text{coker}a = H^0(Z) = Z, \ker a = H^{-1}(Z) = 0$.

$\Rightarrow$. As a injective if and only if $\ker a = 0$ if and only if $H^{-1}(\text{cone}(a)) = 0$ if and only if we have

$$\text{cone}(a) \cong H^0(\text{cone}(a)) = \text{coker}a = Z.$$

Hence we have the distinguished triangle $X \xrightarrow{a} Y \xrightarrow{b} Z \to X[1]$. The uniqueness follows from Lemma 4.2.2. □

Corollary 4.2.11. *Let $X, Z \in \mathscr{D}^\heartsuit$, then*

$$\text{Ext}^1_{\mathscr{D}^\heartsuit}(Z, X) \cong \text{Hom}_{\mathscr{D}}(Z, X[1]) =: \text{Ext}^1_{\mathscr{D}}(Z, X).$$

Here $\text{Ext}^1_{\mathscr{D}^\heartsuit}(Z, X)$ means extension.

Proof. Pick $(0 \to Y \to Y \to Z \to 0) \in \text{Ext}^1_{\mathscr{D}^\heartsuit}(Z, X)$, by the previous corollary we get an element in $\text{Hom}_{\mathscr{D}}(Z, X[1])$; conversely pick $\delta \in \text{Hom}_{\mathscr{D}}(Z, X[1])$, filled as:

$$Z \xrightarrow{\delta} X[1] \to \text{cone}(\delta) \to Z[1].$$

Hence we have $X \to \text{cone}(\delta)[-1] \to Z \xrightarrow{\delta} X[1]$. Since $X, Z \in \mathscr{D}^\heartsuit$, by Lemma 4.2.6(ii) we get $\text{cone}(\delta)[-1] \in \mathscr{D}^\heartsuit$. By the previous corollary we get the result. □

Remark 4.2.12. *(i) In general $\mathbf{D}(\mathscr{D}^\heartsuit) \neq \mathscr{D}$, but Beilinson in perverse sheaf and constructable sheaf we have $D(\mathscr{D}^\heartsuit) \cong \mathscr{D}$.*

(ii) This false for higher Ext*. For example consider $X = S^2$ and $\mathscr{D} := \mathbf{D}_{\text{Loc}}(X)$. Then the canonical $\mathscr{D}^{\leq 0}$ and $\mathscr{D}^{\geq 0}$ forms a t-structure. Then $\mathscr{D}^\heartsuit \cong \text{Loc}(X)$. As $\pi_1(X) = 0$, by monodromy representation we know that it is equivalent to the category of abelian groups. Hence*

$$\begin{aligned} \text{Ext}^2_{\mathscr{D}^\heartsuit}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) &= \text{Ext}^2_{\text{Ab}}(\mathbb{Z}, \mathbb{Z}) = 0, \\ \text{Ext}^2_{\mathscr{D}}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}) &= H^2(X, \underline{\mathbb{Z}}) = \mathbb{Z}, \end{aligned}$$

Well done.

(iii) Furthermore, whether $\mathscr{D}^\heartsuit \hookrightarrow \mathscr{D}$ can be extended to a exact functor $D^b(\mathscr{D}^\heartsuit) \to \mathscr{D}$? It's unknown in general, but in special we have more: Beilinson shows this is right for fibred derived categories; Lurie shows this is right for ∞- categories. The situation of perverse t-structure follows from Beilinson fundamental lemma.

Theorem 4.2.13. *The functor $H^0 : \mathscr{D} \to \mathscr{D}^\heartsuit$ is a cohomology functor.*

Sketch of the proof. Pick a distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathscr{D}$, we just need to show $H^0(X) \to H^0(Y) \to H^0(Z)$ is exact. We omit the diagram chase and consider the main diagrams. For details we refer [28] Proposition 8.1.11.

•**Step 1.** If $X, Y, Z \in \mathscr{D}^{\geq 0}$, then $0 \to H^0(X) \to H^0(Y) \to H^0(Z)$ is exact.

For any $A \in \mathscr{D}^{\heartsuit}$, acting $\operatorname{Hom}(A, -)$ on $X \to Y \to Z \to X[1]$ we get the long exact sequence:

$$0 = \operatorname{Hom}(A, Z[1]) \to \operatorname{Hom}(A, H^0(X)) \to \operatorname{Hom}(A, H^0(Y)) \to \operatorname{Hom}(A, H^0(Z)).$$

By Yoneda's lemma we get the result.
•**Step 2.**If $Z \in \mathscr{D}^{\geq 0}$,then $0 \to H^0(X) \to H^0(Y) \to H^0(Z)$ is exact; if $X \in \mathscr{D}^{\leq 0}$, then $H^0(X) \to H^0(Y) \to H^0(Z) \to 0$ is exact.

Just need to consider the first case. By the definnition (b) of t-structure we have $\tau^{<0}(X) = \tau^{<0}(Y)$. By octahedral axiom we get

$$\begin{tikzcd}
\tau^{<0}(X) \arrow[r] \arrow[d] & \tau^{<0}(Y) \arrow[r] \arrow[d] & 0 \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
\tau^{\geq 0}(X) \arrow[r] & \tau^{\geq 0}(Y) \arrow[r] & Z
\end{tikzcd}$$

Now using Step 1 at the bottom row.
•**Step 3.**Finish the proof.

By octahedral axiom again we get

$$\begin{tikzcd}
\tau^{\leq 0}(X) \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & W \arrow[d] \\
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
\tau^{>0}(X) \arrow[r] & 0 \arrow[r] & Q
\end{tikzcd}$$

Hence $Q \cong (\tau^{>0}(X))[1]$. Now using Step 2. □

Definition 4.2.14. *Let $F : \mathscr{D}_1 \to \mathscr{D}_2$ be a triangulated functor. Let $(\mathscr{D}_1^{\leq 0}, \mathscr{D}_1^{\geq 0})$ and $(\mathscr{D}_2^{\leq 0}, \mathscr{D}_2^{\geq 0})$ are their t-structures.*

(a) We call F is left t-exact if $F(\mathscr{D}_1^{\geq 0}) \subset \mathscr{D}_2^{\geq 0}$.

(b) We call F is right t-exact if $F(\mathscr{D}_1^{\leq 0}) \subset \mathscr{D}_2^{\leq 0}$.

(c) We call F is t-exact if it is both left t-exact and right t-exact.

We some times let ${}^pF := H^0 \circ F \circ \iota : \mathscr{D}_1^{\heartsuit} \to \mathscr{D}_2^{\heartsuit}$.

4.3 Moduli Stack of Objects in t-Structures

Through out this section we will assume that k is a field and X is a projective scheme over k.

4.3.1 Families of t-Structures

Definition 4.3.1. *Given a t-structure on $\mathbf{D}^b(X)$ and a k-algebra R, the induced t-structure on $\mathbf{D}_{qc}(X_R)$ is the unique t-structure for which*

$$\mathbf{D}_{qc}(X_R)^{\leq 0} := \left\{ \begin{array}{c} \textit{The smallest full subcategory of } \mathbf{D}_{qc}(X_R) \\ \textit{containing } R \boxtimes E, \forall E \in \mathbf{D}^b(X)^{\leq 0} \textit{ and that} \\ \textit{is closed under small colimits, extensions} \end{array} \right\}$$

and $\mathbf{D}_{qc}(X_R)^{\geq 0}$ is the category of $E \in \mathbf{D}_{qc}(X_R)$ such that $\mathrm{Hom}(F, E) = 0, \forall F \in \mathbf{D}_{qc}(X_R)^{\leq 0}$. The truncation functors commute with filtered colimits.

Proof. This holds since the category $\mathbf{D}_{qc}(X_R)$ is a presentable stable ∞-category and the Proposition 1.4.4.11 and Proposition 1.4.4.13 in [40]. □

Remark 4.3.2. *Hence R generated by $\mathbf{D}_{qc}(\mathrm{Mod}_R)^{\leq 0}$ under colimits and $\mathbf{D}^b(X)^{\leq 0}$ generates $\mathbf{D}_{qc}(X)^{\leq 0}$ under extensions and colimits. So $\mathbf{D}_{qc}(X)^{\leq 0} = \mathrm{Ind}(\mathbf{D}^b(X)^{\leq 0})$. Hence $\mathbf{D}_{qc}(X)^{\heartsuit} = \mathrm{Ind}(\mathbf{D}^b(X)^{\heartsuit})$.*

Lemma 4.3.3. *For any ring map $R \to S$, the induced map $\phi : X_S \to X_R$ has the following properties with respect to the t-structure we have constructed:*

(i) $\phi^* : \mathbf{D}_{qc}(X_R) \to \mathbf{D}_{qc}(X_S)$ is right t-exact,

(ii) $\phi_* : \mathbf{D}_{qc}(X_S) \to \mathbf{D}_{qc}(X_R)$ is t-exact,

(iii) any $E \in \mathbf{D}_{qc}(X_S)$ lies in $\mathbf{D}_{qc}(X_S)^{\heartsuit}$ (respectively $\mathbf{D}_{qc}(X_S)^{\leq 0}$ or $\mathbf{D}_{qc}(X_S)^{\geq 0}$) if and only if $\phi_*(E)$ does,

(iv) if $R \to S$ is flat then ϕ^* is t-exact, and

(v) if $\{R \to S_\alpha\}_{\alpha \in I}$ is a flat cover of $\mathrm{Spec}(R)$ then $E \in \mathbf{D}_{qc}(X_R)$ lies in the heart if and only if $\phi_\alpha^*(E) \in \mathbf{D}_{qc}(X_{S_\alpha})^{\heartsuit}$ for all $\alpha \in I$.

Proof. (i) is trivial and (ii) using an equivalence of stable ∞-categories which we omitted. (iii) follows formally from the fact that ϕ_* is t-exact and conservative. (iv) follows from (ii) and the fact that $\phi_* \phi^* E \simeq S \otimes_R^{\mathbb{L}} E$, so if $R \to S$ is flat then S is a filtered colimit of free R-modules. (v) follows from (iv) and the fact that $\prod_\alpha \phi_\alpha^*$ is conservative. □

Remark 4.3.4. *Hence $\mathbf{D}_{qc}(X_R)^{[a,b]} = \{E \in \mathbf{D}_{qc}(X_R) : p_*(E) \in \mathbf{D}_{qc}(X)^{[a,b]}\}$ where $p : X_R \to X$.*

Remark 4.3.5. *By some theory of simplicial scheme and ∞-categories, we have:*

For any algebraic k-stack $\mathscr{Y}$, there is a canonical t- structure induced on $\mathbf{D}_{qc}(X_{\mathscr{Y}})$ in which $\mathbf{D}_{qc}(X_{\mathscr{Y}})^{\leq 0}$ (respectively $\mathbf{D}_{qc}(X_{\mathscr{Y}})^{\geq 0}$) is the full subcategory of complexes E such that for any smooth map $\operatorname{Spec} R \to \mathscr{Y}$ we have $E|_{X_R} \in \mathbf{D}_{qc}(X_R)^{\leq 0}$ (respectively $\mathbf{D}_{qc}(X_R)^{\geq 0}$). It suffices to check if $E \in \mathbf{D}_{qc}(X_{\mathscr{Y}})^{\leq 0}$ or $E \in \mathbf{D}_{qc}(X_{\mathscr{Y}})^{\geq 0}$ after restricting to a smooth cover of $\mathscr{Y}$ by affine schemes. See [25] Corollary 6.1.3 for the proof.

Proposition 4.3.6. *Assume the t-structure on $\mathbf{D}^b(X)$ is noetherian and nondegenerate, and let R be an algebra that is essentially of finite type over k. Then the truncation functors on $\mathbf{D}_{qc}(X_R)$ preserve $\mathbf{D}^b(X_R)$, and the induced t-structure on $\mathbf{D}^b(X_R)$ is noetherian.*

Proof. See Lemma 6.1.5 and Proposition 6.1.4 in [25]. □

4.3.2 Moduli Stack of Objects in t-Structures

Definition 4.3.7. *Given a t-structure on $\mathbf{D}^b(X)$ and a k-algebra R, we say that a complex $E \in \mathbf{D}_{qc}(X_R)$ is R-flat if $E \otimes_R^{\mathbb{L}} M \in \mathbf{D}_{qc}(X_R)^{\heartsuit}$ for all R-module M.*

We define the moduli of flat families of objects in $\mathbf{D}^b(X)^{\heartsuit}$ to be the fibred category $\mathscr{M}_{\mathrm{Ind}(\mathbf{D}^b(X)^{\heartsuit})}$ that assigns to an affine k-scheme $\operatorname{Spec} R$ the groupoid

$$\{E \in \mathbf{D}^b(X_R) : E \text{ is } R\text{-perfect and } R\text{-flat}\}.$$

Lemma 4.3.8. *For $E \in \mathbf{D}_{qc}(X_R)$ the following are equivalent:*

(i) *E is R-flat.*

(ii) *$\phi^* E \in \mathbf{D}_{qc}(X_S)^{\heartsuit}$ for any map $\phi : X_S \to X_R$ induced by a map of k-algebras $R \to S$.*

(iii) *$E \otimes_R^{\mathbb{L}} R/I \in \mathbf{D}_{qc}(X_R)^{\heartsuit}$ for all finitely generated ideals $I \subset R$.*

If the t-structure on $\mathbf{D}^b(X)$ is non-degenerate, these are equivalent to

(iv) *$E \in \mathbf{D}_{qc}(X_R)^{\heartsuit}$ and the functor*

$$E \otimes_R (-) = H^0(E \otimes_R^{\mathbb{L}} (-)) : \mathrm{Mod}_R \to \mathbf{D}_{qc}(X_R)^{\heartsuit}$$

is exact.

Furthermore, if R is Noetherian and $E \in \mathbf{D}_{qc}(X_R)^{\leq 0}$ and is pseudo-coherent, then these are equivalent to

(v) *$E|_{R/\mathfrak{m}} \in \mathbf{D}_{qc}(X_{R/\mathfrak{m}})^{\heartsuit}$ for all maximal ideals $\mathfrak{m} \subset R$.*

Proof. By Lemma 4.3.3, $\phi^*E \in \mathbf{D}_{qc}(X_S)^\heartsuit$ if and only if $\phi_*(\phi^*E) = E\otimes_R^{\mathbb{L}} S \in \mathbf{D}_{qc}(X_R)^\heartsuit$. So $(i) \Rightarrow (ii) \Rightarrow (iii)$ tautologically. For $(iii) \Rightarrow (i)$, as $\mathbf{D}_{qc}(X_R)^\heartsuit$ is closed under filtered colimits and Mod_R is compactly generated by finitely presented modules, we just need to show $E \otimes_R^{\mathbb{L}} M \in \mathbf{D}_{qc}(X_R)^\heartsuit$ for all finitely presented R-module M. Actually this is the same as the case in commutative algebra (Tag 00HD).

We omit the proof about (v) althrough the proof is easy, but it needs the Grothendieck existence theorem for the stable ∞-category (see the proof in Lemma 6.2.2 in [25]). We now show that $(i) \Leftrightarrow (iv)$.

We can easy to see that $E \otimes_R^{\mathbb{L}} (-) : \mathbf{D}_{qc}(R) \to \mathbf{D}_{qc}(X_R)$ is right t-exact for any $E \in \mathbf{D}_{qc}(X_R)^\heartsuit$, by presenting any connective complex of R-modules as a complex of free modules M^* in cohomological degree ≤ 0, then observing that $E \otimes_R^{\mathbb{L}} M^*$ lies in the category generated by E under extensions, left shifts, and filtered colimits. The implication $(i) \Rightarrow (iv)$ follows from this observation and the long exact sequence for the cohomology of an exact triangle.

To show $(iv) \Rightarrow (i)$, one considers for any $M \in \mathrm{Mod}_R$ a presentation $0 \to K \to R^S \to M \to 0$. The exactness of $H^0(E\otimes_R^{\mathbb{L}}(-))$ and the long exact cohomology sequence implies that $H^{-1}(E \otimes_R^{\mathbb{L}} M) = 0$, and $H^{-i}(E \otimes_R^{\mathbb{L}} M) \cong H^{-i-1}(E \otimes_R^{\mathbb{L}} M)$ for all $i > 0$. Because this holds for all R-modules simultaneously, it follows that $H^{-i}(E \otimes_R^{\mathbb{L}} M) = 0$ for all $i > 0$. Assuming the t-structure is non-degenerate, i.e., $\bigcap_{n\geq 0} \mathbf{D}_{qc}(X)^{\leq -n} = 0$, this implies that $E \otimes_R^{\mathbb{L}} M \in \mathbf{D}_{qc}(X_R)^\heartsuit$ by Lemma 4.2.6(i). □

Corollary 4.3.9 (Open Heart Property)**.** *Let R be a finite type k-algebra and let $E \in \mathbf{D}^b(X_R)$. The set of prime ideals*

$$U := \{\mathfrak{p} \in \operatorname{Spec} R : E|_{R_\mathfrak{p}} \in \mathbf{D}^b(X_{R_\mathfrak{p}})^\heartsuit\}$$

is open, and it contains those primes for which $E|_{\kappa(\mathfrak{p})} \in \mathbf{D}_{qc}(X_{\kappa(\mathfrak{p})})^\heartsuit$.

Proof. As the restriction along the map $X_{R_\mathfrak{p}} \to X_R$ is t-exact, the subset U is the complement of the image under the projection $X_R \to \operatorname{Spec} R$ of the closed subsets $\mathrm{supp}(\tau^{\leq -1}(E))$ and $\mathrm{supp}(\tau^{\geq 1}(E))$. Therefore $\operatorname{Spec} R\backslash U$ is closed since the projection $X_R \to \operatorname{Spec} R$ is proper. Finally, by Lemma 4.3.8(v), if $E|_{\kappa(\mathfrak{p})} \in \mathbf{D}_{qc}(X_{\kappa(\mathfrak{p})})^\heartsuit$ then $E|_{R_\mathfrak{p}} \in \mathbf{D}_{qc}(X_{R_\mathfrak{p}})^\heartsuit$ (???). □

Definition 4.3.10. *A t-structure on $\mathbf{D}^b(X)$ has the generic flatness property if given a domain R of finite type over k with fraction field K and an object $E \in \mathbf{D}^b(X_R)$ such that $E_K \in \mathbf{D}^b(X_R)^\heartsuit$, there is an $f \in R$ such that $E|_{\operatorname{Spec} R_f} \in \mathbf{D}^b(X_{R_f})$ is flat.*

Example 4.3.1. *By Tag 052A the usual t-structure of $\mathbf{D}^b(X)$ has the generic flatness property.*

Remark 4.3.11. *Note that when $\mathrm{char}(k) = 0$, then generic flatness is equivalent to the following condition: for every smooth k-algebra R and every $E \in \mathbf{D}^b(X_R)^\heartsuit$, there is a dense open subset $U \subset \operatorname{Spec} R$ such that $E|_U$ is flat.*

Indeed, as $\mathbf{D}^b(X_R) \to \mathbf{D}^b(X_K)$ *is* t*-exact, then* $E \in \mathbf{D}^b(X_R)^\heartsuit$ *implies* $E \in \mathbf{D}^b(X_K)^\heartsuit$*. Hence the result follows from the generic flatness. Conversely, consider an integral* k*-algebra* R *and* $E \in \mathbf{D}^b(X_R)$*. If* $E_K \in \mathbf{D}^b(X_R)^\heartsuit$*, then by Corollary 4.3.9 we can find a dense open* $U \subset \operatorname{Spec} R$ *such that* $E|_U \in \mathbf{D}^b(X_U)^\heartsuit$*. By generic smoothness for reduced* k*-algebras we can pass to a smaller open subset* $U'' \subset U$ *that is smooth over* k*. Hence the result follows.*

Note that the only step we use $\operatorname{char}(k) = 0$ *is the generic smoothness. This holds for perfect fields. We refer Tag 020I and Tag 056V.*

Theorem 4.3.12. *Given a noetherian* t*-structure on* $\mathbf{D}^b(X)$ *that satisfies the generic flatness condition, the stack* $\mathscr{M}_{\operatorname{Ind}(\mathbf{D}^b(X)^\heartsuit)}$ *is an open substack of* $\mathscr{D}^b_{\text{pug}}(X)$*, hence it is an algebraic stack locally of finite type over* k *with affine diagonal.*

Proof. By the definition of t-structure, any object in the heart of a t-structure is gluable. Hence by Lemma 4.3.8(ii), any complex $E \in \mathscr{M}_{\operatorname{Ind}(\mathbf{D}^b(X)^\heartsuit)}(R)$ is universally gluable. Hence $\mathscr{M}_{\operatorname{Ind}(\mathbf{D}^b(X)^\heartsuit)}$ is a full substack of $\mathscr{D}^b_{\text{pug}}(X)$. We just need to show that for any k-algebra R and any $E \in \mathscr{D}^b_{\text{pug}}(X)(R)$, there is an open subset $U \subset \operatorname{Spec} R$ such that for any homomorphism of k-algebras $\phi : R \to S$, we have $\phi^*(E) \in \mathscr{M}_{\operatorname{Ind}(\mathbf{D}^b(X)^\heartsuit)}(S)$ if and only if the image of $\operatorname{Spec} S \to \operatorname{Spec} R$ lies in U. Now we use a finiteness result as follows in [38] Proposition 2.2.1:

- As E is relatively perfect, there is a subalgebra $R' \subset R$ of finite type over k and a relatively perfect complex $E' \in \mathbf{D}^b(X_{R'})$ such that $E = E' \otimes_{R'} R$.

Hence we may assume that R is of finite type over k by taking preimage.

Now we will show that if R is finite type and the generic flatness property holds, then the set of prime ideals

$$U := \{\mathfrak{p} \in \operatorname{Spec} R : E|_{R/\mathfrak{p}} \in \mathbf{D}^b(X_{R/\mathfrak{p}})^\heartsuit\}$$

is open and satisfies the desired condition.

A simple inductive argument reduces one to the case where R is integral, so we will assume this. By Corollary 4.3.9, the property $E|_{R/\mathfrak{p}} \notin \mathbf{D}^b(X_{R/\mathfrak{p}})^\heartsuit$ is closed under specialization, hence if K is the field of fractions of R and $E_K \notin \mathbf{D}^b(X_K)^\heartsuit$ then $Z = \emptyset$. On the other hand, if $E_K \in \mathbf{D}^b(X_K)^\heartsuit$ then by generic flatness we know that there is an f such that $E|_{R_f}$ is flat, hence $Z \subset \operatorname{Spec} R/(f)$ which is closed by noetherian induction. Hence U is open.

Finally by Lemma 4.3.8(v), the restriction $E|_U$ is U-flat (???) and relatively perfect, so $\phi^*(E)$ is S-flat for any morphism $\phi : \operatorname{Spec} S \to \operatorname{Spec} R$ landing in U. Conversely for any morphism $\phi : \operatorname{Spec} S \to \operatorname{Spec} R$ such that there is some point $p \in \operatorname{Spec} S$ lying over Z, $\phi^*(E)|_p \notin \mathbf{D}_{qc}(X_{\kappa(p)})^\heartsuit$, so $\phi^*(E)$ is not flat by Lemma 4.3.8(ii). □

Remark 4.3.13. *Actually in this case the stack* $\mathscr{M}_{\operatorname{Ind}(\mathbf{D}^b(X)^\heartsuit)}$ *agrees with other similar descriptions of moduli functors in the following ways:*

(a) *On finite type k-algebras R,* $\mathscr{M}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}(R)$ *is naturally equivalent to the moduli functor:*

$$\{E \in \mathbf{D}^b(X_R) : \mathbf{L}i^*_{\mathfrak{m}}E \in \mathbf{D}^b(X_{\kappa(\mathfrak{m})}) \in \mathbf{D}^b(X_{\kappa(\mathfrak{m})})^\heartsuit\}.$$

(b) *If the t-structure on* $\mathbf{D}^b(X)$ *is bounded with respect to the usual t structure, then for any k-algebra R,* $\mathscr{M}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}(R)$ *is naturally equivalent to*

$$\{E \in \mathbf{D}_{qc}(X_R) : E \text{ is pseudo-coherent and } R\text{-flat}\}.$$

(c) *If the t-structure on* $\mathbf{D}^b(X)$ *is noetherian and bounded with respect to the usual t-structure, then* $\mathscr{M}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}(R)$ *is naturally equivalent to the moduli functor associated to* $\mathbf{D}_{qc}(X)^\heartsuit$*:*

$$\{E \in \mathrm{Mod}_R(\mathbf{D}_{qc}(X)^\heartsuit) : \text{ finitely presented and } R\text{-flat}\}$$

where R-flatness here means that the (non-derived) tensor product functor $E \otimes_R (-) : \mathrm{Mod}_R \to \mathrm{Mod}_R(\mathbf{D}_{qc}(X)^\heartsuit)$ *is exact.*

See the proof of the second part of Proposition 6.2.7 in [25]. Then all these stacks are algebraic stacks locally of finite type over k with affine diagonal. Note that (c) is our special case of Definition 2.1.11 when $\mathcal{A} = \mathbf{D}_{qc}(X)^\heartsuit = \mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)$*.*

Corollary 4.3.14. *Fix a t-structure on* $\mathbf{D}^b(X)$ *that is noetherian, bounded with respect to the usual t-structure, and satisfies generic flatness. Then the stack* $\mathscr{M}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}$ *is* Θ*-complete and* S*-complete with respect to DVR's essentially of finite type over k.*

Proof. This follows from Remark 4.3.13(c), Proposition 2.2.7 and Proposition 2.2.8. □

4.4 Bridgeland Stability Condition

We will follow the paper [19] and the lecture note [43].

4.4.1 Stability of Abelian Categories

Definition 4.4.1. *Let* $\mathcal{A}$ *be an abelian category. Let* $Z : K_0(\mathcal{A}) \to \mathbb{C}$ *be an additive homomorphism which is called a* stability function *if for all nonzero* $E \in \mathcal{A}$ *we have:*

(a) $\Im(Z(E)) \geq 0$.

(b) $\Im(Z(E)) = 0$ *implies* $\Re(Z(E)) < 0$.

Define $R(E) := \Im(Z(E))$ *to be the* generalized rank of E, *and* $D(E) := -\Re(Z(E))$ *is called the* generalized degree of E. *Then* $M(E) = R(E)/D(E)$ *is called the* generalized slope of E.

Remark 4.4.2. *We call $Z : K_0(\mathcal{A}) \to \mathbb{C}$ a weak stability function if it satisfies $Z(E) \in \mathbb{H} \cup \mathbb{R}_{\leq 0}$ instead of $\mathbb{H} \cup \mathbb{R}_{<0}$.*

Definition 4.4.3. *An object $E \in \mathcal{A}$ is Z-stable (resp. Z-semistable) if for all nonzero $F \subsetneq E$, $M(F) < M(E)$ (resp. $M(F) \leq M(E)$).*

Definition 4.4.4. *The pair $(\mathcal{A}, Z)$ as above is called a stability condition if any nonzero object has a Harder-Narasimhan filtration much like before: a filtration $0 = E_0 \subset \cdots \subset E_n = E$ such that E_i/E_{i-1} is Z-semistable and $M(E_{i+1}/E_i) > M(E_i/E_{i-1})$ for all i.*

Remark 4.4.5. *Similar as before, the Harder-Narasimhan filtration is unique up to unique isomorphism if it exists. We also call E is torsion if $Z(E) \in \mathbb{R}_{\leq 0}$ and torsion free if $\Im(Z(E)) > 0$.*

As the stability of sheaves before we have:

Lemma 4.4.6. *Let $A, B \in \mathcal{A}$ be Z-semistable objects with $M(A) > M(B)$. Then $\mathrm{Hom}_{\mathcal{A}}(A, B) = 0$.*

Proof. The same proof of Lemma 3.2.4. ☐

Proposition 4.4.7. *Let $\mathcal{A}$ be an noetherian abelian category. Let $Z : K_0(\mathcal{A}) \to \mathbb{C}$ be a stability function. Assume that the generalized rank $R : K_0(\mathcal{A}) \to R$ has discrete image. Then for any $E \in \mathcal{A}$, the generalized degrees of subobjects of E are bounded above. Finally the Harder-Narasimhan filtrations exist, i.e. (A, Z) is a stability condition.*

Proof. We refer [M392cBrSt] Lemma 8.10 and Proposition 8.18 for the proof. ☐

4.4.2 Basic Properties of Bridgeland Stability

Definition 4.4.8. *A slicing $\mathcal{P}$ of a triangulated category $\mathscr{D}$ is a collection of full additive subcategories $\mathcal{P}(\phi)$ for each $\phi \in \mathbb{R}$ satisfying:*

(a) $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.

(b) For all $\phi_1 > \phi_2$ we have $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$.

(c) For each $0 \neq E \in \mathscr{D}$ there is a sequence $\phi_1 > \cdots > \phi_n$ of real numbers and a sequence of distinguished triangles

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[d] \\
& A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & & A_n \arrow[ul, dashed]
\end{tikzcd}$$

where $A_i \in \mathcal{P}(\phi_i)$ for each i (Harder-Narasimhan filtration).

Remark 4.4.9. *We have the following remarks:*

(a) *We call the objects in $\mathcal{P}(\phi)$ semistable of phase ϕ.*

(b) *Given the slicing $\mathcal{P}$, the sequence of ϕ and the Harder-Narasimhan filtration are automatically unique! Sometimes we set $\phi^{+}_{\mathcal{P}}(E) = \phi_1$ and $\phi^{1}_{\mathcal{P}}(E) = \phi_n$.*

(c) *If $\mathcal{P}(\phi) \neq 0$ only for $\phi \in \mathbb{Z}$, then the slicing is equivalent to the datum of a bounded t-structure with heart $\phi(0)$.*

(d) *More generally, given a slicing $\mathcal{P}$, it's easy to see that $(\mathcal{P}(> a), \mathcal{P}(\leq a+1))$ and $(\mathcal{P}(\geq a), \mathcal{P}(< a+1))$ are t-structures. Their hearts are $\mathcal{P}((a, a+1])$ and $\mathcal{P}([a, a+1))$. In other words, a slicing is always a refinement of a bounded t-structure.*

(e) *Let $\mathscr{D}^{\heartsuit_1}, \mathscr{D}^{\heartsuit_2}$ be two hearts of bounded t-structures, if $\mathscr{D}^{\heartsuit_1} \subset \mathscr{D}^{\heartsuit_2}$ then $\mathscr{D}^{\heartsuit_1} = \mathscr{D}^{\heartsuit_2}$; similarly if $\mathcal{P}_1$ and $\mathcal{P}_2$ are two slicings such that $\mathcal{P}_1(\phi) \subset \mathcal{P}_1(\phi)$ for all $\phi \in \mathbb{R}$, then $\mathcal{P}_1 = \mathcal{P}_2$.*

Definition 4.4.10. *For a triangulated category $\mathscr{D}$. We fix a finite-rank lattice Λ and a surjective group homomorphism $v : K_0 \twoheadrightarrow \Lambda$. Then the Bridgeland stability condition on $\mathscr{D}$ with respect to Λ and v is a pair $\sigma = (\mathcal{P}, Z)$ where $\mathcal{P}$ is a slicing and $Z : \Lambda \to \mathbb{C}$ is a group homomorphism called the central charge such that:*

(a) *For every $0 \neq E \in \mathcal{P}(\phi)$ we have*

$$Z(v(E)) \in \mathbb{R}_{>0} e^{i\pi\phi}.$$

(b) *(support property)*

$$C_\sigma := \inf\left\{ \frac{|Z(v(E))|}{\|v(E)\|} : E \in \mathcal{P}(\phi)\backslash 0, \phi \in \mathbb{R} \right\} > 0.$$

Here the objects in $\mathcal{P}(\phi)$ the σ-semistable of phase ϕ .

Remark 4.4.11. (a) *The stability condition without the support property we will call it the Bridgeland prestability condition. But in the original [19] this property wasn't part of the definition. But was added by Kontsevich-Soibelman in [31].*

(b) *The support property is equivalent to Bridgeland's notion of a full locally-finite stability condition in [20] Definition 4.2. There is an equivalent formulation: There is a symmetric bilinear form Q on $\Lambda_{\mathbb{R}}$ such that*

- *all σ-semistable objects E satisfy the inequality $Q(v(E), v(E)) \geq 0$;*
- *all non zero vectors $v \in \Lambda_{\mathbb{R}}$ with $Z(v) = 0$ satisfy $Q(v, v) < 0$.*

The first condition can be viewed as some generalization of the classical Bogomolov inequality for vector bundles. We refer [M392cBrSt] Lemma 13.15 for the detailed proof.

(c) *Bridgeland stability came out of physics, more precisely, homological mirror symmetry. Douglas wrote some stuff about it, and Bridgeland stability came out of Bridgeland's work to make everything mathematically precise.*

Remark 4.4.12. *Let $\mathscr{D}$ be a triangulated category with a heart of a bounded t-structure $\mathscr{D}^\heartsuit \subset \mathscr{D}$. Consider homomorphisms (Corollary 4.2.10)*

$$i : K_0(\mathscr{D}^\heartsuit) \to K_0(\mathscr{D})$$

induced by inclusion and

$$F : K_0(\mathscr{D}) \to K_0(\mathscr{D}^\heartsuit), \quad [X] \mapsto \sum_{n\in\mathbb{Z}} (-1)^n [H^n_{\mathscr{D}^\heartsuit}(X)].$$

Then we can show that these are inverse to each other.

In a special case, let $\mathcal{A}$ be an abelian category and $\mathbf{D}^b(\mathcal{A})$ be its bounded derived category. Then for any heart of a bounded t-structure $\mathbf{D}^b(\mathcal{A})^\heartsuit \subset \mathbf{D}^b(\mathcal{A})$ there is a natural identification between Grothendieck groups

$$K_0(\mathbf{D}^b(\mathcal{A})^\heartsuit) = K_0(\mathbf{D}^b(\mathcal{A})) = K_0(\mathcal{A})$$

as $\mathcal{A}$ here is just a special heart.

Remark 4.4.13. *Here we discuss a typical choice of Λ.*

Let $\mathscr{D}$ be a triangulated category linear over a field k. Let $\mathscr{D}$ is of finite type, that is, for every pair of objects E and F of $\mathscr{D}$ the vector space $\bigoplus_i \operatorname{Hom}_{\mathscr{D}}(E, F[i])$ is finite-dimensional. We define the Euler pairing *on $K_0(\mathscr{D})$ defined as $\chi(v, w) := \sum_i (-1)^i \dim_k \operatorname{Hom}_{\mathscr{D}}(v, w[i])$.*

We define the numerical Grothendieck group *$K_0^{\mathrm{num}}(\mathscr{D})$ is defined as $K_0(\mathscr{D})/T$ where $T \subset K_0(\mathscr{D})$ consist of $v \in K_0(\mathscr{D})$ such that $\chi(v, w) = 0$ for all $w \in K_0(\mathscr{D})$.*

We call $\mathscr{D}$ is numerically finite *if $K_0^{\mathrm{num}}(\mathscr{D})$ has finite rank. In this case the Bridgeland stability condition on $\mathscr{D}$ with respect to $K_0^{\mathrm{num}}(\mathscr{D})$ is called the* numerical Bridgeland stability.

Even $\mathscr{D}$ is not numerically finite, we sometimes will consider the surjection $K_0(\mathscr{D}) \twoheadrightarrow K_0^{\mathrm{num}}(\mathscr{D}) \twoheadrightarrow \Lambda$ for a finite rank Λ.

Remark 4.4.14. *If we consider $\mathscr{D} := \mathbf{D}^b(X)$ for a projective smooth variety X of dimension d over an algebraically closed field k, we claim that $\mathbf{D}^b(X)$ is numerically finite.*

By HRR formula we have

$$\chi(v,w)=\int_X \mathrm{ch}(v^\vee)\mathrm{ch}(w)\mathrm{td}(X).$$

By [22] Example 15.2.16 we know that $\mathrm{ch}: K_0(X)_{\mathbb{Q}} \to \mathrm{CH}^*(X)_{\mathbb{Q}}$ *is an isomorphism. Then we get* $\mathrm{ch}: K_0^{\mathrm{num}}(X)_{\mathbb{Q}} \cong \mathrm{CH}^*_{\mathrm{num}}(X)_{\mathbb{Q}}$ *by HRR again. Hence we just need to show* $\mathrm{CH}^*_{\mathrm{num}}(X)$ *is of finite rank.*

Pick a Weil cohomoloty theory H^**, for example, take algebraic de Rham cohomology for characteristic zero and take crystalline cohomology for positive characteristic. We just need to prove that* $\dim_{\mathbb{Q}} \mathrm{CH}^i_{\mathrm{num}}(X)_{\mathbb{Q}} \leq \dim H^{2i}(X) =: b_{2i}(X)$*. For simplicity we take étale cohomology* $H^*_{\text{ét}}(X,\mathbb{Q}_\ell)$ *of* $\mathbb{Q}_\ell$*-coefficient with* $\ell \neq \mathrm{char}(k)$*. This is classical.*

Choose $\alpha_1,...,\alpha_m \in Z^{d-i}(X)$ *whose classes in* $H^{2d-2i}_{\text{ét}}(X,\mathbb{Q}_\ell)$ *form a maximal set of* $\mathbb{Q}_\ell$*-linearly independent elements in the image of the cycle class map* $\mathrm{cl}_X : Z^{d-i}(X) \to H^{2d-2i}_{\text{ét}}(X,\mathbb{Q}_\ell)$*. Clearly* $m \leq b_{2d-2i}(X) = b_{2i}(X)$*. Consider the linear map*

$$\lambda : \mathrm{CH}^i(X) \to \mathbb{Z}^m, \beta \mapsto \left(\int_X \beta\cdot\alpha_1, ..., \int_X \beta\cdot\alpha_m\right).$$

We claim $\ker\lambda = Z^i_{\mathrm{num}}(X)$*. Trivially* $Z^i_{\mathrm{num}}(X) \subset \ker\lambda$*. Conversely, set* $\alpha \in \mathrm{CH}^{d-i}(X)$ *and* $\mathrm{cl}_X(\alpha) = \sum \nu_j \mathrm{cl}_X(\alpha_j)$ *where* $\nu_j \in \mathbb{Q}_\ell$*. Then*

$$\begin{aligned}\int_X \beta\cdot\alpha &= \mathrm{tr}(\mathrm{cl}_X(\alpha)\cup\mathrm{cl}_X(\beta))\\ &= \sum_j \nu_j \mathrm{tr}(\mathrm{cl}_X(\alpha_j)\cup\mathrm{cl}_X(\beta))\\ &= \sum_j \nu_j \int_X \beta\cdot\alpha_j.\end{aligned}$$

Hence if $\beta \in \ker\lambda$*, then* $\beta \in Z^i_{\mathrm{num}}(X)$*. Hence we get the claim. By the claim we get* $B^i(X) \hookrightarrow \mathbb{Q}^m$*. Hence well done.*

Proposition 4.4.15. *If* $\sigma = (\mathcal{P}, Z)$ *is a Bridgeland stability condition, then* $\mathcal{P}(\phi)$ *is a finite-length abelian category, that is, it is both noetherian and artinian.*

Proof. As $\mathcal{P}(\phi) \subset \mathcal{P}((a,a+1])$ for some $a \in \mathbb{R}$, then we just need to show that $\mathcal{P}(\phi)$ is closed under kernels and cokernels. This is trivial.

Now we will just show that $\mathcal{P}(\phi)$ is noetherian since there is a similar proof about $\mathcal{P}(\phi)$ is artinian. Given $E \in \mathcal{P}(\phi)$ and an ascending chain $E_0 \subset E_1 \subset \cdots \subset E$, let $A_i := E_i/E_{i-1}$, then for each $n \geq 0$ we have

$$Z(v(E)) = Z(v(E/E_n)) + \sum_{i=1}^n Z(v(A_i)).$$

Since all $Z(v(A_i))$ and $Z(E/E_n)$ lie on the same ray, then

$$\begin{aligned}|Z(v(E))| &= |Z(v(E/E_n))| + \sum_{i=1}^{n} |Z(v(A_i))| \\ &\geq \sum_{i=1}^{n} |Z(v(A_i))|.\end{aligned}$$

Set $s_n := \sum_{i=1}^{n} |Z(v(A_i))|$. As s_n is monotonically increasing in n and bounded, so it converges. Hence $\lim_{i\to\infty} |Z(v(A_i))| = 0$. The support property says that there's a constant C such that $|Z(v(A_i))| \geq C\|v(A_i)\|$ for all i, and therefore $\lim_{i\to\infty} \|v(A_i)\| = 0$ too. But $v(A_i) \in \Lambda \subset \Lambda_{\mathbb{R}}$, so since it converges in a discrete space, we have $v(A_i) = 0$ for i large enough, which means $Z(v(A_i)) = 0$ for i large enough, which means $A_i = 0$ for i large enough. □

Definition 4.4.16. *A σ-stable object of phase ϕ is a simple object of $\mathcal{P}(\phi)$.*

Hence in this case we can define the Jordan-Hölder filtrations. Two objects $E, E' \in \mathcal{P}(\phi)$ are S*-equivalent if their Jordan-Hölder filtrations have the same factors.*

Although the definition we gave is short and good for abstract argumentation, but it is not very practical for finding concrete examples. The following result will give us a nice equivalent formulation.

Proposition 4.4.17. *Let $\mathscr{D}$ be a triangulated category. Then, specifying a Bridgeland stability condition $\sigma = (\mathcal{P}, Z_1)$ on $\mathscr{D}$ is equivalent to specifying a stability condition $Z_2 : K_0(\mathscr{D}^\heartsuit) \to \mathbb{C}$ of $\mathscr{D}^\heartsuit$ which is a heart of a bounded t-structure on $\mathscr{D}$ as in Definition 4.4.4, such that*

$$\inf\left\{\frac{|Z(v(E))|}{\|v(E)\|} : E \in \mathscr{D}^\heartsuit \backslash 0 \text{ is } Z_2\text{-semistable}\right\} > 0.$$

Proof. We follows [19] Proposition 5.3.

By Remark 4.4.9(d), $\mathcal{P}((0,1])$ is the heart of a bounded t-structure on $\mathscr{D}$, so call it $\mathscr{D}^\heartsuit$. By definition, Z_1 maps $\mathscr{D}^\heartsuit$ to complex numbers with argument $\pi/2 \leq \theta \leq \pi$, so Z_1 restricts to a stability condition on $\mathscr{D}^\heartsuit$.

Coversely, suppose $Z_2 : K_0(\mathscr{D}^\heartsuit) \to \mathbb{C}$ is a stability condition on $\mathscr{D}^\heartsuit$. For $\phi \in (0, 1]$, define

$$\mathcal{P}(\phi) = \{E \in \mathscr{D}^\heartsuit : E \text{ is } Z_2\text{-semistable of phase } \phi\}.$$

Then for $\phi \in \mathbb{R}$, let $n := \lceil \phi \rceil - 1$ and $\mathcal{P}(\phi - n)[n]$. Then (and we can check here) $(\mathcal{P}, Z_2)$ is a Bridgeland stability condition. □

Example 4.4.1. *When we consider a nonsingular projective curve X over an algebraically closed field k of characteristic zero. Then define a stability function $Z(E) =$*

$-\deg E + i \operatorname{rank} E$ on $\operatorname{Coh}(X)$. Hence we defined gives a Bridgeland stability condition on the $\mathbf{D}^b(X)$.

Example 4.4.2. *The semistability of quiver representation is also a special case.*

Finally we will give some famous results about deformations of Bridgeland stability. The first step is to equip topology on its space.

Definition 4.4.18. *Let $\mathscr{D}$ be a triangulated category. We define $\operatorname{Slice}(\mathscr{D})$ the set of slicings on $\mathscr{D}$ and $\operatorname{Stab}_\Lambda(\mathscr{D})$ denote the set of stability conditions on $\mathscr{D}$ factoring through a map $v : K_0(\mathscr{D}) \to \Lambda$ (chosen but not specified in the notation).*

Here we need to equip $\operatorname{Stab}(\mathscr{D})$ and $\operatorname{Slice}(\mathscr{D})$ with some canonical topologies. Define a pre-metric $d : \operatorname{Slice}(\mathscr{D}) \times \operatorname{Slice}(\mathscr{D}) \to [0, \infty]$ as:

$$d(\mathcal{P}_1, \mathcal{P}_2) := \sup\{|\phi^+_{\mathcal{P}_1}(E) - \phi^+_{\mathcal{P}_2}(E)|, |\phi^-_{\mathcal{P}_1}(E) - \phi^-_{\mathcal{P}_2}(E)| : E \in \mathscr{D}\}$$

which is reflexive and symmetric, and satisfies the triangle inequality, so the only reason this isn't an actual metric is that it can attain ∞. But this is enough to define a topology on $\operatorname{Slice}(\mathscr{D})$.

We also define the topology on $\operatorname{Stab}_\Lambda(\mathscr{D})$ is the coarsest topology such that two projections $\operatorname{Stab}_\Lambda(\mathscr{D}) \to \operatorname{Slice}(\mathscr{D})$ and $\operatorname{Stab}_\Lambda(\mathscr{D}) \to \operatorname{Hom}(\Lambda, \mathbb{C})$ are continuous, that is, its topology defined by pre-metric $d : \operatorname{Stab}_\Lambda(\mathscr{D}) \times \operatorname{Stab}_\Lambda(\mathscr{D}) \to [0, \infty]$ as:

$$d(\sigma_1, \sigma_2) := \sup\{|\phi^+_{\sigma_1}(E) - \phi^+_{\sigma_2}(E)|, |\phi^-_{\sigma_1}(E) - \phi^-_{\sigma_2}(E)|, \|Z_1 - Z_2\| : E \in \mathscr{D}\}$$

where $\sigma_i = (\mathcal{P}_i, Z_i)$ for $i = 1, 2$.

Theorem 4.4.19 (Bridgeland's Deformation Theorem)**.** *The projection $\operatorname{Stab}_\Lambda(\mathscr{D}) \to \operatorname{Hom}(\Lambda, \mathbb{C})$ is a local homeomorphism, and in particular induces the structure of a complex manifold on (the all connected components of) $\operatorname{Stab}_\Lambda(\mathscr{D})$.*

Proof. We omit this boring proof and we refer [19] or the sketch [43] Theorem 5.15 or more shorter proof [12]. □

Remark 4.4.20. *Note that in this proof we will use a group actions on $\operatorname{Stab}_\Lambda(\mathscr{D})$, here we will give two canonical group actions on it:*

(a) The universal cover $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ of $\mathrm{GL}^+(2, \mathbb{R})$ which consist of (T, f) where $f : \mathbb{R} \to \mathbb{R}$ increasing with $f(\phi + 1) = f(\phi) + 1$ and $T \in \mathrm{GL}^+(2, \mathbb{R})$ such that $f|_{\mathbb{R}/2\mathbb{Z}} = T|_{(R^2 \backslash 0)/\mathbb{R}_{>0}}$. Then (T, f) acts on (P, Z) as

$$(T, f) \cdot (\mathcal{P}, Z) = (\mathcal{P} \circ f, T^{-1} \circ Z).$$

The proof will use this action.

(b) The group of exact autoequivalences $\mathrm{Aut}_\Lambda(\mathscr{D})$*, whose action on* $K_0(\mathscr{D})$ *is compatible with the map* $v : K_0(\mathscr{D}) \to \Lambda$*, acts on the left of* $\mathrm{Stab}_\Lambda(\mathscr{D})$*.*

Proposition 4.4.21. *Assume that* $\sigma = (\mathcal{P}, Z) \in \mathrm{Stab}_\Lambda(\mathscr{D})$ *satisfies the support property with respect to a quadratic form* Q *on* $\Lambda_\mathbb{R}$ *as Remark 4.4.11(b). Consider the open subset of* $\mathrm{Hom}(\Lambda, \mathbb{C})$ *consisting of central charges on whose kernel* Q *is negative definite, and let* U *be the connected component containing* Z*. Let* $\mathcal{U} \subset \mathrm{Stab}_\Lambda(\mathscr{D})$ *be the connected component of the preimage* $Z^{-1}(U)$ *containing* σ*. Then:*

(i) The map $\mathcal{U} \to U$ *is a covering map.*

(ii) Any stability condition $\sigma' \in \mathcal{U}$ *satisfies the support property with respect to the same quadratic form* Q*.*

Proof. See [14] Proposition A.5. □

4.5 Good Moduli Space of Bridgeland Stability

Here we give a introduction of abstract moduli space of Bridgeland (Pre-)Stability. We will follows [8] Example 7.26 and Example 7.29 and section 6.5 in [25].

Recall that we have a special case of Proposition 4.4.17:

Proposition 4.5.1. *Let* $\mathscr{D}$ *be a triangulated category. Then, specifying a Bridgeland pre-stability condition* $\sigma = (\mathcal{P}, Z_1)$ *on* $\mathscr{D}$ *is equivalent to specifying a stability condition* $Z_2 : K_0(\mathscr{D}^\heartsuit) \to \mathbb{C}$ *of* $\mathscr{D}^\heartsuit$ *which is a heart of a bounded* t*-structure on* $\mathscr{D}$ *as in Definition 4.4.4.*

Now we will fix a projective scheme X over an algebraically closed field k of characteristic 0. We will also consider $\mathbf{D}^b(X)$ with a heart of a bounded t-structure $\mathbf{D}^b(X)^\heartsuit$ with $\mathbf{D}_{qc}(X)^\heartsuit = \mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)$ in Definition 4.3.1.

Hence we have defined the moduli stack $\mathscr{M}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}$ in Definition 4.3.7 which is an algebraic stack locally of finite type over k with affine diagonal. By Remark 4.3.13, we know that $\mathscr{M}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}$ is just special case of Definition 2.1.11 when $\mathcal{A} = \mathbf{D}_{qc}(X)^\heartsuit = \mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)$.

Hence we may wish to using the Bridgeland pre-stability in the sense of Proposition 4.5.1 to find a proper good moduli space of components of $\mathscr{M}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}$ since we have the Theorem 2.3.12.

Lemma 4.5.2. *Let* X *be a projective scheme over a field* k*. Let* T *be a connected* k*-scheme of finite type, and let* $E \in \mathbf{D}^b(X_T)$ *be relatively perfect. For any finite type point* $t \in T$*, consider the class*

$$v := \frac{1}{\deg(\kappa(t)/k)}[E_{\kappa(t)}] \in K_0^{\mathrm{num}}(X)_\mathbb{Q}$$

where we regard $E_{\kappa(t)}$ as a complex on X via pushforward along the map $X_{\kappa(t)} \to X$. Then v is independent of the choice of t, and we may write $\mathscr{D}^b_{\mathrm{pug}}(X)$ (respectively $\mathscr{M}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}$) as a disjoint union of open and closed substacks $\mathscr{D}^b_{\mathrm{pug}}(X)^v$ (respectively $\mathscr{M}^v_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}$) parameterizing families of a fixed class $v \in K_0^{\mathrm{num}}(X)_{\mathbb{Q}}$.

Proof. Let $p : X_T \to T$ denotes the projection, by the semicontinuity theorem we find that $p_*((\mathscr{O}_T \boxtimes F) \otimes E) \in \mathrm{Perf}(T)$ for any $F \in \mathrm{Perf}(X)$. Hence the Euler characteristic (hence the mukai pairing)

$$\chi(X_{\kappa(t)}, E_{\kappa(t)} \otimes (\kappa(t) \otimes_k F)) = \deg(\kappa(t)/k)\chi(X, E_{\kappa(t)} \otimes_k F)$$

does not depend on the finite type point $t \in T$, and the claim follows. Finally, this direct;y implies $\mathscr{D}^b_{\mathrm{pug}}(X)^v = \coprod_v \mathscr{D}^b_{\mathrm{pug}}(X)^v$ and hence $\mathscr{M}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)} = \coprod_v \mathscr{M}^v_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}$. Well done. □

If $\mathbf{D}^b(X)^\heartsuit$ is noetherian, then for any finite extension k'/k, one can define a Bridgeland pre-stability condition $(\mathbf{D}^b(X)^\heartsuit_{k'}, Z_{k'})$ where $\mathbf{D}^b(X)^\heartsuit_{k'} \subset \mathbf{D}^b(X_{k'})$ defined as in Definition 4.3.1. And $Z_{k'}(E) = Z(p_*(E))$, where $p_* : \mathbf{D}^b(X_{k'}) \to \mathbf{D}^b(X)$ which satisfied Harder-Narasimhan property by Theorem 6.4.13(3) in [25].

Proposition 4.5.3. *Let X be a projective scheme over field k, and let $\sigma = (\mathbf{D}^b(X)^\heartsuit, Z)$ be a Bridgeland pre-stability condition on $\mathbf{D}^b(X)$ such that Z factored as $Z : K_0(X) \to K_0^{\mathrm{num}}(X) \to \Lambda$ for a finite rank Λ. Assume that:*

(a) $Z(\mathbf{D}^b(X)^\heartsuit) \subset \mathbb{Q} + \mathbb{Q}i$ (that is, σ is an algebraic stability).

(b) $\mathbf{D}^b(X)^\heartsuit$ satisfies the generic flatness property in Definition 4.3.10.

(c) σ satisfies the Boundedness of Quotients condition: if for any $E \in \mathbf{D}^b(X)^\heartsuit$ and any $\phi \in (0, 1)$, the set of points of $\mathscr{M}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}$ that parameterize a torsion-free object $E' \in \mathcal{F}_{k'} \subset \mathbf{D}^b(X)^\heartsuit_{k'}$ over a finite extension k'/k of phase $\le \phi$ that admits a surjection $E \otimes_k k' \to E'$ is bounded.

We define that for any finite type k-scheme T, the groupoid of flat families of torsion-free objects of class $v \in \Lambda \in K_0^{\mathrm{num}}(X)$ in $\mathbf{D}^b(X)^\heartsuit$:

$$\mathscr{M}^v_{\mathcal{F}}(T) := \{E \in \mathscr{M}^v_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)} | E_p \in \mathcal{F}_{\kappa(p)} \subset \mathbf{D}^b(X)^\heartsuit_{\kappa(p)}, \forall \textit{ closed } p \in T\};$$

and the groupoid of flat families of semistable objects of class v in $\mathbf{D}^b(X)^\heartsuit$:

$$\mathscr{M}^{v,\mathrm{ss}}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}(T) := \{E \in \mathscr{M}^v_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)} | E_p \in \mathbf{D}^b(X)^\heartsuit_{\kappa(p)} \textit{ is semistable}, \forall \textit{ closed } p \in T\}.$$

Then these can extend uniquely to open substacks

$$\mathscr{M}^{v,\mathrm{ss}}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)} \subset \mathscr{M}^v_{\mathcal{F}} \subset \mathscr{M}^v_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}$$

on the étale site of all k-schemes.

Moreover, there is a numerical invariant μ induce a Θ-stratification of the algebraic stack $\mathscr{M}^v_{\mathcal{F}}$ such that

- *(i)* $\mathscr{M}^{v,\mathrm{ss}}_{\mathrm{Ind}(\mathbf{D}^b(X)^{\heartsuit})} \subset \mathscr{M}^v_{\mathcal{F}}$ *is the μ-semistable locus.*
- *(ii) the μ-HN filtration of any unstable point in the sense of Definition 1.4.3 corresponds to a canonical weighting of the Harder-Narasimhan filtration in our sense.*

In this case, one can extend the Θ-stratification of $\mathscr{M}^v_{\mathcal{F}}$ to a Θ-stratification of $\mathscr{M}^v_{\mathrm{Ind}(\mathbf{D}^b(X)^{\heartsuit})}$ which is also satisfies the similar properties.

Proof. We omit it and we refer [25] Theorem 6.5.3 and Remark 6.5.6. □

Now we will construct the moduli space. For any $v \in K_0^{\mathrm{num}}(X)$, we define the function:

$$p_v(E) := \deg(E)\,\mathrm{rank}(v) - \deg(v)\,\mathrm{rank}(E)$$

of generalized degree and rank correspond to the stability function Z. Hence the p_v-semistable if and only if Bridgeland pre-semistable!

Theorem 4.5.4. *Let X be a projective scheme over an algebraically closed field k of characteristic zero. Let $\sigma_0 = (\mathbf{D}^b(X)_0^{\heartsuit}, Z_0)$ be a Bridgeland pre-stability condition on $\mathbf{D}^b(X)$ such that Z factored as $Z_0 : K_0(X) \to K_0^{\mathrm{num}}(X) \to \Lambda$ for a finite rank Λ (that is, $\sigma_0 \in \mathrm{PreStab}_\Lambda$). Assume that:*

- *(a) σ_0 is an algebraic stability.*
- *(b) $\mathbf{D}^b(X)_0^{\heartsuit}$ satisfies the generic flatness property in Definition 4.3.10.*
- *(c) σ_0 satisfies the Boundedness of Quotients condition, hence $\mathscr{M}^{v,\mathrm{ss}}_{\mathrm{Ind}(\mathbf{D}^b(X)_0^{\heartsuit})}$ is bounded.*

Let $\mathrm{PreStab}^*_\Lambda$ be a connected component of $\mathrm{PreStab}_\Lambda$ containing σ_0, then for any $\sigma = (\mathbf{D}^b(X)^{\heartsuit}, Z) \in \mathrm{PreStab}^*_\Lambda$ the corresponding moduli stack $\mathscr{M}^{v,\mathrm{ss}}_{\mathrm{Ind}(\mathbf{D}^b(X)^{\heartsuit})}$ has a proper good moduli space.

Proof. We know that by [1] Proposition 5.0.2 we know that if a Bridgeland pre-stability condition $\sigma = (\mathbf{D}^b(X)^{\heartsuit}, Z)$ on $\mathbf{D}^b(X)$ is algebraic, then $\mathbf{D}^b(X)^{\heartsuit}$ is noetherian.

Next, by Proposition 4.12 in [50] these (b)(c) are true for any algebraic stability condition in $\mathrm{PreStab}^*_\Lambda$. Hence for any algebraic stability condition $\sigma = (\mathbf{D}^b(X)^{\heartsuit}, Z) \in \mathrm{PreStab}^*_\Lambda$ the corresponding moduli stack $\mathscr{M}^{v,\mathrm{ss}}_{\mathrm{Ind}(\mathbf{D}^b(X)^{\heartsuit})}$ has a proper good moduli space by Theorem 2.3.12 and Proposition 4.5.3.

Finally we claim that for arbitrary pre-stability condition $\sigma \in \mathrm{PreStab}^*_\Lambda$, one can find an algebraic pre-stability condition σ' which defines the same moduli stack $\mathscr{M}^{v,\mathrm{ss}}_{\mathrm{Ind}(\mathbf{D}^b(X)^{\heartsuit})}$.

To show this claim, fix a class $v \in \Lambda$. For any $v' \in \Lambda$ which is linearly independent of v over $\mathbb{Q}$, consider the real codimension 1 subset (numerical walls, by Theorem 4.4.19)

$$\mathcal{W}_{v'} := \{\sigma = (\mathbf{D}^b(X)^\heartsuit, Z) \in \mathrm{PreStab}^*_\Lambda : Z(v') \in \mathbb{R}_{>0} \cdot Z(v)\}.$$

If one restricts to a small compact neighborhood $\mathcal{B} \subset \mathrm{PreStab}^*_\Lambda$ containing σ, then there is a finite subset $S \subset \Lambda$ such that for any $S' \subset S$ the moduli stack $\mathscr{M}^{v,\mathrm{ss}}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit)}$ is constant for all $\sigma \in \mathcal{C}_{S'} \cap \mathcal{B}$ where

$$\mathcal{C}_{S'} := \left(\bigcup_{v' \in S'} \mathcal{W}_{v'}\right) \setminus \bigcup_{v' \notin S'} \mathcal{W}_{v'}$$

by the similar proof of Wall-Chamber structure which we will see (such as [53] Proposition 2.8 or original [20] Proposition 9.3). Now $\sigma \in \bigcup_{v' \in S'} \mathcal{W}_{v'}$ if and only if that if $W \subset \Lambda_{\mathbb{Q}}$ is the span of v and the $v' \in S'$, then $\dim_{\mathbb{Q}}(Z(W)) = 1$. We can write $Z = Z_1 \oplus Z_2$ under a splitting $\Lambda_{\mathbb{Q}} \cong W \oplus U$, and the condition now amounts to $\mathrm{rank}(Z_1) = 1$. Hence the claim follows from that the rational points are dense in the space of rank 1 real matrices. Well done. □

Conjecture 1. *Let X be a smooth projective variety over $\mathbb{C}$. Let $\sigma_0 = (\mathbf{D}^b(X)^\heartsuit_0, Z_0)$ be a numerical Bridgeland stability condition on $\mathbf{D}^b(X)$ (that is, $\sigma_0 \in \mathrm{Stab}(X)$). Then the moduli stack $\mathscr{M}^{v,\mathrm{ss}}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit_0)}$ is an Artin stack of finite type, and is an open substack of $\mathscr{D}^b_{\mathrm{pug}}(X)^v$.*

If it is, we can again get the proper good moduli space such that the stable locus is a $\mathbb{G}_m$-gerbe.

Remark 4.5.5. *This is true for smooth projective surfaces, smooth projective 3-fold with $\rho(X) = 1$ satisfying Conjecture 3.*

Note that in Proposition 3.26 of [47], they shows that if X is a smooth projective 3-fold with $\rho(X) = 1$ satisfying Conjecture 3 and fix a chern character v, then $\mathscr{M}^{v,\mathrm{ss}}_{\mathrm{Ind}(\mathbf{D}^b(X)^\heartsuit_0)} \cong \underline{\mathrm{Coh}}^{\mathrm{H-ss}}_v(X)$ and similar for stable locus.

4.6 Wall and Chamber Structure

Theorem 4.6.1 (Wall-Chamber). *Let $\mathscr{D}$ be a triangulated category with surjective group homomorphism $K_0(\mathscr{D}) \twoheadrightarrow \Lambda$ is a finite-dimensional lattice. Fix a primitive class $v_0 \in \Lambda$, and an arbitrary set $S \subset \mathscr{D}$ of objects of class v_0. Then there exists a collection of walls $W^S_v(v_0)$ for $v \in \Lambda$, with the following properties:*

(i) Every wall $W^S_v(v_0)$ is a closed submanifold with boundary of real codimension one.

- *(ii) The collection $\{W_v^S(v_0)\}$ is locally finite, that is, every compact subset meets only a finite number of walls.*
- *(iii) For every stability conditions $(Z, \mathcal{P}) \in W_v^S(v_0)$, there exists a phase ϕ and an inclusion $F_v \hookrightarrow E_{v_0}$ in $\mathcal{P}(\phi)$ with $[F_v] = v$ and some $E_{v_0} \in S$.*
- *(iv) If $C \subset \mathrm{Stab}(\mathscr{D})$ is a connected component of the complement of $\bigcup_{v \in \Lambda} W_v^S(v_0)$ and $\sigma_1, \sigma_2 \in C$, then an object $E_{v_0} \in S$ is σ_1-stable if and only if it is σ_2-stable.*

Proof. For a class $v \in \Lambda$, define $V_v^S(v_0)$ be the set of stability conditions satisfies the condition (iii). Define the numerical wall $W_v^{\mathrm{num}}(v_0)$ be the locus of $(Z, \mathcal{P})$ such that $\Im\left(\frac{Z(v)}{Z(v_0)}\right) = 0$. Since v_0 is primitive, we can easy to see $V_v^S(v_0) \subset W_v^{\mathrm{num}}(v_0)$ where $W_v^{\mathrm{num}}(v_0)$ is of real codimension 1 by Theorem 4.4.19.

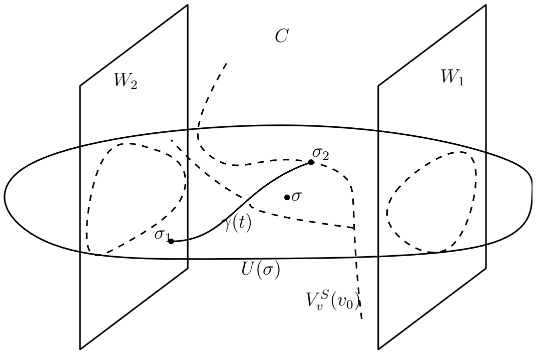
The first step is to show that there only finitely many v for which $V_v^S(v_0)$ intersects a small open ball $U(\sigma)$ of small diameter d around a stability condition $\sigma = (Z, \mathcal{P})$. This follows the support property and we refer [13] Proposition 3.3 or [43] Proposition 5.27. More precisely, let $I_\sigma(S) \subset \Lambda$ be the set of all classes v for which there exists $\phi \in \mathbb{R}$ with $Z(v_0) \in \mathbb{R}_{>0}e^{i\pi\phi}$ and a strict inclusion $F_v \hookrightarrow E$ in the quasi-abelian category $\mathcal{P}((\phi - d, \phi + d))$ with $[F_v] = v$ and $E \in S$. They showed $I_\sigma(S)$ is finite and if $V_v^S(v_0) \cap U(\sigma) \neq \emptyset$, then $v \in I_\sigma(S)$.

Now we know that an object E of class v_0 is $(Z', \mathcal{P}')$-semistable for $(Z', \mathcal{P}') \in U(\sigma)$ if and only if $\Im\left(\frac{Z'(v)}{Z'(v_0)}\right) \leq 0$ for every $v \in I_\sigma(\{E\})$ and it is stable if and only if the inequalities are strict. Repeating this argument for every possible subobject F_v, it follows that inside the codimension one subset $\Im\left(\frac{Z'(v)}{Z'(v_0)}\right) = 0$, the set $V_v^S(v_0)$ is a finite union of subsets, each of which is cut out by a finite number of inequalities of the form $\Im\left(\frac{Z'(v')}{Z'(v_0)}\right) \leq 0$ for some $v' \in I_\sigma(S)$. We let $W_v^S(v_0)$ be the union of all codimension-one components of $V_v^S(v_0)$.

It remains to prove claim (iv) and we only need to consider the situation appear in $U(\sigma) \cap C$. Pick any $\sigma_1, \sigma_2 \in U(\sigma) \cap C$ and we let $E \in S$ is σ_1-stable but not σ_2-stable. Pick a path $\gamma : [0, 1] \to U(\sigma) \cap C$ connecting σ_1, σ_2, then there is a point $\gamma(t)$ on which E is strictly semistable, i.e., $\gamma(t) \in V_v^S(v_0) \cap C$ for some $v \in I_\sigma(S)$ and $t \in (0, 1]$. Now since the codimension of $V_v^S(v_0) \cap C$ at least two, we can choose γ such that for $t \in (0, 1)$, it avoids all of the finitely many non-empty subsets $V_v^S(v_0) \cap C$ for $v \in I_\sigma(S)$, in other words we have that E is $\gamma(t)$-stable for $t \in (0, 1)$, and $\sigma_2 \in V_v^S(v_0) \cap C$ for some $v \in I_\sigma(S)$.

Hence σ_2 is contained in the set $\Im\left(\frac{Z(v)}{Z(v_0)}\right) = 0$ and E will not be stable in the subset of $U(\sigma) \cap C$ with $\Im\left(\frac{Z(v)}{Z(v_0)}\right) \geq 0$. On the other hand, the set $C \backslash \bigcup V_v^S(v_0)$ is path-connected, and by the previous argument E is stable on all of it. This is a contradiction. □

Remark 4.6.2. *We have some remarks:*



(a) In this proof we showed in the last step that higher-codimension components of $V_v^S(v_0)$ always come from objects E_{v_0} that are semistable on this component, and unstable at any nearby point.

(b) If v_0 is not primitive, then these $W_v^S(v_0)$ may have higher codimension! In this case the wall-crossing may not work since we can connect some lines in chambers without passing through the walls as in the proof of (iv).

(c) Some easy arguments can show that the the semistablility can be divided by the numerical walls $W_v^{\text{num}}(v_0)$ as in [20] Proposition 9.3.

This is used in the proof of Theorem 4.5.4, which shows that if one restricts to a small compact neighborhood $\mathcal{B} \subset \operatorname{Stab}_\Lambda^*$ containing σ, then there is a finite subset $S \subset \Lambda$ such that for any $S' \subset S$ the moduli stack $\mathscr{M}^{v,\text{ss}}_{\operatorname{Ind}(\mathbf{D}^b(X)^\heartsuit)}$ is constant for all $\sigma \in C_{S'}(v) \cap \mathcal{B}$ where

$$C_{S'}(v) := \left(\bigcup_{v' \in S'} W_{v'}^{\text{num}}(v) \right) \setminus \bigcup_{v' \notin S'} W_{v'}^{\text{num}}(v).$$

This gives us a nice intuitive explanation.

4.7 Tilting of t-structures

Definition 4.7.1. Let $\mathscr{A}$ be an abelian category and for two additive full subcategories $T, F \subset \mathscr{A}$, we call $\alpha = (T, F)$ is a torsion pair if:

(a) $\operatorname{Hom}(T, F) = 0$.

(b) For any $X \in \mathscr{A}$ there exists $Y \in T, Z \in F$ such that we have the exact sequence $0 \to Y \to X \to Z \to 0$.

Remark 4.7.2. Using (a) one can show that the objects in (b) are unique.

Example 4.7.1. *Let X be a smooth projective variety, then $T = \{\textit{torsion sheaves}\}$ and $F = \{\textit{torsion-free sheaves}\}$ forms a torsion pair of* $\mathrm{Coh}(X)$.

Proposition 4.7.3 (Tilting of t-structures). *Let $\mathscr{D}$ be a triangulated category and $\mathscr{D}^{\heartsuit}$ be the heart of a bounded t-structure on $\mathscr{D}$. Let $\alpha = (T, F)$ be a torsion pair for $\mathscr{D}^{\heartsuit}$. Then*

$$^{\alpha}\mathscr{D}^{\heartsuit} := \{K \in \mathscr{D} : \textit{for } i \neq 0, -1 \textit{ we have } H^i_{\mathscr{D}^{\heartsuit}}(K) = 0, H^0_{\mathscr{D}^{\heartsuit}}(K) \in T, H^{-1}_{\mathscr{D}^{\heartsuit}}(K) \in F\}$$

is the heart of a bounded t-structure on $\mathscr{D}$.

Proof. See the proof of Lemma 6.3 in [43] or Lemma 20.28 in [M392cBrSt] or [17] Claim 9.3. □

Remark 4.7.4. *Some remarks:*

- *(a) Actually $^{\alpha}\mathscr{D}^{\heartsuit}$ is the smallest extension-closed additive subcategory of $\mathscr{D}$ containing both T and $F[1]$, that is, $^{\alpha}\mathscr{D}^{\heartsuit} = \langle T, F[1]\rangle$.*
- *(b) Let A and B be hearts of two t-structures on a triangulated category $\mathscr{D}$, and suppose $A \subset \langle B, B[1]\rangle$. If we define $T := A \cap B$ and $F := B \cap A[-1]$, then (T, F) is a torsion pair of B Hence A is a tilt of B.*

4.8 Construction I–Surfaces

Here we just consider the projective smooth varieties over $\mathbb{C}$.

4.8.1 Twisted Chern Character and Twisted Stability

Let X be a smooth projective variety over $\mathbb{C}$ of dimension $n \geq 2$. We fix an ample divisor class $\omega \in N^1(X)$ and another divisor class $B \in N^1(X)$. Here $N^1(X) := \mathrm{NS}(X)_{\mathbb{R}}$.

Definition 4.8.1. *We define the B-twisted Chern character as*

$$\mathrm{ch}^B := \mathrm{ch} \cdot e^{-B}.$$

Remark 4.8.2. *Hence* $\mathrm{ch}^B_0 = \mathrm{rank}$, $\mathrm{ch}^B_1 = \mathrm{ch}_1 - B \cdot \mathrm{ch}_0$ *and* $\mathrm{ch}^B_2 = \mathrm{ch}_2 - B \cdot \mathrm{ch}_1 + \frac{B^2}{2}\mathrm{ch}_0$.

Definition 4.8.3 (Gieseker-Maruyama-Simpson Stability). *Let $\mathscr{E} \in \mathrm{Coh}(X)$ be a pure sheaf of dimension d.*

- *(a) The B-twisted Hilbert polynomial is*

$$P(\mathscr{E}, B; t) = \int_X \mathrm{ch}^B(\mathscr{E}) \cdot e^{t\omega} \cdot \mathrm{td}(T_X) = \sum_i a_i(\mathscr{E}, B) t^i.$$

(b) *We say that $\mathscr{E}$ is B-twisted Gieseker (semi)stable if, for any proper non-trivial subsheaf $\mathscr{F} \subset \mathscr{E}$, the inequality $\frac{P(\mathscr{F},B;t)}{a_d(\mathscr{F},B)} < (\leq) \frac{P(\mathscr{E},B;t)}{a_d(\mathscr{E},B)}$ holds for $t \gg 0$ as before.*

Definition 4.8.4 (Twisted Slope and Stability)**.** *Let $\mathscr{E} \in \mathrm{Coh}(X)$.*

(a) *The B-twisted slope is*

$$\mu_{\omega,B}(\mathscr{E}) := \frac{\omega^{n-1} \cdot \mathrm{ch}_1^B(\mathscr{E})}{\omega^n \cdot \mathrm{ch}_0^B(\mathscr{E})} = \frac{\omega^{n-1} \cdot \mathrm{ch}_1(\mathscr{E})}{\omega^n \cdot \mathrm{ch}_0(\mathscr{E})} - \frac{\omega^{n-1} \cdot B}{\omega^n}$$

where dividing by 0 is interpreted as $+\infty$.

(b) *is called B-twisted slope (semi)stable if for all proper subsheaves $\mathscr{F} \subset \mathscr{E}$ the inequality $\mu_{\omega,B}(\mathscr{F}) < (\leq) \mu_{\omega,B}(\mathscr{E}/\mathscr{F})$ holds.*

Remark 4.8.5. *Note that:*

(a) *Our definition of slope (semi)stability coincides with the classical definition if $\mathscr{E}$ has positive rank.*

(b) *The B-twisted slope function satisfies the weak see-saw property, i.e., for any exact sequence $0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{E}/\mathscr{F} \to 0$ in $\mathrm{Coh}(X)$ with $\mathscr{F}, \mathscr{E}/\mathscr{F} \neq 0$, one of the following conditions holds:*

$$\mu_{\omega,B}(\mathscr{F}) \leq \mu_{\omega,B}(\mathscr{E}) \leq \mu_{\omega,B}(\mathscr{E}/\mathscr{F});$$
$$\mu_{\omega,B}(\mathscr{F}) \geq \mu_{\omega,B}(\mathscr{E}) \geq \mu_{\omega,B}(\mathscr{E}/\mathscr{F}).$$

which follows from that if $\mathrm{ch}_0^B(\mathscr{F}) = 0$, then ${}_{n-1}\,\mathrm{ch}_1^B(\mathscr{F}) \geq 0$, and similarly for $\mathscr{E}/\mathscr{F}$.

(c) *For B-twisted Gieseker stability and B-twisted slope stability, we also have the Harder-Narasimhan filtration.*

4.8.2 Main Construction and Results for Surfaces

Coherent sheaves will never work in dimension ≥ 2 as follows.

Proposition 4.8.6. *Let X be a smooth projective variety with $d = \dim X \geq 2$. Then there is no numerical Bridgeland stability condition $(Z, \mathrm{Coh}(X))$ on $\mathbf{D}^b(X)$.*

Proof. We will follows [54] Lemma 2.7 with some corrections. If there is such stability condition $Z : K_0(X) \to \mathbb{C}$, we may assume that it is of form

$$Z(\mathscr{E}^*) = \sum_{j=0}^{d} \int_X (u_j + iv_j)\, \mathrm{ch}_j(\mathscr{E}^*)$$

where $u_j + iv_j \in \mathrm{CH}^{2d-2j}(X)_{\mathbb{C}}$. After choose a smooth subvariety $S \subset X$ of dimension 2, we may assume $d = 2$. Let C be a smooth curve over X such that it has positive self-intersection (by Bertini's theorem) and consider a divisor $D = mC$ support on C, then as Z is a stability function we have

$$\begin{aligned}\Im Z(\mathscr{O}_C(D)) &= \int_X (v_1 \operatorname{ch}_1(\mathscr{O}_C) + v_2(\operatorname{ch}_2(\mathscr{O}_C) + \operatorname{ch}_1(\mathscr{O}_C)c_1(\mathscr{O}_X(D))))\\ &= \int_X v_1 c_1(\mathscr{O}(C)) + \int_X v_2(\operatorname{ch}_2(\mathscr{O}_C) + mc_1(\mathscr{O}(C))^2)\\ &= v_1 \cdot [C] + \int_X v_2(\operatorname{ch}_2(\mathscr{O}_C) + mc_1(\mathscr{O}(C))^2) \geq 0.\end{aligned}$$

Let $m \to \pm\infty$ we get $v_2 = 0$. Again as $\Im Z(\mathscr{O}_X(D)) = m\int_X v_1 \cdot c_1(\mathscr{O}_X(C)) + \int_X v_0 \geq 0$ we have

$$\int_X v_1 \cdot c_1(\mathscr{O}_X(C)) = 0.$$

Hence $\Im Z(\mathscr{O}_C(D)) = 0$. Hence

$$\Re Z(\mathscr{O}_C(D)) = u_1 \cdot [C] + \int_X u_2(\operatorname{ch}_2(\mathscr{O}_C) + mc_1(\mathscr{O}(C))^2) < 0.$$

By $m \to \pm\infty$ again we get $u_2 = 0$.

Then for any $\iota : x \in X$, by Grothendieck-Riemann-Roch we get

$$\operatorname{ch}(\iota_*\mathscr{O}_x)\operatorname{td}(T_X) = \iota_*(\operatorname{ch}(\mathscr{O}_x)\operatorname{td}(T_{\{x\}})) = [x].$$

As $\operatorname{ch}_0(\iota_*\mathscr{O}_x) = 0$ by trivial reason, we find that $\operatorname{ch}(\iota_*\mathscr{O}_x) = [x]$. Hence

$$Z(\iota_*\mathscr{O}_x) = \int_X (u_2 + iv_2) \cap [x] = 0$$

which is impossible as $\operatorname{Im}(Z) \in \mathbb{H} \cup \mathbb{R}_{<0}$. □

Remark 4.8.7. *Some interesting remarks:*

- (a) *Hence we find that skyscraper sheaves are the very strong obstruction to find a stability condition.*
- (b) *In the final step we may not using the GRR. We can show it directly as in the proof of GRR: Pick a very ample divisor H on X we define two sections which determined $c_1(\mathscr{O}_X(H))^2 \cap [X] = [S]$ for $S \in Z_0(X)$. Hence we have the Koszul resolution*

$$0 \to \bigwedge^2 \mathscr{O}_X(-H)^{\oplus 2} \to \mathscr{O}_X(-H)^{\oplus 2} \to \mathscr{O}_X \to \mathscr{O}_S \to 0.$$

Hence $\mathrm{ch}(\mathscr{O}_S) = c_1(\mathscr{O}_X(H))^2$ *and we get*

$$Z(\mathscr{O}_S) = \int_X (u_2 + iv_2)c_1(\mathscr{O}_X(H))^2 = 0$$

which is impossible as $\mathrm{Im}(Z) \in \mathbb{H} \cup \mathbb{R}_{<0}$.

Hence instead, we will look for other abelian categories inside $\mathbf{D}^b(X)$ using tilt of hearts above. Fix an ample divisor class $\omega \in N^1(X)$ and another divisor class $B \in N^1(X)$ on the smooth projective surface X over $\mathbb{C}$. For $\mathrm{Coh}(X)$ we define: B-twisted slope stability

$$\begin{aligned} T_{\omega,B} &= \{\mathscr{E} \in \mathrm{Coh}(X) : \text{ any semistable factor } \mathscr{F} \text{ of } \mathscr{E} \text{ satisfies } \mu_{\omega,B}(\mathscr{F}) > 0\} \\ &= \{\mathscr{E} \in \mathrm{Coh}(X) : \mu^-_{\omega,B}(\mathscr{E}) > 0\}; \\ F_{\omega,B} &= \{\mathscr{E} \in \mathrm{Coh}(X) : \text{ any semistable factor } \mathscr{F} \text{ of } \mathscr{E} \text{ satisfies } \mu_{\omega,B}(\mathscr{F}) \leq 0\} \\ &= \{\mathscr{E} \in \mathrm{Coh}(X) : \mu^+_{\omega,B}(\mathscr{E}) \leq 0\}, \end{aligned}$$

where for any $\mathscr{E} \in \mathrm{Coh}(X)$ we have a unique Harder-Narasimhan filtration with repsected to $\mu_{\omega,B}$:

$$\mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_n = \mathscr{E}$$

where $\mathscr{E}_0$ is the torsion part and $\mathscr{E}_i/\mathscr{E}_{i-1}$ is a torsion-free ω-Gieseker-semistable sheaf of slope $\mu_{\omega,B,i}$ which are strictly descending with respect to i. We hence define $\mu^+_{\omega,B}(\mathscr{E})$ and $\mu^-_{\omega,B}(\mathscr{E})$ as before.

Remark 4.8.8. *(a) Note that* $T_{\omega,B}$ *contains all torsion sheaves.*

(b) For any B*, the* $\mu_{\omega,B}$ *define the* ***same stability*** *which the only difference is a constant* $\frac{\omega\cdot B}{\omega^2}$ *of slopes.*

Proposition 4.8.9. *Now* $(T_{\omega,B}, F_{\omega,B})$ *is a torsion pair of* $\mathrm{Coh}(X)$ *which induce the tilted heart* $\mathrm{Coh}^{\omega,B}(X) = \langle T_{\omega,B}, F_{\omega,B}[1]\rangle$.

Proof. By the samilar argument in Lemma 3.2.4(i) we can see that $\mathrm{Hom}(T_{\omega,B}, F_{\omega,B}) = 0$. Next pick any $\mathscr{E} \in \mathrm{Coh}(X)$, we need to find $\mathscr{T} \in T_{\omega,B}$ and $\mathscr{F} \in F_{\omega,B}$ filling the exact sequence

$$0 \to \mathscr{T} \to \mathscr{E} \to \mathscr{F} \to 0.$$

Actually by the unique Harder-Narasimhan filtration

$$\mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_n = \mathscr{E}$$

we can find i such that $\mu_{\omega,B}(\mathscr{E}_i/\mathscr{E}_{i-1}) > 0$ but $\mu_{\omega,B}(\mathscr{E}_{i+1}/\mathscr{E}_i) \leq 0$. Hence we can choose $\mathscr{T} = \mathscr{E}_i$ and $\mathscr{F} = \mathscr{E}/\mathscr{E}_i$ and well done. $\square$

Now for any $E \in \mathbf{D}^b(X)$ we set

$$Z_{\omega,B}(E) = -\int_X e^{i\omega} \cdot \mathrm{ch}^B(E) = \Big(-\mathrm{ch}_2^B(E) + \frac{\omega^2}{2}\,\mathrm{ch}_0^B(E)\Big) + i\omega \cdot \mathrm{ch}_1^B(E)$$

and the corresponding slope function

$$\nu_{\omega,B}(E) = \frac{\mathrm{ch}_2^B(E) - \frac{\omega^2}{2}\,\mathrm{ch}_0^B(E)}{\omega \cdot \mathrm{ch}_1^B(E)}.$$

We will set $\Lambda = K_0^{\mathrm{num}}(X)$ by Remark 4.4.13 and 4.4.14. Note that in the case of surface here, $\Lambda = K_0^{\mathrm{num}}(X)$ is the image of

$$\mathrm{ch} : K_0(X) \to \mathrm{CH}^*(X) \to H^*(X, \mathbb{Q}),$$

hence $v : K_0(X) \to \Lambda$ is just ch.

Lemma 4.8.10. *Let $\omega \in N^1(X)$ be an ample real divisor class. Then there exists a constant $C_\omega \geq 0$ such that, for every effective divisor $D \subset X$, we have*

$$C_\omega(\omega \cdot D)^2 + D^2 \geq 0.$$

Proof. Fix a norm $\| * \|$ on $N^1(X)$, then there is $A \geq 0$ such that $-(D/\|D\|)^2 \leq A$. Hence $-D^2 \leq A\|D\|^2$. On the other hand, as the ample cone is open we find that there is $B_\omega \geq 0$ such that $\omega \cdot (D/\|D\|) \geq B_\omega$. Let $C_\omega = A/B_\omega^2$ we get

$$C_\omega(\omega \cdot D)^2 \geq A\|D\|^2 \geq -D^2$$

and well done. □

Definition 4.8.11. *Let $\omega, B \in N^1(X)$ with ω ample. We define the discriminant function as*

$$\Delta := (\mathrm{ch}_1^B)^2 - 2\,\mathrm{ch}_0^B\,\mathrm{ch}_2^B = (\mathrm{ch}_1)^2 - 2\,\mathrm{ch}_0\,\mathrm{ch}_2\,.$$

We define the ω-discriminant as

$$\overline{\Delta}_\omega^B := (\omega \cdot \mathrm{ch}_1^B)^2 - 2\omega^2\,\mathrm{ch}_0^B\,\mathrm{ch}_2^B\,.$$

Choose a rational non-negative constant C_ω as in Lemma 4.8.10 above. Then we define the (ω, B, C_ω)-discriminant as

$$\Delta_{\omega,B}^{C_\omega} := \Delta + C_\omega(\omega \cdot \mathrm{ch}_1^B)^2.$$

Now the following two theorems are our main results in this section.

Theorem 4.8.12 ([20],[13],[9]). *Let X be a smooth projective surface over $\mathbb{C}$. Fix an ample divisor class $\omega \in N^1(X)$ and another divisor class $B \in N^1(X)$. Then the pair $\sigma_{\omega,B} = (\mathrm{Coh}^{\omega,B}(X), Z_{\omega,B})$ gives a Bridgeland stability condition on X. Moreover, the map*

$$\mathrm{Ample}(X)_{\mathbb{R}} \times N^1(X) \to \mathrm{Stab}(X), \quad (\omega, B) \mapsto \sigma_{\omega,B}$$

is a continuous embedding.

The following theorem is just the support property:

Theorem 4.8.13. *Let X be a smooth projective surface over $\mathbb{C}$. Let $\omega, B \in N^1(X)$ with ω ample. Assume that E is $\sigma_{\omega,B}$-semistable. Then*

$$\Delta^{C_\omega}_{\omega,B}(E) \geq 0, \quad \overline{\Delta}^B_\omega(E) \geq 0.$$

4.8.3 Sketch of the Proof

Lemma 4.8.14 (Bogomolov Inequality). *Let X be a smooth projective surface over $\mathbb{C}$. Let $\omega, B \in N^1(X)$ with ω ample, and let $\mathscr{E}$ be a $\mu_{\omega,B}$-semistable torsion-free sheaf. Then*

$$\Delta(\mathscr{E}) = \mathrm{ch}_1^B(\mathscr{E})^2 - 2\,\mathrm{ch}_0^B(\mathscr{E})\,\mathrm{ch}_2^B(\mathscr{E}) \geq 0.$$

Proof. Since the $\mu_{\omega,B}$-stability and Δ are both not depend on B, we may assume $B = 0$. By Lemma 4.C.5 in [29] the slope stability with respect to an ample divisor changes only at integral classes. Hence we can assume $\omega = H$ is a very ample integral divisor. The the remains are classical and we refer [29] Theorem 3.4.1 and [43] Theorem 6.14. □

Remark 4.8.15. *Let $\mathscr{E}$ be a torsion sheaf. Then $\Delta^C_{\omega,B}(\mathscr{E}) \geq 0$. Indeed, in this case $\mathrm{ch}_0(\mathscr{E}) = 0$ and hence this follows from the definition of C_ω and $\Delta^C_{\omega,B}$.*

Lemma 4.8.16. *Let $T \in T_{\omega,B}$ and $F \in F_{\omega,B}$ be nonzero.*

(i) We have $\omega \cdot \mathrm{ch}_1^B(F) \leq 0$ and $\omega \cdot \mathrm{ch}_1^B(T) \geq 0$.

(ii) If $\omega \cdot \mathrm{ch}_1^B(F) = 0$, then F is $\mu_{\omega,B}$-semistable and $\mathrm{ch}_2^B(F) \leq 0$.

(iii) If $\omega \cdot \mathrm{ch}_1^B(T) = 0$, then $\dim \mathrm{supp}(T) = 0$; in particular, $\mathrm{ch}_0^B(T) = \mathrm{ch}_1^B(T) = \mathrm{ch}_1(T) = 0$, and $\mathrm{ch}_2^B(T) = \mathrm{ch}_2(T) > 0$.

Proof. For (i), as $T \in T_{\omega,B}$ then $\mu_{\omega,B}(T) \leq 0$ and hence $\omega \cdot \mathrm{ch}_1^B(F) \leq 0$ since $\omega^2\,\mathrm{ch}_0^B(T) > 0$. Next we will show $\omega \cdot \mathrm{ch}_1^B(T) \geq 0$. If $\mathrm{ch}_0^B(T) = 0$, then the $\mathrm{supp}(T)$ has class $\mathrm{ch}_1^B(T)$ with some points. In this case $\mathrm{ch}_1^B(T) = \mathrm{ch}_1(T)$ which is effective by GRR, hence $\omega \cdot \mathrm{ch}_1^B(T) \geq 0$. If $\mathrm{ch}_0^B(T) > 0$, then $\mu_{\omega,B}(T) > 0$ by definition. Hence $\omega \cdot \mathrm{ch}_1^B(T) > 0$.

For (iii), if $\omega \cdot \mathrm{ch}_1^B(T) = 0$ then by the proof of (i) we find that $\mathrm{ch}_0^B(T) = \mathrm{ch}_1^B(T) = \mathrm{ch}_1(T) = 0$. Hence in this case $\mathrm{supp}(T)$ are some points on $\mathrm{ch}_2^B(T) = \mathrm{ch}_2(T)$ and hence $\dim \mathrm{supp}(T) = 0$. As $T \neq 0$, then $\mathrm{ch}_2^B(T) = \mathrm{ch}_2(T) > 0$ by GRR for $i : \mathrm{supp}(T) \subset X$.

For (ii), we know that now $\mu_{\omega,B}(F) = 0$. But by definition we find that $\mu_{\omega,B}(F) = 0 \leq \mu^{+}_{\omega,B}(F)$ which shows F is $\mu_{\omega,B}$-semistable. Finally by Bogomolov Inequality 4.8.14 we have

$$\Delta(F) = -2\operatorname{ch}_0^B(F)\operatorname{ch}_2^B(F) \geq 0,$$

hence $\operatorname{ch}_2^B(F) \leq 0$ and we get the results. □

Proposition 4.8.17. *The group homomorphism*

$$Z_{\omega,B}(E) = -\int_X e^{i\omega}\cdot \operatorname{ch}^B(E) = \left(-\operatorname{ch}_2^B(E) + \frac{\omega^2}{2}\operatorname{ch}_0^B(E)\right) + i\omega\cdot\operatorname{ch}_1^B(E)$$

is a stability function on $\operatorname{Coh}^{\omega,B}(X)$.

Proof. By definition we know that $\Im Z_{\omega,B}(E) \geq 0$ for any $E \in \operatorname{Coh}^{\omega,B}(X)$. Now let $E \in \operatorname{Coh}^{\omega,B}(X)$ such that $\Im Z_{\omega,B}(E) = 0$, we will show that $\Re Z_{\omega,B}(E) < 0$.

Actually we have a distinguished triangle

$$\mathcal{H}^{-1}(E)[1] \to E \to \mathcal{H}^0(E) \to$$

where $\mathcal{H}^{-1}(E) \in F_{\omega,B}$ and $\mathcal{H}^0(E) \in T_{\omega,B}$. As $\Im Z_{\omega,B}(E) = 0$ we find that

$$0 \geq \omega\cdot\operatorname{ch}_1^B(\mathcal{H}^{-1}(E)) = \omega\cdot\operatorname{ch}_1^B(\mathcal{H}^0(E)) \geq 0$$

by Lemma 4.8.16(i). Hence by Lemma 4.8.16(ii)(iii) we get $\mathcal{H}^{-1}(E)$ is $\mu_{\omega,B}$-semistable and $\operatorname{ch}_2^B(\mathcal{H}^{-1}(E)) = 0$ and $\dim\operatorname{supp}(\mathcal{H}^0(E)) = 0$; in particular, $\operatorname{ch}_1^B(\mathcal{H}^0(E)) = 0$ and $\operatorname{ch}_2^B(\mathcal{H}^0(E)) > 0$.

To show $\Re Z_{\omega,B}(E) < 0$, we just need to show that $\Re Z_{\omega,B}(\mathcal{H}^{-1}(E)[1]) < 0$ and $\Re Z_{\omega,B}(\mathcal{H}^0(E)) < 0$ by additivity. Now as $\dim\operatorname{supp}(\mathcal{H}^0(E)) = 0$, we have

$$\Re Z_{\omega,B}(\mathcal{H}^0(E)) = -\operatorname{ch}_2^B(\mathcal{H}^0(E)) < 0.$$

On the other hand, by the Hodge Index Theorem and $\omega\cdot\operatorname{ch}_1^B(\mathcal{H}^{-1}(E)) = 0$ we have $\operatorname{ch}_1^B(\mathcal{H}^{-1}(E))^2 \leq 0$. Hence by Bogomolov Inequality 4.8.14 we have $\operatorname{ch}_2^B(\mathcal{H}^{-1}(E)) \leq 0$. Hence

$$\Re Z_{\omega,B}(\mathcal{H}^{-1}(E)) = -\operatorname{ch}_2^B(\mathcal{H}^{-1}(E)) + \frac{\omega^2}{2}\operatorname{ch}_0^B(\mathcal{H}^{-1}(E)) > 0.$$

Hence $\Re Z_{\omega,B}(\mathcal{H}^{-1}(E)[1]) < 0$ and well done. □

Next we need to show the Harder-Narasimhan filtration exist for $Z_{\omega,B}$ in $\operatorname{Coh}^{\omega,B}(X)$ and they satisfies the support property.

For Rational Coefficient Case

We now let $B \in \mathrm{NS}(X)_{\mathbb{Q}}$ and $\omega = \alpha H$ for $\alpha \in \mathbb{R}_{>0}$ and H is an integral ample divisor class. For the existence of Harder-Narasimhan filtration this is pure categorial:

Lemma 4.8.18. *For an abelian category $\mathcal{A}$ with a stability function $Z : K_0(\mathcal{A}) \to \mathbb{C}$ such that $\mathcal{A}$ noetherian. If the image of $\Im Z$ is discrete in $\mathbb{R}$, then Harder-Narasimhan filtrations exist in $\mathcal{A}$ with respect to Z.*

Proof. Pure categorial and we omit it. We refer [43] Proposition 4.10. □

Proposition 4.8.19. *In our case, the category $\mathrm{Coh}^{\omega,B}(X)$ is noetherian and the Harder-Narasimhan filtration exist for $Z_{\omega,B}$ in $\mathrm{Coh}^{\omega,B}(X)$.*

Proof. Actually by definition the image of $\Im Z_{\omega,B}$ is discrete in $\mathbb{R}$. Now the category $\mathrm{Coh}^{\omega,B}(X)$ is noetherian follows from a boring argument. It was first observed in the case of K3 surfaces in Proposition 7.1 of [20] and the proof in general case we refer [50] Lemma 2.17. □

Lemma 4.8.20. *Let $\omega, B \in N^1(X)$ with ω ample. If $E \in \mathrm{Coh}^{\omega,B}(X)$ is $\sigma_{\alpha\cdot\omega,B}$-semistable for all $\alpha \gg 0$, then it satisfies one of the following conditions:*

(a) $\mathcal{H}^{-1}(E) = 0$ and $\mathcal{H}^0(E)$ is a $\mu_{\omega,B}$-semistable torsion-free sheaf.

(b) $\mathcal{H}^{-1}(E) = 0$ and $\mathcal{H}^0(E)$ is a torsion pure sheaf.

(c) $\mathcal{H}^{-1}(E) = 0$ is a $\mu_{\omega,B}$-semistable torsion-free sheaf and $\mathcal{H}^0(E)$ is either 0 or a torsion sheaf supported in dimension zero.

Proof. Note that by the similar proof of Lemma 4.8.16 we can easy to see that if $\omega \cdot \mathrm{ch}_1^B(E) = 0$ this is trivial. For other condition we omitted and we refer Lemma 6.18 in [43]. □

Proof of Rational Case of Theorem 4.8.13 (and hence 4.8.12). Here we will just give a sketch. As $\Im Z_{\omega,B}$ is discrete, we could use the induction on $H \cdot \mathrm{ch}_1^B$ as $\omega = \alpha H$ for an ample integral class H where $\alpha \in \mathbb{R}_{>0}$.

Let $E \in \mathrm{Coh}^{\omega,B}(X)$ be $\sigma_{\alpha_0\omega,B}$-semistable for some $\alpha_0 > 0$ such that $H \cdot \mathrm{ch}_1^B(E) > 0$ is minimal. Use the following result:

- In this case we have

$$c := \min\left\{H \cdot \mathrm{ch}_1^B(F) : F \in \mathrm{Coh}^{\omega,B}(X), H \cdot \mathrm{ch}_1^B(F) > 0\right\} > 0$$

  exists. Let $E \in \mathrm{Coh}^{\omega,B}(X)$ satisfy $H \cdot \mathrm{ch}_1^B(E) = c$ and $\mathrm{Hom}(A, E) = 0$ for all $A \in \mathrm{Coh}^{\omega,B}(X)$ with $H \cdot \mathrm{ch}_1^B(A) = 0$, then E is $\sigma_{\omega,B}$-stable.

Then E is $\sigma_{\alpha H,B}$-stable for $\alpha \gg 0$. Hence by Lemma 4.8.20 and Bogomolov inequality the Theorem holds.

The induction step is not easy. If $E \in \mathrm{Coh}^{\omega,B}(X)$ is also $\sigma_{\alpha H,B}$-stable for $\alpha \gg 0$, then we again get the result. If not, let α' denote the largest value of α such that E is $\sigma_{\alpha' H,B}$-semistable. We get the short exact sequence in $\mathrm{Coh}^{\omega,B}$

$$0 \to A \to E \to B \to 0$$

such that $\nu_{\alpha' H,B}(A) = \nu_{\alpha' H,B}(E)$ and $\nu_{\alpha H,B}(A) > \nu_{\alpha H,B}(E)$ for $\alpha > \alpha'$. But one can show that in this case $H \cdot \mathrm{ch}_1^B(A), H \cdot \mathrm{ch}_1^B(B) < H \cdot \mathrm{ch}_1^B(E)$, hence by induction they satisfy the desired inequality. Then by the similar argument of wall-crossing structure we can get the result. □

For General Coefficient Case

Lemma 4.8.21. *Let X be a quasi-projective scheme of finite type over $\mathbb{C}$ and $E \in \mathbf{D}^b(X)$. If $\mathrm{Ext}^i(E, \mathscr{O}_x) = 0$ for all $x \in X$ and $i < 0$ and $i > s \in \mathbb{Z}$. Then E is quasi-isomorphic to a complex F^* of locally free sheaves such that $F^i = 0$ for $i > 0$ and $i < -s$.*

Proof. One omitted and we refer the original [18] Proposition 5.4. □

Theorem 4.8.22 (Bridgeland)**.** *Let $\sigma = (\mathcal{A}, Z_{\omega,B})$ be a numerical Bridgeland stability condition on X such that all skyscraper sheaves are σ-stable. Then $\mathcal{A} = \mathrm{Coh}^{\omega,B}(X)$.*

Proof. Let $E \in \mathcal{A}$ and we first claim that $E \in \langle \mathrm{Coh}(X), \mathrm{Coh}(X)[1] \rangle$. Notice that E is an iterated extension of σ-stable objects, so we can assume that E is σ-stable. As $\mathscr{O}_x \in \mathrm{Coh}^{\omega,B}(X)$ for all $x \in X$ which is also σ-stable, we may assume $E \neq \mathscr{O}_x$ for any $x \in X$. Hence by Serre duality we get $\mathrm{Ext}^i(E, \mathscr{O}_x) = 0$ for all $i \neq 0, 1$. Hence by Lemma 4.8.21 we get E is quasi-isomorphic to a complex $0 \cdots \to F^{-1} \to F^0 \to 0$ where F^i are all locally free. Hence we get the claim and $\mathcal{A} \subset \langle \mathrm{Coh}(X), \mathrm{Coh}(X)[1] \rangle$. Moreover one shows that $\mathcal{H}^{-1}(E)$ is torsion free.

By Remark 4.7.4(b) we know that $\mathcal{A} = \langle T, F[1] \rangle$ is a tilting of $\mathrm{Coh}(X)$ where $T = \mathcal{A} \cap \mathrm{Coh}(X)$ and $F = \mathcal{A}[-1] \cap \mathrm{Coh}(X)$. To show $\mathcal{A} = \mathrm{Coh}^{\omega,B}(X)$ we just need to show $T_{\omega,B} \subset T$ and $F_{\omega,B} \subset F$.

Let $E \in \mathrm{Coh}(X)$ be $\mu_{\omega,B}$-semistable, then we just need to show that if $\mu_{\omega,B} \leq 0$, then $E \in F$, and if $\mu_{\omega,B} > 0$, then $E \in T$. Choose $T_E \in T$ and $F_E \in F$ such that

$$0 \to T_E \to E \to F_E \to 0.$$

As $F_E[1] \in F[1] \subset \mathcal{A}$, we know that $F_E = \mathcal{H}^{-1}(F_E[1])$ is torsion free.

When E is torsion, then $F_E = 0$ and $E = T_E \in T$.

When E is torsion-free, then $F_E[1] \in \mathcal{A}$, implying $\Im Z_{\omega,B}(F_E[1]) = -\omega \operatorname{ch}_1^B(F_E) \geq 0$. Similarly $T_E \in \mathcal{A}$ implying $\Im Z_{\omega,B}(T_E) = \omega \operatorname{ch}_1^B(T_E) \geq 0$. This implies by definition that $\mu_{\omega,B}(F_E) \leq 0$ and $\mu_{\omega,B}(T_E) \geq 0$. This is a contradiction to E being $\mu_{\omega,B}$-semistable unless either $F_E = 0$ or $T_E = 0$. Hence $E \in T$ or $E \in F$.

Hence we can see that if $\mu_{\omega,B}(E) > 0$, then $E \in T$. If $\mu_{\omega,B}(E) < 0$, then $E \in T$. If $\mu_{\omega,B}(E) = 0$, we claim that $E \in F$. Indeed if not, then $E \in T$. Hence $Z_{\omega,B}(E) \in \mathbb{R}_{<0}$ and E is σ-semistable which is a sheaf. Hence there is a skyscraper sheaf $\mathscr{O}_x$ together with a surjective morphism of coherent sheaves $E \twoheadrightarrow \mathscr{O}_x$ with kernel K. Since $\mathscr{O}_x$ is stable of slope ∞ this morphism is also surjective in $\mathcal{A}$. Hence $K \in \mathcal{A} \cap \operatorname{Coh}(X) = T$. Then $Z_{\omega,B}(K) = Z_{\omega,B}(E) + 1$. Iterating this procedure will lead to an object K with $Z(F) \in \mathbb{R}_{\geq 0}$, a contradiction. Well done. ☐

Proof of Theorem 4.8.12. As $\mathbb{Q}$ is dense in $\mathbb{R}$, then this follows from Theorem 4.6.1 and Theorem 4.8.22. ☐

4.8.4 An Example of Wall-Crossing

Fix H be a integral ample class and $B_0 \in \operatorname{NS}(X)_{\mathbb{Q}}$. Consider the (α, β)-plane consist of $\sigma_{\alpha,\beta} := \sigma_{\alpha H, B_0 + \beta H}$ for all $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$. Then we will consider the wall structure in it.

Proposition 4.8.23. *Fix a class $v \in K_0^{\mathrm{num}}(X)$ with $\Delta_H(v) \geq 0$.*

(i) All numerical walls are either semicircles with center on the β-axis or vertical rays.

(ii) Two different numerical walls for v cannot intersect.

(iii) For a given class $v \in K_0^{\mathrm{num}}(X)$ the hyperbola $\Re Z_{\alpha,\beta}(v) = 0$ intersects all numerical semicircular walls at their top points.

(iv) If $\operatorname{ch}_0(v) \neq 0$, then there is a unique numerical vertical wall defined by the equation $\beta = \frac{H \operatorname{ch}_1^{B_0}(v)}{H^2 \operatorname{ch}_0^{B_0}(v)}$.

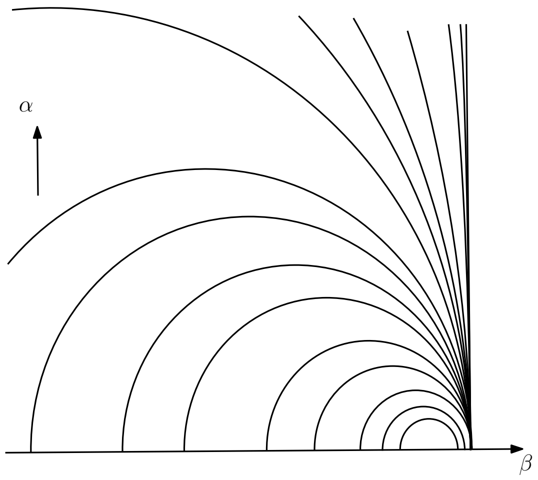
(v) If $\operatorname{ch}_0(v) \neq 0$, then all semicircular walls to either side of the unique numerical vertical wall are strictly nested semicircles.

(vi) If $\operatorname{ch}_0(v) = 0$, then there are only semicircular walls that are strictly nested.

(vii) If a wall is an actual wall at a single point, it is an actual wall everywhere along the numerical wall.

Proof. The proof of (i) is very long but very easy since it doesn't use any deep ideas or indeed any algebraic geometry. The proof of (iii)-(iv) are all high-school algebra. The proof of (ii) will use some linear algebra. For the detailed proof we refer [41]. ☐

Here is a diagram of it:



4.8.5 Some Examples of Bridgeland Stable Objects

Proposition 4.8.24. *For such $\sigma_{\omega,B}$, the skyscraper sheaves and objects with minimal $H \cdot \mathrm{ch}_1^B$ (or with $H \cdot \mathrm{ch}_1^B = 0$) are all $\sigma_{\omega,B}$-stable where $\omega = \alpha H$ for an ample integral class H.*

Proof. As the skyscraper sheaves are minimal objects in the category $\mathrm{Coh}^{\omega,B}(X)$, this is trivial. For the second one, see the proof of the rational case of Theorem 4.8.13. □

Proposition 4.8.25. *Let E be a $\mu_{\omega,B}$-stable vector bundle. If $\Delta^C_{\omega,B}(E) = 0$ or $\overline{\Delta}^B_H(E) = 0$, then E is $\sigma_{\omega,B}$-stable.*

Proof. See [43] Lemma 6.28, follows from wall structure in (α, β)-plane as we discussed above with the following lemma:

- Let $0 \neq v \in K_0^{\mathrm{num}}(X)$ with $\overline{\Delta}^{B_0}_H(E) \geq 0$. For fixed $\beta_0 \in \mathbb{Q}$ there are only finitely many walls intersecting the vertical line $\beta = \beta_0$.

See Lemma 6.24 in [43] for the proof. Note that this lemma can show that a largest wall exists and we will also be able to prove that walls are locally finite in this case. □

4.9 Construction II–Threefolds, an Introduction

For the basic construction we follows [11]. Let X is a smooth projective threefold over $\mathbb{C}$. We first introduce the classical Bogomolov-Gieseker inequality and Hodge index theorem:

Lemma 4.9.1 (Bogomolov-Gieseker)**.** *Let X be a n-dimensional smooth projective variety over $\mathbb{C}$ and let ω be an ample divisor on X. For any torsion free $\mu_{\omega,B}$-semistable sheaf $\mathscr{E}$, we have the following inequality*

$$\omega^{n-2}(\mathrm{ch}_1^B(\mathscr{E})^2 - 2\,\mathrm{ch}_0^B(\mathscr{E})\,\mathrm{ch}_2^B(\mathscr{E})) \geq 0.$$

Lemma 4.9.2 (Hodge Index Theorem)**.** *Let X be a n-dimensional smooth projective variety over an algebraically close k. Let D be a divisor on X and $D_1, ..., D_k$ are nef divisors on X. Let $n_1 + \cdots n_k = n-1 \geq 1$ and $n_1 \geq 1$, then*

$$(D \cdot D_1^{n_1} \cdots D_k^{n_k})^2 \geq (D^2 \cdot D_1^{n_1-1} \cdots D_k^{n_k})(D_1^{n_1+1} \cdot D_2^{n_2} \cdots D_k^{n_k}).$$

Furthermore, if $A_1, ..., A_k$ are nef divisors on X and $m_1 + \cdots m_k = n$, then

$$(A_1 \cdots A_k)^n \geq (A_1^n)^{n_1} \cdots (A_k^n)^{n_k}.$$

Proof. We refer [15] and [35]. □

Now pick again

$$Z_{\omega,B} := -\int_X e^{-i\omega}\,\mathrm{ch}^B = -\,\mathrm{ch}_3^B + \frac{1}{2}\omega^2\,\mathrm{ch}_2^B + i\left(\omega\,\mathrm{ch}_2^B - \frac{1}{6}\omega^3\,\mathrm{ch}_0^B\right)$$

for ample ω and any B in $N^1(X)_{\mathbb{Q}}$. If we again consider the pair $(Z_{\omega,B}(E), \mathrm{Coh}^{\omega,B}(X))$, we can not control the sign of $\Im Z_{\omega,B}(E)$ for $E \in \mathrm{Coh}^{\omega,B}(X)$, hence this does not give a Bridgeland stability condition.

The idea in [11] is to define a now slope and give a second tilt. They find that in this case the tuple $(\omega^2\,\mathrm{ch}_1^B(E), \Im Z_{\omega,B}(E), -\Re Z_{\omega,B}(E))$ on $\mathrm{Coh}^{\omega,B}(X)$ behaves like $(\mathrm{rank}, \mathrm{ch}_1, \mathrm{ch}_2)$ on surfaces (see Remark 4.8.5(b)):

Lemma 4.9.3. *For any nonzero $E \in \mathrm{Coh}^{\omega,B}(X)$, one of the following conditions holds:*

(a) $\omega^2\,\mathrm{ch}_1^B(E) > 0$.

(b) $\omega^2\,\mathrm{ch}_1^B(E) = 0$ *and* $\Im Z_{\omega,B}(E) > 0$.

(c) $\omega^2\,\mathrm{ch}_1^B(E) = \Im Z_{\omega,B}(E) = 0$ *and* $-\Re Z_{\omega,B}(E) > 0$.

Proof. Similar argument in Lemma 4.8.16. By definition of $\mathrm{Coh}^{\omega,B}(X)$ we have $\omega^2\,\mathrm{ch}_1^B(E) \geq 0$. Let $\omega^2\,\mathrm{ch}_1^B(E) = 0$, then $H^0(E) \in \mathrm{Coh}^{\leq 1}X$ and $H^{-1}(E)$ is $\mu_{\omega,B}$-semistable torsion free sheaf with $\mu_{\omega,B}(H^{-1}(E)) = 0$. By Hodge index theorem and Bogomolov-Gieseker inequality above we have

$$0 \geq \omega\,\mathrm{ch}_1^B(H^{-1}(E))^2 \geq 2\omega\,\mathrm{ch}_0^B(H^{-1}(E))\,\mathrm{ch}_2^B(H^{-1}(E))$$

which implies $\omega\,\mathrm{ch}_2^B(H^{-1}(E)) \leq 0$. As $\mathrm{ch}_0^B(E) \leq 0$ and $\omega\,\mathrm{ch}_2^B(H^0(E)) \geq 0$, we get $\Im Z_{\omega,B}(E) \geq 0$.

Finally if $\omega^2\,\mathrm{ch}_1^B(E) = \Im Z_{\omega,B}(E) = 0$, the above argument shows that $H^{-1}(E) = 0$ and $E = H^0(E)$ has zero-dimensional support; hence the inequality $-\Re Z_{\omega,B}(E) > 0$ holds. □

Hence similar as $\mu_{\omega,B}$, we define a new slope on $\mathrm{Coh}^{\omega,B}(X)$:

$$\nu_{\omega,B}(E) := \frac{\Im Z_{\omega,B}(E)}{\omega^2 \operatorname{ch}_1^B(E)} = \frac{\omega \operatorname{ch}_2^B(E) - \frac{1}{6}\omega^3 \operatorname{ch}_0^B(E)}{\omega^2 \operatorname{ch}_1^B(E)}.$$

Definition 4.9.4. *An object $E \in \mathrm{Coh}^{\omega,B}(X)$ is $\nu_{\omega,B}$-(semi)stable if, for any non-zero proper subobject $F \subset E$ in $\mathrm{Coh}^{\omega,B}(X)$, we have $\nu_{\omega,B}(F) < (\leq)\nu_{\omega,B}(E/F)$.*

Remark 4.9.5. *Note that by Lemma 4.9.3 the slope $\nu_{\omega,B}$ also satisfies the weak see-saw property as in Remark 4.8.5(b). Also we have Harder-Narasimhan filtration of $\nu_{\omega,B}$ in $\mathrm{Coh}^{\omega,B}(X)$.*

Definition 4.9.6. *For $\mathrm{Coh}^{\omega,B}(X)$, we define a torsion pair on it:*

$$T'_{\omega,B} = \{E \in \mathrm{Coh}^{\omega,B}(X) : \nu^-_{\omega,B}(E) > 0\};$$
$$F'_{\omega,B} = \{E \in \mathrm{Coh}^{\omega,B}(X) : \nu^+_{\omega,B}(E) \leq 0\}.$$

Hence we get a tilt $\mathscr{A}^{\omega,B}(X) = \left\langle T'_{\omega,B}, F'_{\omega,B}[1]\right\rangle$ of $\mathrm{Coh}^{\omega,B}(X)$.

Remark 4.9.7. *Another construction uses the perverse coherent sheaves rather than sheaves, and uses polynomial stability conditions rather than slope-stability. In [11] they discused this construction and shows that these two construction are the same.*

Conjecture 2. *If X is a smooth projective threefold over $\mathbb{C}$, then $(\mathscr{A}^{\omega,B}(X), Z_{\omega,B}) \in \mathrm{Stab}(X)$.*

Definition 4.9.8. *We say that the tuple (X, ω, B) as above satisfies the BG-type inequality if for any non-zero $\nu_{\omega,B}$-semistable object $E \in \mathrm{Coh}^{\omega,B}(X)$, we have the inequality*

$$(\omega^2 \operatorname{ch}_1^B(E))^2 - 2\omega^3 \operatorname{ch}_0^B(E) \cdot \omega \operatorname{ch}_2^B(E) \\ + 12(\omega \operatorname{ch}_2^B(E))^2 - 18\omega^2 \operatorname{ch}_1^B(E) \cdot \operatorname{ch}_3^B(E) \geq 0.$$

Proposition 4.9.9. *The tuple (X, ω, B) satisfies the BG-type inequality if and only if for any $\nu_{\omega,B}$-semistable object with $\nu_{\omega,B}(-) = 0$ we have $\operatorname{ch}_3^B \leq \frac{1}{18}\omega^2 \operatorname{ch}_1^B$.*

Moreover, if a tuple (X, ω, B) satisfies the BG-type inequality with $\rho(X) = 1$, then the Conjecture 2 holds.

Proof. See [14] Theorem 4.2 and Theorem 8.2. □

Remark 4.9.10. *Some remarks:*

(i) Note that $\operatorname{ch}_3^B(E) \leq \frac{1}{18}\omega^2 \operatorname{ch}_1^B(E)$ implies $Z_{\omega,B}(E) \in \mathbb{H}$.

(ii) *If a tuple (X, ω, B) satisfies the BG-type inequality with $\rho(X) > 1$, then $(Z_{\omega,B}, \mathscr{A}^{\omega,B}(X))$ is just a stability condition which is not strong enough to prove the support property. Moreover, the support property when X is an abelian threefold with $\rho(X) > 1$ was proved using Fourier-Mukai transforms in [47].*

In [11] they conjectured that the BG-type inequality always hold. However we later find that it is not true in general, e.g. a blowing-up at a point of $\mathbb{P}^3$, see [51]. Note that the exceptional divisor is not nef in this case, so we can make the following conjecture:

Conjecture 3. *Let X be a smooth projective threefold such that any effective divisor on it is nef. Then any tuple (X, ω, B) satisfies the BG-type inequality.*

Remark 4.9.11. *Some remarks:*

(a) *If there is a non-nef effective divisor, it seems that we need to add some modification term of the BG-type inequality, for the Fano case we refer [16].*

(b) *This conjecture was proved for Fano 3-fold with $\rho(X) = 1$ ([11][37]), abelian 3-fold ([14][42]), X with nef tangent bundle ([32]) and X is a quintic 3-fold ([36]).*

For the final case, C. Li shows that let $X \subset \mathbb{P}^4$ is a quintic 3-fold and let $\omega = \sqrt{3}\alpha H$ and $B = \beta H$ for $H = \mathscr{O}_X(1)$ for $(\alpha, \beta) \in \mathbb{R}^2$, if

$$\alpha^2 + \left(\beta - \lfloor \beta \rfloor - \frac{1}{2}\right)^2 > \frac{1}{4},$$

then the tuple (X, ω, B) satisfies the BG-type inequality.

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