Moduli Spaces of Algebraic Curves

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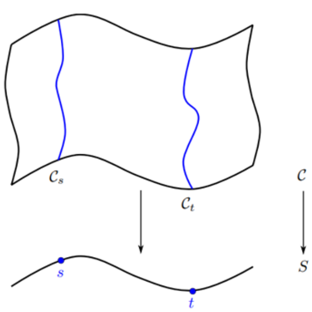
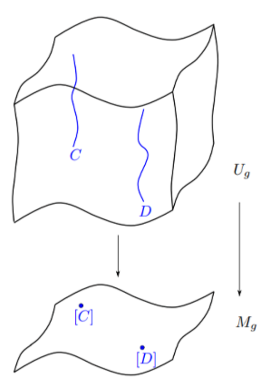
Abstract

In this note we introduced the basic results of algebraic curves and algebraic stacks at the beginning, as the basis for the later theories. Then we mainly discuss the properties and geometry of the moduli space of algebraic curves. We first introduce the moduli stack of smooth curves and stable curves and shows that the moduli stack of stable curves is a smooth proper Deligne-Mumford stack using the stable reduction of stable curves. Then we use the Kollár's Criterion to show that its coarse moduli space is actually projective. Then we discuss the several line bundles on it and calculate some relations of them. And we introduced the generators and its relations in the Picard group of moduli stacks. We also introduced many results about its Kodaira dimension and prove the moduli space of high-genus stable curves is of general type. Then we introduce the Hassett-Keel program of the moduli spaces of curves aiming to find out the log canonical models of them which plays an important role in this area in this few decades. We also give a half of proof of results. Finally we introduced some basic results about the Chow rings and cohomology groups of moduli space of stable curves.





Contents

1 Introduction to the moduli spaces of curves

1.1 What is a moduli space?

One of the characterizing features of algebraic geometry is that the set (or maybe a groupoid) of all geometric objects of a fixed type (such as smooth projective curves, subspaces of a fixed vector space, or coherent sheaves or budnels on a fixed variety or schemes) often itself has some algebraic structure, for example, an algebraic variety or some more general notion of algebro-geometric space, like algebraic space or stacks. Such a space $\mathcal{M}$ is the moduli space classifying objects of the given type and in some sense the study of all objects of the given type is reduced to the studying the geometry of the space $\mathcal{M}$! This self-referential nature of algebraic geometry is a crucial aspect of the field.

More precisely, let $\mathcal{C}$ be some sets of all geometric objects of a fixed type. If there exists a scheme $\mathcal{M}$ such that $\mathcal{M}$ is $1-1$ corresponding to the isomorphisc class of objects in $\mathcal{C}$, then we will call $\mathcal{M}$ the moduli space of $\mathcal{C}$-type objects. But this is not enough, if we have a scheme $\mathcal{M}'$ which corresponding to the objects in $\mathcal{C}$ (then $\mathcal{M} \cong \mathcal{M}'/\sim$), that is, we have a morphism

$$\pi : \mathcal{U} \to \mathcal{M}'$$

such that for any $S \to \mathcal{M}'$ pullback along π will get a family of objects in $\mathcal{C}$ over S. In this case we will call $\mathcal{M}$ a fine moduli space and $\mathcal{U}$ be a universal family.

Example 1.1. *We consider the plane conic curves (both smooth and singular one) in $\mathbb{P}^2_{\mathbb{C}}$. Let x, y, z be its homogeneous coordinates. Then any plane conic curves determind by a degree 2 homogeneous polynomials. So the moduli space of plane conic curves is the moduli space of degree 2 homogeneous polynomials! As a degree 2 homogeneous polynomials has 6 coefficients, up to some invertible multiplicity. Hence this is correspond to $\mathbb{P}^5$! Actually by the basic theory of Hilbert schemes, this is a fine moduli space.*

But not any moduli problem has a fine moduli space, e.g. moduli space of algebraic curves! That is, we can not have universal family in the category of schemes. Actually the main problem is that the objects have non-trivial automorphisms! Hence we now have two choice in this theory:

- To extend the category of schemes (algebraic stacks);
- Consider the coarse moduli space without universal family.

We will work both two sides in moduli space of algebraic curves.

1.2 Why stacks?

To extend the category of schemes, we first need to consider the collections of families geometric objects of a fixed type $\mathcal{C}$ and to find out why it is not a fine moduli problem. This lead us to the notion of moduli functors.

Definition 1.2. *Consider the (pseudo-)functor*

$$\mathscr{M} : \mathrm{Sch}^{op} \to \mathrm{Sets} \text{ (or Groupoids)}$$

sending a scheme S to $\mathscr{M}(S)$, a collection of families $X \to S$ such that the geometric fibers are of a fixed type $\mathcal{C}$.

Then the existence of fine moduli space is equivalent to the representability of $\mathscr{M}$ by schemes! Hence if the objects have non-trivial automorphisms, then the (pseudo-)functor never represented by schemes since the (pseudo-)functor maps to the 2-categories of groupoids! Hence in this case if we consider the moduli of elliptic curves, it is never a fine moduli space.

When we module the isomorphisms, we may get a coarse moduli space M. But this space have no informations of automorphisms and universal family.

Hence now we may give the pseudo-functor $\mathscr{M} : \mathrm{Sch}^{op} \to$ Groupoids some algebraic structure to close to the schemes. Note that since the pseudo-functor is not so natural, we may use a new objects, categories fibered in the groupoids, to replace it with the same meaning. Now a stack is just a categoriy fibered in the groupoids with some descent properties which can do many things better like schemes.

But this is not enough. We may equip some topology over them. If the automorphisms are finite, then we can equip them étale topology to imitate the analytic topology, then we get Deligne-Mumford stack. If the automorphisms are not finite, we can just equip them smooth topology, then we get algebraic stack(or called Artin stack).

By some technical results, we can also define the similar properties and theorems in the standard scheme theory. But this is much more complicated then scheme theory. Fortunately, here we just consider some simple results.

Another good things is that for some good enough algebraic stack $\mathscr{X}$, we have a canonical morphism to its coarse moduli space $\pi : \mathscr{X} \to X$ by Keel-Mori's theory. So we can connect them closer.

1.3 Moduli space of algebraic curves

It has a long history to study the moduli space of algebraic curves. Here we first give a brief history of the development of moduli space M_g of smooth genus g curves:

- Riemann (1857): Riemann surfaces of genus g depend on $3g-3$ parameters;
- Cayley (1862): A new analytic representation of curves in space: Constructs moduli of space curves: $C \mapsto$ (all lines meeting C).
  - General theory: van der Waerden,Chow,Hodge-Pedoe.
- Hilbert (1890):Uber die Theorie der algebraischen Formen, finite generation of rings of invariants ("Theologie" according to Gordan).
  - BUT: nobody seems to have taken its Proj.

- Hurwitz (1891): Uber Riemann'sche Flachen mit gegebenen Verzweigungspunkten: M_g is irreducible.
- Fricke-Klein (1897-1912): T_g exists and is contractible: T_g= discrete, cocompact representations $\pi_1(C) \to \mathrm{PGL}_2(\mathbb{R}) = \mathrm{Aut}(\text{unit disc})$, modulo conjugation. But complex structure not natural, not considered much;
- Severi (1915): Sulla classificazione delle curve algebriche e sul teorema d'esistenza di Riemann: M_g unirational for $g \leq 10$ without showing the existence of moduli space.
- Teichmüller (1940-44): complete theory of T_g with complex structure + functorial aspects;
- Weil, Matsusaka (1946-56): field of definition/field of moduli M_g, A_g should be defined over $\mathbb{Z}$, so k_C :=residue field of $[C] \in M_g$. Aim: finding k_C from C (without knowing M_g);
- Weil (1958): Bourbaki seminar: "As for M_g there is virtually no doubt that it can be provided with the structure of an algebraic variety";
- Grothendieck (1960): Cartan Seminar, T_g represents a functor (based on Teichmüller?) that a projective families over analytic bases;
- Mumford (1965): Construct M_g successfully using GIT.

Here T_g be the Teichmüller space to construct M_g. Hence it took more than 100 years for us to successfully define the structure of curve moduli space algebraically.

After construct M_g, we find that this space is not compact (or proper in algebraic geometry). So a natural question is to find a meaningful compactification $\overline{M}_g$.

Example 1.3. *Actually Satake (1956-60) and Baily-Borel (1966) Compactifying A_g to be a compact space. But the points at infinity are lower dimensional Abelian varieties!*

Hence we may have many non-natural compactifications. But fortunately, in 1969 Deligne-Mumford construct a compactification consist of stable curves with worse-nodal singularities and ample log canonical bundles, so called Deligne-Mumford compactification.

In this paper we will using the tools of algebraic stacks to construct the moduli stacks of smooth curves and stable curves and their coarse moduli spaces. Then we will discover its geometric properties, such as line bundles, Picard groups, Chow rings and the log canonical models of them.

2 The basic facts of curves

2.1 Basic facts of general curves

2.1.1 Standard results

Definition 2.1. *A* curve *over k is a pure one-dimensional scheme C of finite type over k. If C is proper, we define the arithmetic genus (simply the genus) of C as $g(C) := g_a(C) = 1 - \chi(C, \mathscr{O}_C)$. By Review A.10, if C is geometrically connected and geometrically reduced, this is equal to $h^1(C, \mathscr{O}_C)$.*

Theorem 2.2 (St 0B5Y)**.** *Let C be a proper scheme of dimension 1 over a field k and let L be a line bundle on it. Let C_i be the irreducible components of dimension 1. Then L is ample if and only if $\deg(L|_{C_i}) > 0$ for all i.*

Theorem 2.3 (Serre duality for smooth curves)**.** *Let C be a smooth complete curve over k with canonical bundle $\omega_C = \Omega_C$, then for any vector bundle F we get*

$$H^0(C, F^\vee \otimes \omega_C) \cong H^1(C, F)^\vee.$$

If we define the geometrical genus $g_e(C) = h^0(C, \omega_C)$ and if C is smooth projective curve which is geometrically connected and geometrically reduced, then $h^0(C, \mathscr{O}_C) = 1$. Hence by serre-duality we get $g_e(C) = g_a(C)$.

Theorem 2.4 (Riemann-Roch for smooth curves)**.** *Let C be a smooth complete curve over k with a line bundle L, then*

$$\chi(C, L) = h^0(C, L) - h^0(C, \omega_C \otimes L^\vee) = \deg L + 1 - g.$$

Theorem 2.5 (Positivity of divisors on smooth curves)**.** *Let C be a smooth complete curve over k of genus g with a line bundle L, then*
(a) if $\deg L \geq 2g$, then L is base-point-free;
(b) if $\deg L \geq 2g + 1$, then L is very ample;
(c) if $\deg L > 0$, then L is ample.
(d) if $\deg L < 0$, then $h^0(C, L) = 0$.

Proof. See the section IV.3 of [58] for the proof when k is algebraic closed. This is also right when k is not algebraic closed, see section 20.2 in [79].

Here we use another method to show (d) as a special case of [58] Ex.III.7.1. We just consider the case C is integral. If $\deg L < 0$, then L^{-1} is ample. Let $h^0(C, L) > 0$ and take a nonzero $s \in H^0(C, L)$. As $H^0(C, L) = \mathrm{Hom}(\mathscr{O}_C, L)$, we can get $- \times s : \mathscr{O}_C \to L$. As C integral, s must nonzero at the generic point, hence $- \times s : \mathscr{O}_C \to L$ is injective. Hence we get $L^{-1} \subset \mathscr{O}_C$. Let n such that L^{-n} generated by global sections, we get $L^{-n} \subset \mathscr{O}_C$. Hence $H^0(C, L^{-n}) \subset H^0(C, \mathscr{O}_C)$. Consider hilbert polynomial $\chi(L^{-n}) = \alpha n + \beta$ as $\deg \chi(L^{-n}) = \dim \mathrm{supp}(L^{-1}) = \dim C = 1$. By Serre's vanishing theorem, we get for $n \to \infty$, we have $\chi(L^{-n}) = h^0(C, L^{-n}) \to \infty$. This is impossible since $h^0(C, \mathscr{O}_C) < \infty$. $\square$

Theorem 2.6 (Riemann-Hurwtiz Theorem,St 0C1B)**.** *Let $f : X \to Y$ be a separable morphism of smooth projective curves over a field k and if $k = H^0(X, \mathscr{O}_X) = H^0(Y, \mathscr{O}_Y)$ and X and Y have genus g_X and g_Y, then*

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \deg R$$

where R be the ramified divisor. Moreover, $\deg R = \sum_x d_x[\kappa(x) : k]$ where $d_x = \mathrm{length}_{\mathscr{O}_{X,x}} \Omega_{X/Y,x}$. If $\mathscr{O}_{X,x}$ is tamely ramified over $\mathscr{O}_{Y,f(x)}$, then $d_x = e_x - 1$. If not, we only have $d_x > e_x - 1$ where e_x is the ramification index.

2.1.2 Automorphisms of curves

Here we only consider smooth connected projective curves of genus g over some algebraically closed field k.

Proposition 2.7. *For $g = 0$, we get $\mathrm{Aut}(\mathbb{P}^1_k) \cong \mathrm{PGL}_2$. Moreover, if we consider all automorphisms fixed n points, then this group is finite if and only if $n \geq 3$.*

Proof. See [58] Example II.7.1.1, we get $\mathrm{Aut}(\mathbb{P}^1_k) \cong \mathrm{PGL}_2$. Moreover, all automorphisms fixed n points is finite if and only if $n \geq 3$ by easy linear algebra. $\square$

Proposition 2.8. *For curve C with $g = 1$, we get $\mathrm{Aut}(C)$ is infinite group. Moreover, if we consider all automorphisms fixed n points, then this group is finite if and only if $n \geq 1$.*

Proof. In this case C is actually a group scheme of dimension 1 (by Picard varisties) and C can then act on C. Hence $C \subset \mathrm{Aut}(C)$, hence infinite. Moreover, by [58] Corollary IV.4.7 (for $\mathrm{char}(k) \neq 2$), if we fixed one point P_0, then $\mathrm{Aut}(C; P_0)$ is finite. Indeed, let $f : X \to \mathbb{P}^1$ such that $f(P_0) = \infty$, branched over $0, 1, \lambda, \infty$. Let $\sigma \in \mathrm{Aut}(C; P_0)$, then there exists an automorphism $\tau \in \mathrm{Aut}(\mathbb{P}^1; \infty)$ such that $f \circ \sigma = \tau \circ f$. Hence τ sends $\{0, 1, \lambda\}$ to $\{0, 1, \lambda\}$ in some order.

(i) If $\tau = \mathrm{id}$, then $\sigma = \mathrm{id}$ or interchanging two sheets of f;

(ii) If $\tau \neq \mathrm{id}$, then τ permutes $\{0, 1, \lambda\}$ and the orbit of λ is of that six forms.

In both cases is finite, well done. $\square$

Proposition 2.9 (Hurwitz)**.** *For curve C with $g \geq 2$, the group $\mathrm{Aut}(C)$ is finite. Moreover, if k has characteristic 0, we have $\#(\mathrm{Aut}(C)) \leq 84g - 84$.*

Proof. See [58] Ex.V.1.11 and Hurwitz's Automorphism Theorem. $\square$

Lemma 2.10 (St 0E67)**.** *Let X be a smooth, proper, connected curve over k of genus g.*

(a) If $g \geq 2$, then $\mathrm{Der}_k(\mathscr{O}_X, \mathscr{O}_X) = 0$;

(b) If $g = 1$ and $0 \neq D \in \mathrm{Der}_k(\mathscr{O}_X, \mathscr{O}_X)$, then D does not fix any closed point of X;

(c) If $g = 0$ and $0 \neq D \in \mathrm{Der}_k(\mathscr{O}_X, \mathscr{O}_X)$, then D can fix at most 2 closed points of X.

Remark 2.11. *We called an element* $D \in \mathrm{Der}_k(\mathscr{O}_X, \mathscr{O}_X)$ *fixes* x *if* $D(\mathscr{I}) \subset \mathscr{I}_x$ *where* $\mathscr{I}_x$ *be the ideal sheaf of* x.

Sketch. As we have the canonical derivation $d : \mathscr{O}_X \to \Omega_{X/k}$, taking any $D \in \mathrm{Der}_k(\mathscr{O}_X, \mathscr{O}_X)$ we get $D = f \circ d$ where $f \in \mathrm{Hom}_{\mathscr{O}_X}(\Omega_{X/k}, \mathscr{O}_X)$ and $\deg(\Omega_{X/k}) = 2g-2$.

(a) If $g \geq 2$, then $\deg(\Omega_{X/k}) > 0$. Hence

$$\mathrm{Hom}_{\mathscr{O}_X}(\Omega_{X/k}, \mathscr{O}_X) = \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X, T_{X/k}) = \Gamma(X, T_{X/k}) = 0,$$

hence $f = 0$;

(b)(c) We claim that the vanishing of f at $x \in X$ is equivalent to the statement that D fixes x. Indeed, by St 0C1E we get for the uniformizer $z \in \mathscr{O}_{X,x}$, dz is a basis of $\Omega_{X,x}$. Since $D(z) = f(dz)$, we conclude the claim.

If $g = 1$, then a nonzero f does not vanish anywhere. Hence by the claim, D does not fix any closed point of X. If $g = 0$, then a nonzero f vanishes in a divisor of degree 2. Hence by the claim, D can fix at most 2 closed points of X. □

Lemma 2.12. *Let* X *be a proper scheme over a field* k *of dimension* ≤ 1, *then the following are equivalent*

(i) $\underline{\mathrm{Aut}}(X)$ *is geometrically reduced over* k *and has dimension 0;*

(ii) $\underline{\mathrm{Aut}}(X) \to \mathrm{Spec}(k)$ *is unramified;*

(iii) $\mathrm{Der}_k(\mathscr{O}_X, \mathscr{O}_X) = 0$.

Proof. See St 0DSW and St 0E6G. Note that these two lemmas can also gives the results about automorphism groups of smooth connected curves. □

Proposition 2.13. *Let* C *be a curve of genus* g *over a field* k *of characteristic 0, then for any non-trivial automorphism of* C *fixed at most* $2g+2$ *points.*

Proof. See I.F-4 in [9]. Now we give a sketch. Let $p_1, ..., p_{g+1}$ are general points and $\phi \in \mathrm{Aut}(C)$. Then there exists $f \in K(C)$ with divisor $(f)_\infty = p_1 + ... + p_{g+1}$, and counting zeros and poles of $f - \phi^* f$, give that ϕ has at most $2g+2$ fixed points. □

2.2 Families of curves

2.2.1 Families of general curves

Lemma 2.14 (Dualizing sheaves of the families of curves)**.** *Let* $(S, f : C \to S)$ *in* $\mathscr{M}_g$ *(or more general) for* $g \geq 2$.

(i) $f_*\mathscr{O}_C = \mathscr{O}_S$;

(ii) For $k > 1$ *the sheaf* $f_*(\Omega^1_{C/S})^{\otimes k}$ *is locally free of rank* $(2k-1)(g-1)$ *on* S, *and for any* $g : S' \to S$, *we get an isomorphism* $g^* f_*(\Omega^1_{C/S})^{\otimes k} \cong f'_*(\Omega^1_{C'/S'})^{\otimes k}$. *Moreover,* $R^i f_*(\Omega^1_{C/S})^{\otimes k} = 0, i > 0$;

(iii) The sheaf $f_*\Omega^1_{C/S}$ *is locally free of rank* g *on* S, *and for any* $g : S' \to S$, *we get an isomorphism* $g^* f_*\Omega^1_{C/S} \cong f'_*\Omega^1_{C'/S'}$. *Moreover,* $R^1 f_*(\Omega^1_{C/S}) = \mathscr{O}_S$ *and* $R^i f_*(\Omega^1_{C/S}) = 0, i > 1$;

(iv) For $k \geq 3$, $(\Omega^1_{C/S})^{\otimes k}$ *is relative very ample.*

Proof. (i) By definition, for all $s \in S$ the C_s is proper geometrically connected and geometrically reduced, then by Review A.10 we get $H^0(C_s, \mathscr{O}_s) = \kappa(s)$, hence $\phi_s^0 : f_*\mathscr{O}_C \otimes \kappa(s) \to H^0(C_s, \mathscr{O}_s)$ is surjective. By Review A.1 with $i = 0$, we get ϕ_s^0 is an isomorphism and $f_*\mathscr{O}_C$ is a line bundle. Now consider the natural map $\mathscr{O}_S \to f_*\mathscr{O}_C$ induce a surjective fiber map $\kappa(s) \to f_*\mathscr{O}_C \otimes \kappa(s)$ by seen

$$\kappa(s) \to f_*\mathscr{O}_C \otimes \kappa(s) \to H^0(C_s, \mathscr{O}_s) = \kappa(s).$$

Thus $\mathscr{O}_S \to f_*\mathscr{O}_C$ is surjective, hence an isomorphism.

(ii) For all $s \in S$ and $k > 1$ we get $H^1(C_s, (\Omega^1_{C_s/\kappa(s)})^{\otimes k}) = H^0(C_s, (\Omega^1_{C_s/\kappa(s)})^{\otimes(1-k)})^\vee = 0$ as $(\Omega^1_{C_s/\kappa(s)})^{\otimes(1-k)}$ is anti-ample. Hence $H^i(C_s, (\Omega^1_{C_s/\kappa(s)})^{\otimes k}) = 0$ for $i > 0$. Now use Review A.1 we get $R^i f_*(\Omega^1_{C/S})^{\otimes k} = 0, i > 0$.

On the other hand, by Riemann-Roch theorem, we get

$$h^0(C_s, (\Omega^1_{C_s/\kappa(s)})^{\otimes k}) = \deg((\Omega^1_{C_s/\kappa(s)})^{\otimes k}) + 1 - g = (2k-1)(g-1).$$

Use Review A.1 again, we get $f_*(\Omega^1_{C/S})^{\otimes k}$ is locally free of rank $(2k-1)(g-1)$ on S.

(iii) By Review A.1 and the fact $H^i(C_s, \Omega^1_{C/S} \otimes \kappa(s)) = 0, i > 1$ implies $R^i f_*\Omega^1_{C/S} = 0, i > 1$. Now we use the duality $f_*\mathscr{H}om(F, \Omega^1_{C/S}) \cong \mathscr{H}om(R^1 f_*F, \mathscr{O}_S)$, then let $F = \Omega^1_{C/S}$. We get $f_*\mathscr{O}_C \cong (R^1 f_*\Omega^1_{C/S})^*$. Hence $R^1 f_*\Omega^1_{C/S} \cong (f_*\mathscr{O}_C)^* = \mathscr{O}_S^* \cong \mathscr{O}_S$.

By Review A.1(ii) with $i = 1$, we get $\phi_s^0 : f_*\Omega^1_{C/S} \otimes \kappa(s) \to H^0(C_s, \Omega^1_{C_s/\kappa(s)})$ is surjective, hence an isomorphism. Then apply Review A.1(i)-(ii) with $i = 0$ to imply $f_*\Omega^1_{C/S}$ is locally free of rank $h^0(C_s, \Omega^1_{C_s/\kappa(s)}) = g$.

(iv) Easy to see for any $s \in S$ the fiber $(\Omega^1_{C_s/\kappa(s)})^{\otimes k}$ is very ample as $\deg(\Omega^1_{C_s/\kappa(s)})^{\otimes k} = k(2g-2) \geq 2g+1$. Using noetherian approximation, we may let S is noetherian. Then use Review A.11 and well done. □

Remark 2.15. *Note that we can be generalized these statements into more general families of curves, such as nodal curves and so on, without any modification.*

Proposition 2.16 (Flatness Criterion over Smooth Curves)**.** *Let C be an integral and regular scheme of dimension 1 (e.g. the spectrum of a DVR or a smooth curve over a field) and $X \to C$ a qcqs morphism of schemes. A quasi-coherent $\mathscr{O}_X$-module F is flat over C if and only if every associated point of F maps to the generic point of C.*

2.2.2 Families of elliptic curves and j-invariant

This section are some preliminaries of the coarse moduli space of $\mathscr{M}_{1,1}$. Here we follows [74] 13.1 and for the basic theory of single elliptic curves, we refer [58] IV.4.

Definition 2.17. *Let S be a scheme, then an* elliptic curve over S *defined by a pair $(f : E \to S; e)$ where f is a smooth proper morphism and $e : S \to E$ be a section of f, and for every geometric point $\bar{x} : \mathrm{Spec} k \to S$ the pullback $(E_{\bar{x}}, e_{\bar{x}})$ is an elliptic curve over k.*

Proposition 2.18. *Let $(f : E \to S; e)$ be a family of elliptic curves over S, then as f is proper the section $e : S \to E$ is a closed embedding. Let $\mathscr{J}$ be the ideal sheaf of e, then $\mathscr{J}$ defines a Cartier divisor.*

Proof. This is not hard to prove, we refer [74] section 13.1.4. $\square$

Let $\mathscr{O}_E(ne) = \mathscr{J}^{\otimes -n}$, by Review A.1 we know that $f_*\mathscr{O}_E(3e)$ is locally free of rank 3 over S. As we can check over geometric fibers (with Riemann-Roch theorem), the adjunction map $f^*f_*\mathscr{O}_E(3e) \to \mathscr{O}_E(3e)$ is surjective which induce a closed embedding $E \hookrightarrow \mathbb{P}(f_*\mathscr{O}_E(3e))$. Now we will use this to deduce the family-version of Weierstrass forms of elliptic curves.

By Review A.1 again we know that $f_*\mathscr{O}_E(e)$ and $f_*\mathscr{O}_E(2e)$ is locally free of rank $1, 2$ over S, respectively. Then zariski locally we cna choose bases for these bundles:

$$1 \in f_*\mathscr{O}_E(e), \quad 1, x \in f_*\mathscr{O}_E(2e), \quad 1, x, y \in f_*\mathscr{O}_E(3e).$$

Consider $f_*\mathscr{O}_E(6e)$ and surjection $\mathscr{O}_S^7 \twoheadrightarrow f_*\mathscr{O}_E(6e)$ can induced by $1, x, x^2, x^3, y, y^2, xy$ as it can be checked over geometric fibers as in the single case in [58] Theorem IV.4.6.

Pick $\alpha_i \in \mathscr{O}_S, i = 1, ..., 7$ such that $\alpha_1 + \alpha_2 x + ... + \alpha_6 xy + \alpha_7 y^2$ maps to zero under this map. By checking fibers again we get $\alpha_4, \alpha_7 \in \mathscr{O}_S^*$. Hence WLOG we let $\alpha_4 = 1$. Pick $F \in \mathscr{O}_S[X, Y, Z]$ be a homogeneous polynomial:

$$F := \alpha_1 Z^3 + \alpha_2 XZ^2 + \alpha_3 X^2 Z + X^3 + \alpha_5 YZ^2 + \alpha_6 XYZ + \alpha_7 Y^2 Z.$$

Proposition 2.19. *We have $E \cong V_+(F) \subset \mathbb{P}^2_S$ and section e given by $[0 : 0 : 1]$.*

Proof. See [74] (13.1.6.2) – (13.1.6.3). $\square$

Now change $Z \mapsto \alpha_7 Z$ we may let $\alpha_7 = 1$. Then we see that E/S given by equation

$$Y^2 Z + a_1 XYZ + a_3 YZ^2 = X^3 + a_2 X^2 Z + a_4 XZ^2 + a_6 Z^3.$$

Define

$$\begin{aligned} &b_2 = a_1^2 + 4a_2, b_4 = a_1 a_3, b_6 = a_3^2 + 4a_6; \\ &b_8 = -a_1 a_3 a_4 - a_4^2 + a_1^2 + a_6 + a_2 a_3^2 + 4a_2 a_6, \end{aligned}$$

and discriminant $\Delta := -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6$.

Theorem 2.20. *Let $E = V_+(F) \subset \mathbb{P}^2_S$ where F as above.*

(i) E is smooth over S if and only if Δ invertible in S;

(ii) Let (E, e) be the family of elliptic curves and define j-invariant $j_{(E,e)} \in \Gamma(S, \mathscr{O}_S)$ by

$$j_{(E,e)} := \frac{(b_2^2 - 24b_4)^3}{\Delta}.$$

Then two elliptic curves $(E, e), (E', e')$ over S with j-invariants j, j' is isomorphic if and only if $j = j'$.

Proof. See [74] section 13.1.7 and Proposition 13.1.13. $\square$

2.3 Singularities of curves

2.3.1 δ-invariant

The the more details, see St 0C3Q and St 0C3Z.

Lemma 2.21 (St 0C3S)**.** *Let $(A, \mathfrak{m})$ be a 1-dimensional reduced local ring of finite type over a field k. Let A' be the integral closure of A in the total ring of fractions of A. Then A' is a normal with $A \to A'$ is finite, and A'/A has finite length as an A-module.*

Definition 2.22. *Let A be a reduced 1-dimensional local ring of finite type over a field k. The* δ-invariant *of A defined by* $\mathrm{length}_A(A'/A)$ *where A' is as in Lemma.*

Let X be a scheme locally of finite type over k. Let $x \in X$ such that $\mathscr{O}_{X,x}$ is reduced with dimension 1. The δ-invariant of X at x is the δ-invariant of $\mathscr{O}_{X,x}$.

Proposition 2.23 (St 0C3V)**.** *Let A be a reduced 1-dimensional local ring of finite type over a field k. Then $\widehat{A}$ has the same δ-invariant as A and $A' \otimes_A \widehat{A}$ is the integral closure of $\widehat{A}$ in its total ring of fractions.*

Proposition 2.24 (St 0C1R)**.** *Let X be a reduced scheme locally finite type over a field of dimension 1 with normalization $f : \widetilde{X} \to X$. Then $\mathscr{O}_X \subset f_*\mathscr{O}_{\widetilde{X}}$ and $f_*\mathscr{O}_{\widetilde{X}}/\mathscr{O}_X$ is a direct sum of skyscraper sheaves $\mathscr{Q}_x$ in the singular points x and $\mathscr{Q}_x = (f_*\mathscr{O}_{\widetilde{X}})_x/\mathscr{O}_{X,x}$ has finite length equal to the δ-invariant of X at x.*

2.3.2 Some singularities of curves

Definition 2.25. *Let C be a curve over k. Here we let k algebraically closed and if not, we condier the base-change.*

(a) We say that $p \in C(k)$ is a node *is we have $\widehat{\mathscr{O}}_{C,p} \cong k[[x,y]]/(xy)$;*

(b) We say that $p \in C(k)$ is a cusp *is we have $\widehat{\mathscr{O}}_{C,p} \cong k[[x,y]]/(y^2 - x^3)$;*

(c) We say that $p \in C(k)$ is a tacnode *is we have $\widehat{\mathscr{O}}_{C,p} \cong k[[x,y]]/(y^2 - x^4)$.*

Definition 2.26. *A curve C has* locally planar singularities *at p if $\widehat{\mathscr{O}}_{C,p} \cong k[[x,y]]/(f)$ for a reduced series $f \in k[[x,y]]$.*

Proposition 2.27. *If a curve have worst locally planar singularities, then it is Gorenstein. Hence nodal, cuspidal and tacnodal curves are all Gorenstein.*

2.4 Ramification and Plücker Formula

We will follows the sequences of exercises in [9] as Exercise I.C. Let C be a smooth projective curve of genus g over $\mathbb{C}$ and we will describe the notion of ramification of a map $C \to \mathbb{P}^r$, or more generally, of a linear series on C, fixed as $L = (\mathscr{L}, V)$ be a $\mathfrak{g}^r_d$. We also fix a point $p \in C(\mathbb{C})$.

Lemma 2.28. *We have $\sharp\{\mathrm{ord}_p\sigma : \sigma \in V\backslash\{0\}\} = r + 1$.*

Proof. There exists a basis for V consisting of sections with distinct orders of vanishing at p. To construct this basis, replace a pair of sections with the same vanishing order by two sections, one with the same order, and one with one higher order. □

Definition 2.29. *(i) If we let these $r+1$ numbers as $0 \leq a_0^L(p) < \cdots < a_r^L(p) \leq d$, then the sequence $\{a_0^L(p), \cdots, a_r^L(p)\}$ is called* ***vanishing sequence*** *of L at p;*

(ii) Let $\alpha_i^L(p) := a_i^L(p) - i$, then the sequence $\{\alpha_0^L(p), \cdots, \alpha_r^L(p)\}$ is called ***ramification sequence*** *of L at p. The weight $w^L(p)$ of p with respect to L is defined by*

$$w^L(p) = \sum_{i=0}^{r} \alpha_i^L(p) = \sum_{i=0}^{r} a_i^L(p) - \binom{r+1}{2};$$

(iii) We say that L is unramified at p if $\{\alpha_0^L(p), \cdots, \alpha_r^L(p)\} = \{0, ..., 0\}$, else that p is a ramification point of L. If we consider the canonical series $(K_C, |K_C|)$, then the ramification points are called Weierstrass points.

Remark 2.30. *These can be also defined over some singuler curves and p be a smooth point.*

Lemma 2.31. *There are only finitely many ramification points of L on C.*

Theorem 2.32 (Plücker Formula)**.** *We have*

$$\sum_{p \in C} w^L(p) = (r+1)d + \binom{r+1}{2}(2g+2).$$

Proof. See [31] Propositon 1.1. □

3 The basic facts of stacks

As the most of the proofs of the fundamental theory of stacks are very complicated, we just give an basic introduction and without proofs. Here we follows [74].

3.1 Sites and Grothendieck topos

Definition 3.1 (Grothendieck topology and sites)**.** *Let $\mathcal{C}$ be a category. A Grothendieck topology over $\mathcal{C}$ consist of sets $\{\{U_i \to U\}_{i \in I}\} = \mathrm{Cov}(U)$ where U be any object such that*

(i) If $V \to X$ is an isomorphism, then $\{V \to X\} \in \mathrm{Cov}(X)$;

(ii) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $Y \to X$ be any morphism, then the fiber product $X_i \times_X Y$ exists and

$$\{X_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}(Y);$$

(iii) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and for any $i \in I$ with $\{V_{ij} \to X_i\}_{j \in J_i}$, then

$$\{V_{ij} \to X_i \to X\}_{i \in I, j \in J_i} \in \mathrm{Cov}(X).$$

Category $\mathcal{C}$ and its Grothendieck topology called a site.

Example 3.2 (Small Zariski site). *Let X be a scheme. Consider the category* $\mathrm{Op}(X)$ *consist of open subschemes with inclutions. Then $\{U_i \to U\}_{i\in I}$ is a covering if $U = \bigcup_i U_i$. We denote this site X_{Zar}.*

Example 3.3 (Big Zariski site). *Let X be a scheme. Consider the category* Sch/X, *then $\{U_i \to U\}_{i\in I}$ is a covering if $U_i \to U$ are open immersions and $U = \bigcup_i U_i$. We denote this site X_{ZAR}.*

Example 3.4 ((Small) étale site). *Let X be a scheme. Consider the category* Et/X *consist of étale X-schemes with étale morphisms. Then $\{U_i \to U\}_{i\in I}$ is a covering if $\coprod_{i\in I} U_i \to U$ is surjective. We denote this site $X_{\text{ét}}$.*

Example 3.5 (Big étale site). *Let X be a scheme. Consider the category* Sch/X, *then $\{U_i \to U\}_{i\in I}$ is a covering if $U_i \to U$ is étale and $\coprod_{i\in I} U_i \to U$ is surjective. We denote this site X_{Et}.*

Example 3.6 (Fppf site). *Let X be a scheme. Consider the category* Sch/X, *then $\{U_i \to U\}_{i\in I}$ is a covering if $U_i \to U$ is flat and locally of finite presentation and $\coprod_{i\in I} U_i \to U$ is surjective. We denote this site X_{fppf}.*

Definition 3.7. *A presheaf of a site $\mathcal{C}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$.*

Definition 3.8. *Fix a site $\mathcal{C}$.*

(i) A presheaf F is called a separated presheaf if for any $U \in \mathcal{C}$ and covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, the map $F(U) \to \prod_{i\in I} F(U_i)$ is injective;

(ii) A presheaf F is called a sheaf if for any $U \in \mathcal{C}$ and covering $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(U)$, we have the following equalizer:

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)$$

where the morphisms induced by $U_i \times_U U_j \to U_i$ and $U_i \times_U U_j \to U_j$.

Definition 3.9. *A category is called Grothendieck Topos if it is equivalent to a category of sheaves over some site.*

Definition 3.10 (Sheafification). *Fix a site $\mathcal{C}$ and let $\mathscr{P}$ be a presheaf, we call a sheaf $\mathscr{P}^\sharp$ such that $\mathscr{P} \to \mathscr{P}^\sharp$ is the sheafification of $\mathscr{P}$ if for any sheaf $\mathscr{G}$ and $\mathscr{P} \to \mathscr{G}$ we have:*

$$\begin{tikzcd} \mathscr{P} \arrow[r] \arrow[d] & \mathscr{P}^\sharp \arrow[dl, dashed, "\exists!"] \\ \mathscr{G} & \end{tikzcd}$$

Theorem 3.11. *Fix a site $\mathcal{C}$ and let $\mathscr{P}$ be a presheaf. For some covering $\mathfrak{U} = \{U_i \to U\} \in \mathrm{Cov}(U)$, we define*

$$\check{H}^0(\mathfrak{U}, \mathscr{P}) := \ker\left(\prod_i \mathscr{P}(U_i) \rightrightarrows \prod_{i,j} \mathscr{P}(U_i \times_U U_j)\right).$$

Hence we have a canonical map $\mathscr{P}(U) \to \check{H}^0(\mathfrak{U}, \mathscr{P})$. It's not hard to show that this map is well defined (Tag 03NQ), so we define:

$$\mathscr{P}^+ : U \mapsto \varinjlim_{\mathfrak{U}} \check{H}^0(\mathfrak{U}, \mathscr{P}).$$

(i) The presheaf $\mathscr{P}^+$ is separated;
(ii) If a presheaf $\mathscr{P}$ is separated, then $\mathscr{P}^+$is a sheaf and $\mathscr{P} \to \mathscr{P}^+$ is injective;
(iii) If $\mathscr{P}$ be a sheaf, then $\mathscr{P} \to \mathscr{P}^+$is an isomorphism;
(iv) Presheaf $\mathscr{P}^{++}$ is a sheaf and $\mathscr{P}^{++} \cong \mathscr{P}^\sharp$.

Proof. We refer Tag 00WB. □

Remark 3.12. *The sheafification functor $\sharp$ and forgetful functor i forms an adjoint pair $(\sharp, i)$ and $\sharp$ is exact.*

3.2 Algebraic spaces

3.2.1 Basic definitions

Roughly speaking, an algebraic spaces are the étale-version of schemes. You can stick affine schemes together over étale topology to get the algebraic space. The definition is similar, first we have the following result:

Proposition 3.13. *Let Aff_S be the categories of affine S-schemes where S is a fixed scheme, consider a functor $F : \mathrm{Aff}_S^{op} \to \mathrm{Sets}$ and it is representable by a S-scheme if and only if*
(i) F is a sheaf over big Zariski site;
(ii) The diagonal $\Delta : F \to F \times F$ is representable be separated schemes;
(iii)There exists a family $\{X_i\}$ in Aff_S and morphisms $\pi_i : h_{X_i} \to F$ which are open embeddings such that the map of Zariski sheaves $\coprod_i h_{X_i} \to F$ is surjective.

Proof. See [74] Proposition 1.4.11 and 1.4.13. □

Similarly, we define algebraic space as follows:

Definition 3.14. *Fix a scheme S. Then an algebraic space X over S is a functor $X : (\mathrm{Sch}/S)^{op} \to \mathrm{Sets}$ such that:*
(i) X is a big étale sheaf;
(ii) $\Delta : X \to X \times_S X$ is representable by schemes;
(iii) there exists a surjective étale morphism $U \to X$ of S-schemes.

Remark 3.15. *(a) Note that the first two conditions shows that if T is a scheme, then any morphism $f : T \to X$ are representable by schemes by some easy pure categorial argument (see [74] Lemma 5.1.9). Hence the condition (iii) make sense.*

(b) Many books will define algebraic space is a fppf sheaf instead of a big étale sheaf. But one can show that these two objects are equivalent with some difficult arguments (see [74] Theorem 5.5.2).

Here we give another popular definition of algebraic space which can explain what I said before (stick affine schemes together over étale topology): fixe a scheme S, we call an étale equivalence relation over some S-scheme X is a monomorphism $R \hookrightarrow X \times_S X$ such that:

(a) for every S-scheme T, the T-valued points $R(T) \subset X(T) \times X(T)$ is an equivalence relation;

(b) maps $s, t : R \to X$ induced by two projections are étale.

Proposition 3.16. *Consider $X/R : (\mathrm{Sch}/S)^{op} \to \mathrm{Sets}$ be an étale sheafification of the presheaf $T \mapsto X(T)/R(T)$. Then*

(i) X/R is an algebraic space;

(ii) If Y is an algebraic space with an étale surjective morphism $X \to Y$ from a scheme X. Let $R := X \times_Y X \hookrightarrow X \times_S X$, then R is an étale equivalence relation with isomorphism $X/R \cong Y$.

Proof. See the proof of Proposition 5.2.5 in [74]. □

3.2.2 Basic properties

Proposition 3.17. *The objects of algebraic spaces are closed under finite limits.*

Proof. Pure categorial arguments, we refer [74] Proposition 5.4.6. □

Lemma 3.18. *Let S be a scheme and we work over étale topology over* Sch/S.

(i) The following properties of schemes are locally on the étale topology: locally northerian, reduced, normal, regular, Cohen-Macaulay;

(ii) The following properties of morphisms are locally on target: proper, separated, quasi-separated, surjective, quasi-compact, universally closed, universally injective, open immersion, closed immersion, integral, finite;

(iii) The following properties of morphisms are locally on sourse and target: locally of finite type, locally of finite presentation, flat, étale, universally open, locally quasi-finite, uniramified, smooth.

Proof. This is part of descent theory, we refer Tag 0238. □

Definition 3.19. *Let P be the property of schemes locally on the étale topology (such as Lemma 3.18 (i)), then we call an algebraic space X has property P if there exists an étale surjection $U \to X$ such that U has property P.*

Definition 3.20. *Let P be the property of morphisms locally on the target (such as Lemma 3.18 (ii)), then we call an morphism of algebraic spaces $f : X \to Y$ has property P if there is an étale cover $V \to Y$ such that $V \times_Y X \to V$ has property P.*

Moreover, let $f : X \to Y$ be an morphism of algebraic spaces with diagonal $\Delta_{X/Y} : X \to X \times_Y X$. We say f is separated (quasi-separated, locally separated) if $\Delta_{X/Y}$ is a closed immersion (quasi-compact, immersion).

Remark 3.21. *(i) Let $f: X \to Y$ be an morphism of algebraic spaces, its diagonal $\Delta_{X/Y}: X \to X \times_Y X$ is automatically representable! Indeed, for T a S-scheme with morphism $T \to X \times_Y X$, since $X \times_Y X \to X \times_S X$ is monomorphism, we get*

$$X_{\Delta_{X/Y}, X \times_Y X} T \cong X_{\Delta_X, X \times_S X} T$$

and hence $\Delta_{X/Y}$ is representable.

(ii) The definition of locally separated told us that the different between schemes and algebraic spaces since the diagonals of schemes always be an immersion.

Definition 3.22. *Let P be the property of morphisms locally on the sourse and target (such as Lemma 3.18 (iii)), then we call an morphism of algebraic spaces $f: X \to Y$ has property P if there is an étale cover $v: V \to Y$ and $u: U \to X$ such that $U \times_Y V \to V$ has property P.*

Remark 3.23. *These two definitions are not conflit by trivial diagram:*

$$\begin{tikzcd} U \arrow[r, "u"'] & X \times_Y V \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & V \arrow[d, "v"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

3.3 Fibered categories and stacks

Definition 3.24. *(i) Let $p: F \to C$ be a functor (called F a category over C). Then $\phi: u \to v$ in F is called cartesian if for any $w \in F$ with $\psi: w \to v$ and factorization of $p(\psi)$:*

$$p(w) \xrightarrow{h} p(u) \xrightarrow{p(\phi)} p(v)$$

such that:

$$\begin{tikzcd} w \arrow[d, mapsto] \arrow[drr, dashed, "\exists!\lambda"] \arrow[drrr, "\psi"] & & & \\ p(w) \arrow[drr, "h"] & & u \arrow[r, "\phi"] \arrow[d, mapsto] & v \arrow[d, mapsto] \\ & & p(u) \arrow[r, "p(\phi)"] & p(v) \end{tikzcd}$$

(ii) For a category $p: F \to C$ over C, for any $U \in C$ we define $F(U)$ be a subcategory of F consist of $u \in F$ such that $p(u) = U$ and whose morphisms consist of $f: u' \to u$ such that $p(f) = \mathrm{id}_U$.

Definition 3.25. *(i) We call a category $p: F \to C$ over C to be a fibered category over C if for any morphism $f: U \to V$ in C and $v \in F(V)$, there exists a cartesian morphism $\phi: u \to v$ such that $p(\phi) = f$;*

(ii) A morphism of fibered categories $p: F \to C, q: G \to C$ over C is a functor $g: F \to G$ such that $q \circ g = p$ and g sends cartesian maps to cartesian maps.

Definition 3.26. *(i) A fibered category $p : F \to C$ is called a category fibered in sets if for any $U \in C$ then all morphisms in $F(U)$ are identity (that is, $F(U)$ is a set);*

(ii) a fibered category $p : F \to C$ is called a category fibered in groupoids if for any $U \in C$ the sub categories $F(U)$ is a groupoids.

Remark 3.27. *Let $X \in C$ and we define $\underline{\mathrm{Aut}}_x : (C/X)^{op} \to \mathrm{Groups}$ as*

$$(f : Y \to X) \mapsto \mathrm{Aut}_{F(Y)}(f^*x)$$

where f^*x are some pullbacks and this defined up to some canonical isomorphism. Similarly we can define $\underline{\mathrm{Isom}}(x, x')$.

Proposition 3.28. *The functor*

$$(\textit{presheaves on } C) \to (\textit{categories fibered in sets over } C)$$

sending a presheaf $\mathscr{P}$ to $\mathcal{P}$ where $\mathcal{P}$ has objects of pairs $(U, x), x \in \mathscr{P}(U)$ and morphisms $(U', x') \to (U, x)$ such that $g : U' \to U$ in C and $g^*x = x'$ in $F(U')$, is an equivalent.

Proof. An application of 2-Yonada lemma, we refer [74] Proposition 3.2.8. □

Note that as all fibered categories over a category C form a 2-category, the diagrams or fiber products of them are all 2-diagrams, that is, there is always commute up to a isomorphism of functors! For details we refer [74]. This also well used in the diagrams of stacks.

To define the notions of stacks, we need to learn the descent theory of fibered categories (as in chapter 4 in [74]). Here we just give a definition and some remarks. Now let C be a category with finite fiber products and $p : F \to C$ be a fibered category.

Definition 3.29. *Let $\{X_i \to Y\}_{i \in I}$ be a set of morphisms in C and we define $F(\{X_i \to Y\})$ to be the category consist of: $(\{E_i\}_{i \in I}, \{\sigma_{ij}\}_{i,j \in I})$ where $E_i \in F(X_i)$ and isomorphism $\sigma_{ij} : \mathrm{pr}_1^* E_i \to \mathrm{pr}_2^* E_j$ in $F(X_i \times_Y X_j)$ such that for any $i, j, k \in I$ the following diagram in $F(X_i \times_Y X_j \times_Y X_k)$ commutes (cocycle condition):*

$$\begin{tikzcd}
\mathrm{pr}_{12}^*\mathrm{pr}_1^*E_i \arrow[r, "\mathrm{pr}_{12}^*\sigma_{ij}"] \arrow[d, equal] & \mathrm{pr}_{12}^*\mathrm{pr}_2^*E_j \arrow[r, equal] & \mathrm{pr}_{23}^*\mathrm{pr}_1^*E_j \arrow[d, "\mathrm{pr}_{23}^*\sigma_{jk}"'] \\
\mathrm{pr}_{13}^*\mathrm{pr}_1^*E_i \arrow[r, "\mathrm{pr}_{13}^*\sigma_{ik}"] & \mathrm{pr}_{13}^*\mathrm{pr}_2^*E_k \arrow[r, equal] & \mathrm{pr}_{23}^*\mathrm{pr}_2^*E_k
\end{tikzcd}$$

Then we call $\{\sigma_{ij}\}$ a descent data on $\{E_i\}$. Moreover, we have the canonical functor

$$\epsilon : F(Y) \to F(\{X_i \xrightarrow{f_i} Y\})$$

by sending $E_0 \in F(Y)$ to $\{f_i^* E_0, \sigma_{ij,\mathrm{can}}\}$ where $\sigma_{ij,\mathrm{can}} : \mathrm{pr}_1^* f_i^* E_0 \to \mathrm{pr}_2^* f_j^* E_0$ is the canonical isomorphism.

The collection of morphisms $\{X_i \to Y\}$ is of effective descent for F if the functor ϵ is an equivalence. We call it is only effective if ϵ essential surjective.

Definition 3.30. *Let C be a site and $F \to C$ be a category fibered in groupoids. We call $F \to C$ is a stack if for any $X \in C$ and any covering $\{X_i \to X\}$ they forms an effective descent.*

Remark 3.31. *A category fibered in groupoids is a stack if and only if any covering $\{X_i \to X\}$ effective and $\underline{\mathrm{Isom}}(x, y)$ is a sheaf.*

Proposition 3.32. *(i) The fiber product of stacks is also a stack;*

(ii) For any category fibered in groupoids $F \to C$ has a stackification F^a over C which is a stack and a morphism $q : F \to F^a$ induce

$$\mathrm{Hom}_C(F^a, G) \to \mathrm{Hom}_C(F, G)$$

is an equivalence.

Proof. See [74] Proposition 4.6.4 and Theorem 4.6.5. $\square$

3.4 Algebraic stacks and Deligne-Mumford stacks

3.4.1 Basic definitions

Here we may let a scheme U to be the category $(\mathrm{Sch}/U)_{\text{ét}}$ and use Proposition 3.28 such that an algebraic space X to be the corresponding category fibered in sets.

Definition 3.33. *A morphism of stacks $f : \mathscr{X} \to \mathscr{Y}$ is called representable (by algebraic space) if for any scheme V and morphism $y : V \to \mathscr{Y}$, the stack $\mathscr{X} \times_{\mathscr{Y},y} V$ is an algebraic space.*

Remark 3.34. *For any representable $f : \mathscr{X} \to \mathscr{Y}$ of stacks and for any algebraic space V with $y : V \to \mathscr{Y}$, the fiber product $\mathscr{X} \times_{\mathscr{Y},y} Y$ is an algebraic space. This can be checked after an étale base change. Then this follows from Tag 02YS.*

Definition 3.35. *(i) A stack $\mathscr{X}/S$ is called an algebraic stack (or sometimes they called Artin stack) if $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable and there exists a scheme X and $\pi : X \to \mathscr{X}$, a smooth surjective morphism.*

(ii) An algebraic stack $\mathscr{X}/S$ is called a Deligne-Mumford stack if there exists an étale cover $\pi : X \to \mathscr{X}$ with X a scheme, instead of smooth one.

Remark 3.36. *(i) The condition $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable holds if and only if for any S-scheme U and $u_1, u_2 \in \mathscr{X}(U)$ such that the space $\underline{\mathrm{Isom}}(u_1, u_2)$ is an algebraic space. Indeed this is almost trivial as*

$$\underline{\mathrm{Isom}}(u_1, u_2) \cong \mathscr{X} \times_{(u_1,u_2),\mathscr{X}\times_S\mathscr{X}} U.$$

Note that this can be replaced by algebraic spaces, using similar Tag 02YS;

(ii) The condition $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable implies any morphism from a scheme to an algebraic stack is representable. Indeed we have the following 2-fiber product:

$$\begin{tikzcd} U \times_{u,\mathscr{X},t} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \times_S T \arrow[d] \\ \mathscr{X} \arrow[r, "\Delta"] & \mathscr{X} \times_S \mathscr{X} \end{tikzcd}$$

where $t : T \to \mathscr{X}, u : U \to \mathscr{X}$ are morphisms from schemes;

(iii) For any S-algebraic spaces X, Y maps to an algebraic stack $\mathscr{X}$, by (i) we can find that $X \times_{\mathscr{X}} Y \cong \underline{\mathrm{Isom}}(\mathrm{pr}_1^* x, \mathrm{pr}_2^* y)$, which is an algebraic space.

Proposition 3.37. *(i) Let X/S is an algebraic space and G/S be a smooth group scheme which act on X, define the quotient stack $[X/G]$ to be a stack consist of $(T, \mathscr{P}, \pi)$ where T is an S-scheme, $\mathscr{P}$ is a $G \times_S T$ torsor over the big étale site of T and $\pi : \mathscr{P} \to X \times_S T$ is a $G \times_S T$-equivariant morphism of sheaves on* Sch/T. *Then stack $[X/G]$ is an algebraic stack;*

(ii) Every algebraic stack $\mathscr{X}$ can be written as a smooth groupoid $s, t : R \rightrightarrows U$ with $U \to \mathscr{X}$ be the smooth surjection;

(iii) Fiber product of algebraic stacks is also an algebraic stack.

Proof. See [74] Example 8.1.12 and Proposition 8.1.16. $\square$

Remark 3.38. *If G/S be a smooth group scheme which acts trivially on S, then we define classifying stack of G is BG.*

3.4.2 Basic properties

Definition 3.39. *Let P be some property of S-schemes which is local in the smooth topology (such as locally noetherian, regular, locally of finite type over S, locally of finite presentation over S). We say that an Artin stack $\mathscr{X}/S$ has property P if there exists a scheme X and a smooth cover $f : X \to \mathscr{X}$ having property P (By [74] Lemma 8.2.4, this can be replaced by algebraic space).*

Definition 3.40. *(i) Let $f : \mathscr{X} \to \mathscr{Y}$ be any morphism (may NOT representable) of algebraic stacks over S. A call a chart for f is:*

$$\begin{tikzcd}
X \arrow[dr, dashed, "g"] \arrow[drr, "h"] \arrow[ddr] & & \\
& \mathscr{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "p"] \\
& \mathscr{X} \arrow[r] & \mathscr{Y}
\end{tikzcd}$$

where X, Y are algebraic spaces and g, p are smooth and surjective.

(ii) Let $f : \mathscr{X} \to \mathscr{Y}$ be any morphism (may NOT representable) of algebraic stacks over S. Let P be a property of morphisms of schemes which is locally on the sourse and target with respect to the smooth topology (such as smooth, locally of finite presentation, surjective). We say f has property P if there exists a chart of f by schemes (means X, Y above are schemes, but by [74] Proposition 8.2.8, we can replace this by any charts) such that the morphism h has property P.

(iii) Let $f : \mathscr{X} \to \mathscr{Y}$ be any representable morphism of algebraic stacks over S. Let P be a property of morphisms of schemes which is locally on the target with respect to the smooth topology over the category of algebraic spaces over S (such as étale, smooth of relative dimension d, separated, quasi-separated, proper, affine, finite, unramified,

a closed embedding, an open embedding, an embedding). We say f has property P if for any Y an algebraic space and $Y \to \mathscr{Y}$, the morphism $\mathscr{X} \times_{\mathscr{Y}} Y \to Y$ has property P.

Remark 3.41. *As in the case of algebraic spaces, the diagonal $\Delta_{\mathscr{X}/\mathscr{Y}} : \mathscr{X} \to \mathscr{X} \times_{\mathscr{Y}} \mathscr{X}$ is always representable.*

Definition 3.42. *Let $f : \mathscr{X} \to \mathscr{Y}$ be any morphism (may NOT representable) of algebraic stacks over S.*

(i) We say f is quasi-separated if $\Delta_{\mathscr{X}/\mathscr{Y}}$ is quasi-compact and quasi-separated;

(ii) We say f is separated if $\Delta_{\mathscr{X}/\mathscr{Y}}$ is proper.

Remark 3.43. *Note that if f is representable, then this definition is not conflit to the previous definitions. This follows that now $\Delta_{\mathscr{X}/\mathscr{Y}}$ is proper if and only if it is closed immersion (Tag 04YS); $\Delta_{\mathscr{X}/\mathscr{Y}}$ is quasi-compact if and only if it is quasi-compact and quasi-separated (Tag 04YT).*

Definition 3.44. *(i)Let $\mathscr{X}$ be a noetherian algebraic stack with smooth cover $U \to \mathscr{X}$ and corresponding smooth groupoid $s, t : R \rightrightarrows U$, and let $u \in U$ be a preimage of $x \in |\mathscr{X}|$ (that is, some $x : \mathrm{Spec} K \to \mathscr{X}$). We define the dimension of $\mathscr{X}$ at x to be*

$$\dim_x \mathscr{X} = \dim_u U - \dim_{e(u)} R_u \in \mathbb{Z} \cup \{\infty\}$$

where R_u is the fiber of $s : R \to U$ over u and $e : U \to R$ denotes the identity morphism in the groupoid. We define the dimension of $\mathscr{X}$ is

$$\dim \mathscr{X} := \sup_{x \in |\mathscr{X}|} \dim_x \mathscr{X} \in \mathbb{Z} \cup \{\infty\};$$

(ii) If $\mathscr{X}$ is an algebraic stack and $x : \mathrm{Spec} k \to \mathscr{X}$, Then Zariski tangent space $T_{\mathscr{X},x}$ of $\mathscr{X}$ at x defined as all 2-commutative diagrams module isomorphisms:

$$\begin{tikzcd} \mathrm{Spec} k \arrow[d, hook] \arrow[dr, "x"] \arrow[d, phantom, shift left=3ex, "\Leftarrow"] & \\ \mathrm{Spec} k[\varepsilon] \arrow[r] & \mathscr{X} \end{tikzcd}$$

Theorem 3.45. *Let $\mathscr{X}$ be a smooth northerian algebraic stack over k and $x \in \mathscr{X}(k)$ be a point with smooth stabilizer. Then*

$$\dim_x \mathscr{X} = \dim T_{\mathscr{X},x} - \dim G_x.$$

3.4.3 Results of Deligne-Mumford stacks

Here is a very important characteristic of Deligne-Mumford stacks and we will use this many times:

Theorem 3.46. *(See [74] 8.3.3) Let $\mathscr{X}/S$ be an algebraic stack, then the following statement*

(a) $\mathscr{X}$ is a Deligne-Mumford stack;

(b) the diagonal $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is formally unramified;

(c) for any algebraic closed field k and any point $x \in \mathscr{X}(k)$, the group scheme $\underline{\mathrm{Aut}}_x$ is reduced finite k-scheme.

Then (a)⇔(b), and if $\mathscr{X}$ noetherian, then (a)⇔(b)⇔(c).

Proof. See [74] Theorem 8.3.3. □

Proposition 3.47. *Let $\mathscr{X}$ be a Deligne-Mumford stack with an étale cover $X \to \mathscr{X}$.*
***(A)** Let $\mathrm{Qcoh}(\mathscr{X})$ be the category of the quasi-coherent sheaves over $\mathscr{X}$, an object F of it defined as:*

(A1) For any $f : S \to \mathscr{X}$ where S be a scheme, a quasi-coherent sheaf F_f on S_{zar};

(A2) An isomorphism $\rho_H : h^*F_g \cong F_f$ for any 2-diagram

$$\begin{tikzcd} S \arrow[r, "h"] \arrow[d, "f"'] \arrow[r, phantom, shift right=3, "\Searrow"] & T \arrow[dl, "g"] \\ \mathscr{X} & \end{tikzcd}$$

of schemes;

(A3) For any pair of morphisms $H_1 : f_1 \to f_2, H_2 : f_2 \to f_3$ where $f_i : S_i \to \mathscr{X}$ are schemes, the diagram

$$\begin{tikzcd} h_1^*(h_2^*(F_{f_3})) \arrow[r, "\cong"] \arrow[d, "h_1^*(\rho_{H_2})"] & (h_2 \circ h_1)^*(F_{f_3}) \arrow[d, "\rho_{H_2 \circ H_1}"] \\ h_1^*(F_{f_2}) \arrow[r, "\rho_{H_1}"] & F_{f_1} \end{tikzcd}$$

of isomorphisms of sheaves over S_1 commutes.
***(B)** Let $\mathrm{Eqcoh}(\mathscr{X})$ be the category of the extended quasi-coherent sheaves over $\mathscr{X}$, an object F of it defined as:*

(B1) A quasi-coherent sheaf F_f on S_{zar} for any étale map $f : S \to \mathscr{X}$ from a scheme;

(B2) An isomorphism $\rho_H : h^*F_g \cong F_f$ for any 2-diagram

$$\begin{tikzcd} S \arrow[r, "h"] \arrow[d, "f"'] \arrow[r, phantom, shift right=3, "\Searrow"] & T \arrow[dl, "g"] \\ \mathscr{X} & \end{tikzcd}$$

of étale maps of schemes;

(B3) For any pair of étale morphisms $H_1 : f_1 \to f_2, H_2 : f_2 \to f_3$ where $f_i : S_i \to \mathscr{X}$ are schemes, the diagram

$$\begin{tikzcd} h_1^*(h_2^*(F_{f_3})) \arrow[r, "\cong"] \arrow[d, "h_1^*(\rho_{H_2})"] & (h_2 \circ h_1)^*(F_{f_3}) \arrow[d, "\rho_{H_2 \circ H_1}"] \\ h_1^*(F_{f_2}) \arrow[r, "\rho_{H_1}"] & F_{f_1} \end{tikzcd}$$

of isomorphisms of sheaves over S_1 commutes.
(C) *Let $\mathrm{Qd}_X(\mathscr{X})$ be the category of the quasi-coherent sheaves over X with descent data related to $X \to \mathscr{X}$.*
Conclusion. *Then there are equivalence*

$$\mathrm{Qcoh}(\mathscr{X}) \cong \mathrm{Eqcoh}(\mathscr{X}) \cong \mathrm{Qd}_X(\mathscr{X})$$

and their composition in any three orders is isomorphic to the appropriate identity functor.

Proof. See [8] Proposition XIII.2.9. □

Theorem 3.48 (Local structure of DM-stacks)**.** *Let $\mathscr{X}$ be a separated Deligne-Mumford stack. Let $x \in \mathscr{X}(k)$ be a geometric point with stabilizer G_x. Then exists an affine étale map*

$$f : ([\mathrm{Spec}A/G_x], w) \to (\mathscr{X}, x)$$

where $w \in (\mathrm{Spec}A)(k)$ such that f induces an isomorphism of the stabilizer groups at w. Moreover, it can be arranged that $f^{-1}(BG_x) \cong BG_w$.

Proof. See [1] Theorem 4.2.1. □

3.5 Some fundamental theorems

Theorem 3.49 (Valuative Criteria)**.** *Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of noetherian algebraic stacks. Assume f is of finite type and with separated diagonals. Then consider any DVR and its fraction field K with a 2-commutative diagram*

$$\begin{tikzcd}
\mathrm{Spec}K \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\
\mathrm{Spec}R \arrow[r] \arrow[ur, dashed] & \mathscr{Y}
\end{tikzcd}$$

Then

(1) f is proper if and only if there exists an extension of DVRs $R \to R'$ and $K \to K'$ of fraction fields having finite transcendence degree and a lifting unique up to unique isomorphism

$$\begin{tikzcd}
\mathrm{Spec}K' \arrow[r] \arrow[d] & \mathrm{Spec}K \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\
\mathrm{Spec}R' \arrow[r] \arrow[urr, dashed] & \mathrm{Spec}R \arrow[r] & \mathscr{Y}
\end{tikzcd}$$

(2) f is separated if and only if every two liftings of the first diagram are uniquely isomorphic.

(3) f is universally closed if for the first diagram, there exists an extension of DVRs $R \to R'$ and $K \to K'$ of fraction fields having finite transcendence degree and a lifting as in the second diagram.

Proof. See [1] Theorem 3.8.5 or St 0CLY. □

Here is a heart definition in the moduli theory:

Definition 3.50 (Coarse moduli space)**.** *A morphism $f : \mathscr{X} \to X$ from an algebraic stack to an algebraic space is a coarse moduli space if*

(i) for every algebraically closed field k, the morphism $\mathscr{X}(k)/\sim \to X(k)$, from the set of isomorphism classes of objects of $\mathscr{X}$ over k, is bijective;

(ii) for every map $g : \mathscr{X} \to Y$ to an algebraic space factors uniquely as

$$\begin{tikzcd} \mathscr{X} \arrow[d, "f"] \arrow[dr, "g"] & \\ X \arrow[r, dashed, "\exists!"'] & Y \end{tikzcd}$$

Now we have the famous theorem due to Keel-Mori:

Theorem 3.51 (Keel-Mori theorem)**.** *Assume S is locally noetherian and that X is an algebraic stack separated and locally of finite presentation over S. Then there exists a coarse moduli space $f : \mathscr{X} \to X$ such that*

(i) X/S is separated and locally of finite type;

(ii) f is proper, and the map $\mathscr{O}_X \to f_*\mathscr{O}_{\mathscr{X}}$ is an isomorphism;

(iii) If $X' \to X$ is a flat morphism of algebraic spaces, then $f' : \mathscr{X} \times_X X' \to X'$ is a coarse moduli space for X'.

Proof. This is very complicated, we refer [74] Theorem 11.1.2. □

Remark 3.52. *Actually for the existence of the coarse moduli space, we can remove the noetherian hypothesis and separateness and let the inertia stack $I_S(\mathscr{X})$ finite (or has finite diagonal). This is due to B. Conrad. For non-separated algebraic stack, we may not have the coarse moduli space. But J. Alper develops the theory of good moduli spaces in this case and use it into several aspects, such as the Hassett-Keel program of $\overline{M}_g$ and K-moduli of Fano varieties.*

Theorem 3.53 (Local structure of coarse moduli space)**.** *Let $\mathscr{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Let $\pi : \mathscr{X} \to X$ be its coarse moduli space. For any closed point $x \in |\mathscr{X}|$ with geometric stabilizer G_x, there exists a cartesian*

$$\begin{tikzcd} {[\mathrm{Spec} A/G_x]} \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "\pi"] \\ \mathrm{Spec} A^{G_x} \arrow[r, "s"] & X \end{tikzcd}$$

such that s is an étale neighborhood of $\pi(x) \in |X|$.

Proof. Follows from the construction in the proof of the Keel-Mori theorem. See [1] Corollary 4.3.23. □

3.6 Miscellany

Theorem 3.54 (Le Lemme de Gabber)**.** *Let $\mathscr{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian scheme S. Then there exists a finite, generically étale and surjective morphism $Z \to \mathscr{X}$ where Z be a scheme.*

Proposition 3.55. *Let $\mathscr{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Let $\pi : \mathscr{X} \to X$ be the coarse moduli space. If $\mathscr{L}$ is a line bundle on $\mathscr{X}$, then for N sufficiently divisible $\mathscr{L}^{\otimes N}$ descends to X.*

Proof. See [1] Proposition 4.3.37. □

Proposition 3.56. *If G be an algebraic group acting on some scheme H, hence we get a quotient stack $[H/G]$. Then we have $\mathrm{Qcoh}([H/G]) \cong \mathrm{Qcoh}(H, G)$ where the latter is the category of the G-equivariant quasi-coherent sheaf over H.*

Proof. See [8] Proposition XIII.2.19. □

Corollary 3.57. *We have a group isomorphism $\mathrm{Pic}([H/G]) \cong \mathrm{Pic}(H, G)$.*

Proposition 3.58. *A morphism $X \to T$ of schemes is called a Brauer-Severi scheme of relative dimension r if there exists an étale cover $T' \to T$ such that $X \times_T T' \cong \mathbb{P}^r_{T'}$. Then:*

(i) The groupoids {Brauer-Severi schemes of relative dimension r over T} and {principal PGL_r-bundles over T} defines an equivalence of groupoids given by $X \mapsto \underline{\mathrm{Isom}}_T(\mathbb{P}^r_T, X)$ and conversely $P \mapsto (P \times \mathbb{P}^r)\backslash \mathrm{PGL}_r$;

(ii) Let $X \to S$ be a proper, flat, and finitely presented morphism of schemes. Assume that for every geometric point $\mathrm{Spec} k \to S$, the geometric fiber $X \times_S k$ is isomorphic to $\mathbb{P}^1_k$. Then $X \to S$ is a Brauer-Severi scheme of relative dimension 1.

4 The basic theory of moduli space of curves

4.1 $\mathscr{M}_g$ be a Deligne-Mumford Stack for $g \neq 1$

Here we mainly consider the $g \neq 1$ curves.

Definition 4.1. *Let $\mathscr{M}_g$ be the fibered category over schemes with objects of form $(S, f : C \to S)$ where S be a scheme and f be a proper smooth morphism such that every geometric fiber of S is a connected genus g curve. The morphisms are base-change.*

Our main result of this section is to prove that $\mathscr{M}_g$ is a Deligne-Mumford stack for $g \geq 2$. For $g = 0$ we can run the same argument and we can get $\mathscr{M}_0 \cong B\mathrm{PGL}_2$ and some results of $\mathscr{M}_{0,n}$. Here we follows [1].

4.1.1 $\mathscr{M}_g$ be a stack for $g \neq 1$

Lemma 4.2 (Descent for polarized schemes). *Let $\mathscr{P}ol$ be the category consist of objects $(f : X \to Y, L)$ where f is a proper flat morphism and L is a relatively ample invertible sheaf. The morphism are diagrams of cartesian with isomorphic pullback of line bundles. Consider the fibered category $\mathscr{P}ol \to (Sch)$, then it has effective fppf-descent.*

Proof. See [74] 4.4.10. □

Theorem 4.3. *For $g \neq 1$, the fibered category $\mathscr{M}_g$ is a stack.*

Proof. Consider $\mathscr{M}_g \to \mathscr{P}ol$ sends $C \to S$ to $(C \to S, \Omega^1_{C/S})$ when $g \geq 2$ and $(C \to S, \Omega^{1,\otimes -1}_{C/S})$ when $g = 0$.

For a fppf covering $S' \to S$, then we get

$$\begin{tikzcd} \mathscr{M}_g(S) \arrow[r] \arrow[d] & \mathscr{P}ol(S) \arrow[d, "\cong"] \\ \mathscr{M}_g(S' \to S) \arrow[r] & \mathscr{P}ol(S' \to S) \end{tikzcd}$$

Hence every object of $\mathscr{M}_g(S' \to S)$ is in the essential image of $\mathscr{M}_g(S)$. By the descent of sheaves used in $h_- \to h_=$ making it fully faithful. □

4.1.2 For $g \geq 2$, $\mathscr{M}_g$ be a Deligne-Mumford stack

Now let $L_{C/S} = (\Omega^1_{C/S})^{\otimes 3}$. By Lemma 2.14 (iv), for any family of smooth curves $p : D \to S$ we get a closed immersion $D \hookrightarrow \mathbb{P}(p_* L_{D/S})$ where $p_* L_{D/S}$ is locally free of rank $5g - 5$. Let $H = \underline{\mathrm{Hilb}}^P_{\mathbb{P}^{5g-6}}$ where $P(t) = \deg(L^{\otimes t}_{C/S}) + 1 - g = (6g - 6)t + 1 - g$ be the Hilbert polynomial of $D_s \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(s)}$. Let the universal closed subscheme:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[dr, "\pi"'] & H \times \mathbb{P}^{5g-6} \arrow[d] \\ & H \end{tikzcd}$$

► **Claim 1.** There is a unique subscheme $H' \subset H$ consist of $h \in H$ such that

(a) $C_h \to \mathrm{Spec}(\kappa(h))$ is smooth and geometrically connected;

(b) $C_h \hookrightarrow \mathbb{P}^{5g-6}_{\kappa(h)}$ is embedded by complete linear system $|L_{C_h/\kappa(h)}|$;

(c) the line bundles $L_{C_{H'}/H'}$ and $\mathscr{O}_{C_{H'}}(1)$ differ by a pullback of a line bundle from H' (that is, there exists a line bundle N over H' such that $L_{C_{H'}/H'} \otimes p^* N = \mathscr{O}_{C_{H'}}(1)$). Moreover, if $T \to H$ be a morphism such that (a)-(c) hold for the family $C_T \to T$, then $T \to H$ factors through H'.

Since the condition that a fiber of a proper morphism (of finite presentation) is smooth is an open condition on the target, the condition on H that C_h is smooth is open. Consider the Stein factorization (St 03H0) $C \to \widetilde{H} := \underline{\mathrm{Spec}}_H \pi_* \mathscr{O}_C \to H$

where $C \to \widetilde{H}$ is proper with geometrically connected fibres and $\widetilde{H} \to H$ is finite. As $\mathscr{O}_H \to \pi_*\mathscr{O}_C$ is a morphism between coherent sheaves, then the kernel and cokernel of it have closed supports. Hence $\widetilde{H} \to H$ is an isomorphism over an open subscheme of H, which is precisely where the fibers of $C \to H$ are geometrically connected. Hence the points satiefies (a) be a open subscheme of H, denoted by $H_1 \subset H$.

By Review A.2, there exists a locally closed subscheme $H_2 \subset H_1$ such that a morphism $T \to H_1$ factor through H_2 if and only if $L_{C_T/T}$ and $\mathscr{O}_{C_T}(1)$ differ by a pullback of a line bundle from T. In particular, (c) holds and for all $h \in H_2$, $L_{C_h/\kappa(h)} \cong \mathscr{O}_{C_h}(1)$.

For (b), let $\pi_2 : C_2 := C_{H_2} \to H_2$. Consider $\alpha : H^0(\mathbb{P}^{5g-6}_{\mathbb{Z}}, \mathscr{O}(1)) \otimes \mathscr{O}_{H_2} \to \pi_{2,*}\mathscr{O}_{C_2}(1)$ of vector bundles of rank $5g-5$ on H_2 with fiber $\alpha_h : H^0(\mathbb{P}^{5g-6}_{\kappa(h)}, \mathscr{O}(1)) \to H^0(C_h, \mathscr{O}_{C_h}(1)) \cong H^0(C_h, L_{C_h/\kappa(h)})$. As they have the same rank, α_h is an isomorphism if and only if h is not in supp(coker(α)). Let $H' = H_2 \backslash (\text{supp}(\text{coker}(\alpha)))$ and it satisfies (a)-(c) with that universal property.

▶ **Claim 2.** The group scheme $\mathrm{PGL}_{5g-5} = \underline{\mathrm{Aut}}(\mathbb{P}^{5g-6}_{\mathbb{Z}})$ act on H as: for $g \in \mathrm{Aut}(\mathbb{P}^{5g-6}_S)$ and $[D \subset \mathbb{P}^{5g-6}_S] \in H(S)$, we let $g \cdot [D \subset \mathbb{P}^{5g-6}_S] = [g(D) \subset \mathbb{P}^{5g-6}_S]$. As H' is PGL_{5g-5}-invariant, we claim that $\mathscr{M}_g \cong [H'/\mathrm{PGL}_{5g-5}]$ be an algebraic stack. (See St 044O, St 04UV for quot stacks)

Consider $H' \to \mathscr{M}_g$ as $[D \subset \mathbb{P}^{5g-6}_S] \mapsto (D \to \mathbb{P}^{5g-6}_S \to S)$ is well defined by **Claim 1**. This morphism is PGL_{5g-5}-invariant, hence descends to $[H'/\mathrm{PGL}_{5g-5}]^{pre} \to \mathscr{M}_g$. We claim that this map is fully faithful. Indeed, for a family $p : D \to S$ in H' given by $D \subset \mathbb{P}^{5g-6}_S$, we get $\mathscr{O}_D(1) \cong L_{D/S} \otimes p^*M$ for some line bundle M on S. Use (b) we get

$$H^0(\mathbb{P}^{5g-6}_{\mathbb{Z}}, \mathscr{O}(1)) \otimes \mathscr{O}_S \to p_*\mathscr{O}_D(1) \cong p_*(L_{D/S} \otimes p^*M) \cong p_*L_{D/S} \otimes M$$

be an isomorphism. Then any automorphism of $D \to S$ induces an automorphism of $L_{D/S}$ and thus an automorphism of $p_*L_{D/S} \otimes M$, which induce an automorphism of $\mathbb{P}^{5g-6}_S$ preserving D. By Theorem 4.3, $\mathscr{M}_g$ be a stack, hence induce $[H'/\mathrm{PGL}_{5g-5}] \to \mathscr{M}_g$ which is fully faithful since stackification is fully faithful. Finally we check that $[H'/\mathrm{PGL}_{5g-5}] \to \mathscr{M}_g$ is essentially surjective. As these are all stacks, then they satisfied effective descent of étale covering. Hence we just need to show that for any $p : D \to S$, there exists an étale covering $\{S_i \to S\}$ such that each D_{S_i} is in the image of $H'(S_i) \to \mathscr{M}_g(S_i)$. Actually since $L_{D/S}$ defined $D \hookrightarrow \mathbb{P}(p_*L_{D/S})$ and $p_*L_{D/S}$ is locally free of rank $5g-5$, we let $\{S_i\}$ be open (zariski, hence étale) covering of S such that $(p_*L_{D/S})|_{S_i}$ are all free. Well done.

▶ **Claim 3.** The algebraic stack $\mathscr{M}_g$ is a Deligne-Mumford stack.

By Theorem 3.46, we just need to show for any smooth connected proper curve C over a algebraic closed field k, the group scheme $G := \underline{\mathrm{Aut}}_k(C) = \mathrm{Aut}(C)$ is finite and reduced. We find that $T_{G,e}$ can be identified with the automorphism group of the trivial first order deformation of C. Hence by Proposition 4.43, we get $T_{G,e} = H^0(C, T_C) = 0$, well done.

4.1.3 First properties of $\mathscr{M}_g$ for $g \geq 2$

Proposition 4.4. *As $\mathscr{M}_g \cong [H'/\mathrm{PGL}_{5g-5}]$ and H' is locally of finite type, then $\mathscr{M}_g$ is locally of finite type over $\mathbb{Z}$. As H' is noetherian, so is $\mathscr{M}_g$. So it is finite type over*

$\mathbb{Z}$.

Proposition 4.5. *$\mathscr{M}_g$ have affine diagonal. Indeed, since we have $\mathscr{M}_g \cong [H'/\mathrm{PGL}_{5g-5}]$ which is an algebraic stack, then we have cartesian square*

$$\begin{tikzcd} H' \times \mathrm{PGL}_{5g-5} \arrow[r] \arrow[d] & H' \times H' \arrow[d] \\ \mathscr{M}_g \arrow[r] & \mathscr{M}_g \times \mathscr{M}_g \end{tikzcd}$$

As PGL_{5g-5} affine, then $\mathrm{PGL}_{5g-5} \times H' \to H' \times H' \to H'$ affine, so is $\mathrm{PGL}_{5g-5} \times H' \to H' \times H'$.

4.1.4 Smoothness and dimension of $\mathscr{M}_g$ for $g \geq 2$

Proposition 4.6. *If C is a smooth connected projective curve of genus $g \geq 2$ over k, then $\dim T_{\mathscr{M}_g,[C]} = 3g-3$.*

Proof. By Proposition 4.39, we get $T_{\mathscr{M}_g,[C]} = H^1(C, T_C)$. As $\deg T_C < 0$, we get $H^0(C, T_C) = 0$. So by Riemann-Roch we get

$$\dim T_{\mathscr{M}_g,[C]} = \dim H^1(C, T_C) = -\chi(T_C) = -\deg T_C + g - 1 = 3g - 3,$$

well done. □

Theorem 4.7. *For $g \geq 2$, the Deligne-Mumford stack $\mathscr{M}_g$ is smooth over $\mathbb{Z}$ of relative dimension $3g-3$.*

Proof. Let a field k and a smooth projective connected curve $C \to \mathrm{Spec}(k)$. Consider the following 2-diagram:

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r] \arrow[rr, bend left, "{[C]}"] & \mathrm{Spec}(A_0) \arrow[r] \arrow[d, hook] & \mathscr{M}_g \arrow[d, "f"] \\ & \mathrm{Spec}(A) \arrow[r] & \mathrm{Spec}(\mathbb{Z}) \end{tikzcd}$$

where $A \to A_0$ be a surjective maps of artinian local rings with residue field k with $k = \ker(A \to A_0)$. The map $\mathrm{Spec}(A_0) \to \mathscr{M}_g$ corresponds to a family of curves $C_0 \to \mathrm{Spec}(A_0)$ and a cartesian:

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d] & C_0 \arrow[r, hook, dashed] \arrow[d] & C' \arrow[d, dashed] \\ \mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(A_0) \arrow[r, hook] & \mathrm{Spec}(A) \end{tikzcd}$$

of solid arrows. So to find the lifting, we just need to find the dashed arrows, that is, deformation of C along A. By Propositon 4.43(iii), there exists a cohomology class

$ob_C \in H^2(C, T_C)$ such that this happens if and only if $ob_C = 0$. Hence this is right as C be a curve. Hence $\mathscr{M}_g$ is smooth. By Theorem 3.45, we get

$$\dim_{[C]} \mathscr{M}_g = \dim T_{\mathscr{M}_g,[C]} - \dim \mathrm{Aut}(C).$$

By the final step of the proof of the DM-ness of $\mathscr{M}_g$, we get $\dim \mathrm{Aut}(C) = 0$. Hence $\dim_{[C]} \mathscr{M}_g = \dim T_{\mathscr{M}_g,[C]} = 3g - 3$, well done. $\square$

4.1.5 For $g = 0$

Let $\mathscr{M}_{0,n}$ be a stack parameterizing the genus 0 curves with n sections. By the same argument we can get $\mathscr{M}_{0,n}$ be a stack.

Proposition 4.8. *(i) We have $\mathscr{M}_{0,0} = \mathscr{M}_0 \cong B\mathrm{PGL}_2$ and $\mathscr{M}_{0,1} \cong BU_2$ where $U_2 \subset \mathrm{PGL}_2$ is the two-dimensional subgroup of upper triangular matrices;*

(ii) we have $\mathscr{M}_{0,2} \cong B\mathbb{G}_m$;

(iii) we have $\mathscr{M}_{0,3} \cong \mathrm{Spec}\mathbb{Z}$;

(iv) for $n \geq 3$ we have $\mathscr{M}_{0,n} \cong (\mathbb{P}^1 \backslash \{0, 1, \infty\})^{n-3} \backslash \Delta$ where Δ is the closed subscheme where at least two of the $n - 3$ points are equal.

Proof. (i) Here we need to use some results about Brauer-Severi schemes. Actually $\mathscr{M}_0 \cong B\mathrm{PGL}_2$ is directly from Proposition 3.58. We omitted for the proof of $\mathscr{M}_{0,1} \cong BU_2$;

(ii) By Proposition A.19(ii) that for any T this is equivalent to the line bundles over T which correspond to principal $\mathbb{G}_m$-bundles. Hence $\mathscr{M}_{0,2} \cong B\mathbb{G}_m$;

(iii) This follows directly from Proposition A.19(iii) that all families are trivial;

(iv) Given any curve $(C \to S, s_1, ..., s_n)$ we still have $C \cong S \times \mathbb{P}^1$ and there exists a unique element $B \in \mathrm{PGL}_2$ of the automorphism group of $\mathbb{P}^1$ sending s_1, s_2, s_3 to $0, 1, \infty$. Hence we get

$$(C \to S, s_1, ..., s_n) \cong (S \times \mathbb{P}^1 \to S, 0, 1, \infty, s_4, ..., s_n)$$

and other sections are pairwise distinct and also distinct from $0, 1, \infty$ we get the result. $\square$

Corollary 4.9. *When we consider the stack over Sch/k for any algebraically closed field k, we have $\mathscr{M}_0 \cong B\mathrm{PGL}_2$.*

Proof. In this case we have a more direct and easy proof. Actually here we have invertible sheaves of degree 1. Hence any proper smooth curve of genus 0 be $\mathbb{P}^1_k$. The Hilbert polynomial $p(t) = t+1$ in $\mathbb{P}^1_k$ and can show that it is $\mathbb{P}^1_k$ by some tricks. Hence

$$\mathscr{M}_0 \cong [\underline{\mathrm{Hilb}}^{p(t)}_{\mathbb{P}^1_k}/\mathrm{PGL}_2] = [\mathrm{Grass}_k(2, 2)/\mathrm{PGL}_2] = B\mathrm{PGL}_2,$$

well done. $\square$

4.2 Nodal curves

4.2.1 Basic facts of nodal curves

The the more details, see St 0C46.

Definition 4.10 (Nodes)**.** *Let C be a curve over k.*

(a) If k algebraically closed, we say that $p \in C(k)$ is a node is we have $\widehat{\mathscr{O}}_{C,p} \cong k[[x,y]]/(xy)$;

(b) If k need not be algebraically closed, we say a closed point $p \in C$ is a node if there exists a node $p' \in C_{\overline{k}}$ over p.

We say C be a nodal curve if every closed point is either smooth or nodal.

Proposition 4.11. *Let C be a curve over k. Consider the following statments.*

(a) $p \in C$ is a node;

(b) $\kappa(p)/k$ is separable, there exists a nondegenerate quadratic form $ax^2+bxy+cy^2$ over $\kappa(p)$ such that $\widehat{\mathscr{O}}_{C,p} \cong \kappa(p)[[x,y]]/(ax^2+bxy+cy^2)$ as a k-algebra;

(c) $\kappa(p)/k$ is separable, $\mathscr{O}_{C,p}$ is reduced with δ-invariant 1.

Then we have (a)$\Leftrightarrow$(b)$\Rightarrow$(c).

Proof. See St 0C49 and St 0C4D.

We assume (a)$\Leftrightarrow$(b). Here by Lemma 2.23, we just need to consider the case $\mathscr{O}_{C,p} \cong \kappa(p)[[x,y]]/(ax^2+bxy+cy^2)$ where $Q = ax^2+bxy+cy^2$ is a nondegenerate quadratic form.

Case (I): If Q is split, we may let $\mathscr{O}_{C,p} \cong \kappa(p)[[x,y]]/(xy)$ after some coordinate transformation. Then we get

$$\widetilde{\mathscr{O}}_{C,p} \cong \kappa(p)[[x,y]]/(x) \times \kappa(p)[[x,y]]/(y);$$

Case (II): If not, we have $c \neq 0$ and $b^2-4ac \neq 0$. Hence $\kappa' = \kappa(p)[t]/(a+bt+ct^2)$ is a separable extension of $\kappa(p)$ of degree 2. Then $t = y/x$ is integral over ring $\mathscr{O}_{C,p}$. Hence we conclude that

$$\widetilde{\mathscr{O}}_{C,p} = \kappa'[[x]]$$

with y mapping to tx on the right hand side.

In both cases one verifies by hand that the δ-invariant is 1, well done. □

Remark 4.12. *(i) As for a node $p \in C$ in a nodal curve C, we have $\kappa(p)/k$ is separable. As the two cases above, if p is of case (I), then $f^{-1}(p)$ has two points with residue fields $\kappa(p)$. If p is of case (II), then $f^{-1}(p)$ has only one point with residue field κ', a degree 2 separable extension of $\kappa(p)$;*

(ii) As in (i), all closed points of $\widetilde{C}$ is regular with separable residue fields over k. Hence $\widetilde{C}$ is smooth over k by St 00TV.

Proposition 4.13. *If C is a curve over k and $p \in C$ be a node. Then exists a finite separable field extension K/k, a point $P \in C_K$ over p and $\widehat{\mathscr{O}}_{C_K,P} \cong K[[x,y]]/(xy)$.*

Proof. By Proposition 4.11(b), we get $\kappa(p)/k$ is separable, $\widehat{\mathscr{O}}_{C,p} \cong \kappa(p)[[x,y]]/(ax^2+bxy+cy^2)$ as a k-algebra where $Q = ax^2+bxy+cy^2$ is a nondegenerate quadratic form over $\kappa(p)$. If Q is split, well done. If not, let $K = k[t]/(at^2+bt+c)$ be a separable extension over k with Q split, well done. □

4.2.2 Genus fomula

Let k be algebraically closed field now. Let C be a connected nodal projective curve over k. Let $z_1, ..., z_s$ be its nodes and $C_1, ..., C_t$ be its irreducible components.

By Proposition A.13(1) and (4), we get $\widetilde{C} = \coprod_{i=1}^{t} \widetilde{C}_i$ where $\widetilde{C}, \widetilde{C}_i$ are normalizations. Let $f : \widetilde{C} \to C$. By Proposition 2.24, we get a exact sequence

$$0 \to \mathscr{O}_C \to f_* \mathscr{O}_{\widetilde{C}} \to \bigoplus_{i=1}^{s} \mathscr{Q}_i \to 0$$

where $\mathscr{Q}_i$ supported over z_i. Since by Proposition 4.11(c), we get $\mathscr{Q}_i = \kappa(z_i)$ as the δ-invariant are all 1, hence we get

$$0 \to \mathscr{O}_C \to f_* \mathscr{O}_{\widetilde{C}} \to \bigoplus_{i=1}^{s} \kappa(z_i) \to 0.$$

Hence we get long exact sequence

$$0 \to \underbrace{H^0(C, \mathscr{O}_C)}_{1} \to \underbrace{H^0(\widetilde{C}, \mathscr{O}_{\widetilde{C}})}_{t} \to \underbrace{\bigoplus_{i=1}^{s} \kappa(z_i)}_{s} \to \underbrace{H^1(C, \mathscr{O}_C)}_{g(C)} \to \underbrace{H^1(\widetilde{C}, \mathscr{O}_{\widetilde{C}})}_{\sum_{i=1}^{t} g(\widetilde{C}_i)} \to 0$$

with the labels underneath indicating the dimensions.

Theorem 4.14 (Genus fomula)**.** *With the situation as above, we get*

$$g(C) = \sum_{i=1}^{t} g(\widetilde{C}_i) + s - t + 1.$$

Proof. Trivial by the argument above. □

4.2.3 The dualizing sheaf

We have three way to see this. Consider C be a fixed nodal curve over k.

• **The first way.**

We find that C is locally complete intersection as we can checking locally. As for a node $p \in C$, we have $\widehat{\mathscr{O}}_{C,p} \cong \kappa(p)[[x, y]]/(ax^2 + bxy + cy^2)$ for some nondegenerate quadratic form. By [71] Theorem 21.2(iii), we get $\mathscr{O}_{C,p}$ is a complete intersection over k. Hence by [58] Theorem III.7.11 (adjunction formula for l.c.i), if we embedding it into $\mathbb{P}^N$, then we have $\omega_C \cong \omega_{\mathbb{P}^N} \otimes \bigwedge^{N-1}(\mathscr{I}/\mathscr{I}^2)$ where $\mathscr{I}$ be the ideal sheaf. As this is locally complete intersection, this is a line bundle.

• **The second way.**

This is an abstract way of duality theory, see St 0E31 for more details. As C is locally complete intersection, then by St 0BVA we get C is Gorenstein. By St 0BS2, C must have a dualizing complex ω_C^*. By 0BFQ, as C is Gorenstein, ω_C^* is invertible. By C Cohen-Macaulay, $\omega_C^* = \omega_C[0]$. Hence we win.

• **The third way.**

We can explicit ω_C precisely. Let Σ be the set of nodes of C and let $U = C\backslash\Sigma$. Let the normalization $f : \widetilde{C} \to C$ and $\widetilde{\Sigma} := f^{-1}(\Sigma), \widetilde{U} := f^{-1}(U)$. Now $\widetilde{C}$ is smooth, then we have the dualizing sheaf (line bundle) $\Omega_{\widetilde{C}}$. We get

$$0 \to \Omega_{\widetilde{C}} \to \Omega_{\widetilde{C}}(\widetilde{\Sigma}) \to \mathscr{O}_{\widetilde{\Sigma}} \to 0.$$

Actually the sections of $\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ is the rational sections of $\Omega_{\widetilde{C}}$ with at worst simple poles in $\widetilde{\Sigma}$. Hence for any open $V \subset \widetilde{C}$ and $y \in V \cap \widetilde{\Sigma}$ we have the residue $\mathrm{res}_y : \Gamma(V, \Omega_{\widetilde{C}}(\widetilde{\Sigma})) \to \kappa(y)$.

Definition 4.15. *We define the subsheaf $\omega_C \subset f_*\Omega_{\widetilde{C}}(\widetilde{\Sigma})$ as for any open $V \subset C$ we have*

$$\Gamma(V, \omega_C) = \left\{ \begin{matrix} s \in \Gamma(f^{-1}(V), \Omega_{\widetilde{C}}(\widetilde{\Sigma})) : \text{ for any } z_i \in V \cap \Sigma \\ \text{and } f^{-1}(z_i) = \{p_i, q_i\} \text{ with } \mathrm{res}_{p_i}(s) + \mathrm{res}_{q_i}(s) = 0 \end{matrix} \right\}.$$

Hence we get two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_C & \longrightarrow & f_*\Omega_{\widetilde{C}}(\widetilde{\Sigma}) & \longrightarrow & \bigoplus_{z_i \in \Sigma} \kappa(z_i) & \longrightarrow & 0 \\ & & & & s & \longmapsto & (\mathrm{res}_{p_i}(s) - \mathrm{res}_{q_i}(s)) & & \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f_*\Omega_{\widetilde{C}} & \longrightarrow & \omega_C & \longrightarrow & \bigoplus_{z_i \in \Sigma} \kappa(z_i) & \longrightarrow & 0 \\ & & & & s & \longmapsto & (\mathrm{res}_{p_i}(s)) & & \end{array}$$

Proposition 4.16. *Let C be a nodal curve C over k.*

(a) If $g : C' \to C$ be an étale morphism, then $g^*\omega_C \cong \omega_{C'}$;

(b) Conclude that ω_C be a line bundle.

Proof. (a) As the normalization commutes with étale base change (see St 03GE), we have the cartesian with normalizations

$$\begin{tikzcd} \widetilde{C'} \arrow[r, "g'"] \arrow[d, "f'"] & \widetilde{C} \arrow[d, "f"] \\ C' \arrow[r, "g"] & C \end{tikzcd}$$

By flat base change, we have the process $g^*\omega_C \subset g^*f_*\Omega_{\widetilde{C}}(\widetilde{\Sigma}) \cong f'_*(g')^*\Omega_{\widetilde{C}}(\widetilde{\Sigma}) = f'_*\Omega_{\widetilde{C'}}(\widetilde{\Sigma'})$. By definition and this process, we get $g^*\omega_C \cong \omega_{C'}$.

(b) Use Corollary A.5 and Proposition 4.13, there exists a separable extension K/k such that we get the common étale neighborhood as

$$\begin{tikzcd} & (U, u) \arrow[dl, "F"'] \arrow[dr, "G"] & \\ (C, p) & & (\mathrm{Spec} K[x, y]/(xy), 0) \end{tikzcd}$$

Let $D = \mathrm{Spec}K[x,y]/(xy)$ and normalization $\widetilde{D} \cong \mathbb{A}^1_K \sqcup \mathbb{A}^1_K$. Then $\Gamma(\widetilde{D}, \Omega_{\widetilde{D}}) = \Gamma(\mathbb{A}^1_K, \omega_{\mathbb{A}^1_K}) \times \Gamma(\mathbb{A}^1_K, \omega_{\mathbb{A}^1_K})$ and $(\frac{dx}{x}, -\frac{dy}{y})$ be a section of ω_D. As any section is of form $(f(x)\frac{dx}{x}, -g(y)\frac{dy}{y})$ where $f(0) = g(0)$, which is precisely the condition for $(f,g) \in \Gamma(\widetilde{D}, \mathscr{O}_{\widetilde{D}})$ to descend to a global function on D. In other words, $\omega_D \cong \mathscr{O}_D$ with generator $(\frac{dx}{x}, -\frac{dy}{y})$. By (a), we get $\omega_U = G^*\omega_D$, hence ω_U is a line bundle. As $F^*\omega_C = \omega_U$ be a line bundle, we use the descent theory and we win. $\square$

Proposition 4.17. *Let C be a proper nodal curve C over k, then ω_C be the dualizing line bundle of C.*

Proof. (See [8]) We may assume that k is algebraic closed. Choose a divisor $D = r_1 + \cdots + r_h$ consisting of distinct smooth points of C, with the property that any component of C contains at least one of the r_i's. We first claim that $H^1(\omega_C(D)) = 0$. Indeed, we get an exact sequence

$$0 \to (f_*\omega_{\widetilde{C}}) \otimes \mathscr{O}_C(D) = f_*(\omega_{\widetilde{C}}(D)) \to \omega_C(D) \to \bigoplus_{\text{nodes}} k \to 0.$$

Hence deduce a surjection

$$H^1(C, f_*(\omega_{\widetilde{C}}(D))) = H^1(\widetilde{C}, \omega_{\widetilde{C}}(D)) \twoheadrightarrow H^1(C, \omega_C(D)).$$

As $\widetilde{C}$ is smooth, we get for any irreducible components and Serre duality in smooth case, we get $H^1(\widetilde{C}, \omega_{\widetilde{C}}(D)) = 0$ as D meets every irreducible components. Hence $H^1(\omega_C(D)) = 0$.

Next we deduce an exact sequence by using the claim as

$$H^0(C, \omega_C(D)) \to H^0(C, \omega_C(D)/\omega_C) \to H^1(C, \omega_C) \to H^1(C, \omega_C(D)) = 0.$$

For any $\phi \in H^1(C, \omega_C)$ we have some lifts $\phi' \in H^0(C, \omega_C(D)/\omega_C)$. We define the trace map as

$$\mathrm{tr}_C : H^1(C, \omega_C(D)) \to k, \phi \mapsto 2\pi\sqrt{-1}\sum_{i=1}^{l} \mathrm{res}_{r_i}\phi'$$

and this is well defined by using residue theorem (of definition). Perfect pairing is omitted. $\square$

Proposition 4.18. *Let C be a nodal curve C over k and $T \subset C$ be an irreducible component and D_T be the union of the intersections of T and another irreducible components, then $\omega_C|_T = \omega_T(D_T)$.*

Proof. Trivial by definition of the dualizing sheaves. $\square$

4.2.4 Local structure of nodes

Theorem 4.19 (Local structure of nodes)**.** *Let $\pi : C \to S$ be a flat and finitely presented morphism such that every geometric fiber is a curve. Let $p \in C$ be a node* *in C_s. Thenwe have a following diagram*

$$\begin{tikzcd}
(C,p) \arrow[d] & (U,u) \arrow[l, "ét"'] \arrow[r, "ét"] \arrow[d] & (\mathrm{Spec} A[x,y]/(xy-f),0) \arrow[dl] \\
(S,s) & (\mathrm{Spec} A, s') \arrow[l, "ét"] &
\end{tikzcd}$$

where each horizontal arrow is a residually-trivial pointed étale morphism and $f \in A$ is a function vanishing at s'.

Sketch. See [1] 5.2.12 or St 0CBY for more details.
Step 1. Reduce to S of finite type over $\mathbb{Z}$. Using noetherian approximation.
Step 2. Reduce to the case where $\widehat{\mathscr{O}}_{C_s,p} \cong \kappa(s)[[x,y]]/(xy)$. Just need to use Proposition 4.13 and since separable, we can find a étale neighborhood (S', s') such that $\kappa(s') = K$.
Step 3. Show that $\widehat{\mathscr{O}}_{C,p} \cong \widehat{\mathscr{O}}_{S,s}[[x,y]]/(xy-f)$ where $f \in \widehat{\mathfrak{m}}_s$. Using the Schlessinger's theorem in formal deformation theory to deduce a diagram similar as what we want at the completion level.
Step 4. Apply Artin approximation (Theorem A.4). Using Artin approximation to deduce our diagram from the completion level. $\square$

Corollary 4.20. *Let $\pi : C \to S$ be a flat and finitely presented morphism such that every geometric fiber is a curve, then the locus*

$$C^{\leq nod} = \{p \in C : p \in C_{\pi(p)} \text{ either smooth or node}\} \subset C$$

is open.

Proof. First we know the smooth locus is open. If $p \in C_{\pi(p)}$ is a node, then by Theorem 4.19 we get an étale morphism $g : (U,u) \to (C,p)$. Then $p \in g(U) \subset C^{\leq nod}$ is open. $\square$

Corollary 4.21. *Let $\pi : C \to S$ be a proper flat and finitely presented morphism such that every geometric fiber is a curve, then the locus $S^{\leq nod} = \{s \in S : C_s \text{ is nodal}\} \subset S$ is open.*

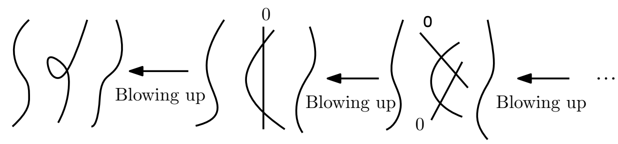
Proof. As we find that

$$S^{\leq nod} = S \backslash \pi(C \backslash C^{\leq nod}).$$

By the previous Corollary and π is proper, then $S^{\leq nod}$ is open. $\square$

Remark 4.22. *Actually later we can prove that the stack $\mathscr{M}_g^{\leq nod}$ is a algebraic stack. But the main problem of $\mathscr{M}_g^{\leq nod}$ is that it is not separated and not of finite type. We can see the figure below for intuitive understanding:*

Corollary 4.23 (Comparison)**.** *Let's compare $\omega_{\mathcal{C}/Y}$ and $\Omega^1_{\mathcal{C}/Y}$ where $\phi : \mathcal{C} \to Y$ are a family of complex nodal curves. We will follows [8] X.2 and more general we can see [69] 6.4.2. Also, we will work on the complex topology.*



Pick a node p in some fiber, then by Theorem 4.19 we get near p we have the composition $\phi|_U : U \hookrightarrow \mathbb{C}^2 \times Y \to Y$ where U defined by $F := xy - f$. By adjunction formula we get the local generator of $\omega_{\mathcal{C}/Y}$ is $F^{-1}dx \wedge dy \pmod F$. Using [69] Lemma 6.4.12, we get a homomorphism

$$\rho : \Omega^1_{\mathcal{C}/Y} \to \omega_{\mathcal{C}/Y}$$

given by id *if it near smooth points and $\rho(\alpha) = F^{-1}\alpha' \wedge dF \pmod F$ if near the nodes where α' is on $\mathbb{C}^2 \times Y \to Y$ restriction is α. Actually near nodes we have $\rho(dx) = xF^{-1}dx \wedge dy$ and $\rho(dy) = -yF^{-1}dx \wedge dy$. Now we consider*

$$0 \to \ker \rho \to \Omega^1_{\mathcal{C}/Y} \xrightarrow{\rho} \omega_{\mathcal{C}/Y} \to \mathrm{coker}\rho \to 0.$$

• ***Claim 1.*** $\rho(\Omega^1_{\mathcal{C}/Y}) = \mathscr{I}\omega_{\mathcal{C}/Y}$ ***where*** $\mathscr{I}$ ***be the ideal locally generated by*** x, y ***(locally ideal of that node).***

Let S be the subspace correspond to $\mathscr{I}$, then for now $\mathrm{coker}\rho = \omega_{\mathcal{C}/Y} \otimes \mathscr{O}_S$. *As locally near nodes we get $xy = f$ and $\rho(\Omega^1_{\mathcal{C}/Y})$ generated by $xF^{-1}dx \wedge dy$ and $yF^{-1}dx \wedge dy$, then $\mathscr{I}$ be the ideal locally generated by x, y.*

• ***Claim 2. When*** Y ***be a single point, then*** $\ker \rho$ ***is the one-dimensional complex vector space generated by the class of*** $xdy = -ydx$***.***

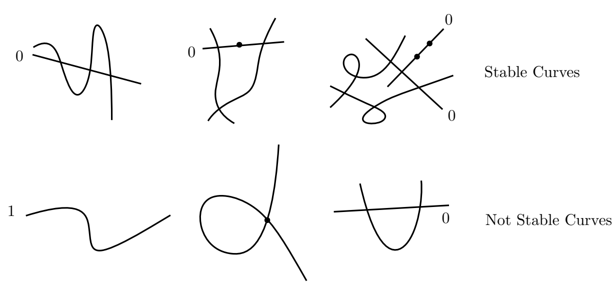
This is trivial by this construction.

• ***Claim 3. When*** Y ***is integral sand generic fiber of*** ϕ ***is smooth, then*** ρ ***is injective.***

4.3 Stable curves

4.3.1 Basic facts of stable curves

An n-pointed curve is a curve C over a field k together with an ordered collection of k-rational points $p_1, ..., p_n \in C$ which we call the marked points. A point $q \in C$ of an n-pointed curve is called special if q is a node or a marked point.



Definition 4.24. *A n-pointed curve $(C, p_1, ..., p_n)$ over k is prestable if it is a geometrically connected, nodal and projective curve, and $p_1, ..., p_n \in C(k)$ are distinct smooth points.*

A n-pointed curve $(C, p_1, ..., p_n)$ over k is semistable if

(a) it is prestable;

(b) every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 2 special points;

(c) C is not of genus 1 without marked points.

A n-pointed curve $(C, p_1, ..., p_n)$ over k is stable if

(a) it is semistable;

(b) every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 3 special points.

Remark 4.25. *(1) Note that there are no n-pointed stable curve of genus g if $2g-2+n \leq 0$ by Proposition 4.27. We will often impose the condition that $2g-2+n > 0$ in order to exclude these special cases;*

(2) An automorphism of a stable curve $(C, p_1, ..., p_n)$ is an automorphism $\alpha : C \to C$ such that $\alpha(p_i) = p_i$. We denote by $\mathrm{Aut}(C, p_1, ..., p_n)$ the group of automorphisms;

(3) For some general Riemann Roch theorem (such as 0BS6) and the fact that the prestable curves are proper geometrically connected and geometrically reduced, then $\deg(\omega_C) = 2g-2$.

Proposition 4.26. *Let $(C, p_1, ..., p_n)$ be an n-pointed nodal projective curve such that the points p_i are distinct and smooth. Let $\pi : \widetilde{C} \to C$ be the normalization and $\widetilde{p}_i \in \widetilde{C}$ be the unique preimage of p_i and $\widetilde{q}_1, ..., \widetilde{q}_m \in \widetilde{C}$ be an ordering of the preimages of nodes. Then*

(a) $(C, p_1, ..., p_n)$ is stable if and only if every connected component of $(\widetilde{C}, \{\widetilde{p}_i\}, \{\widetilde{q}_j\})$ is stable.

(b) The group scheme $\underline{\mathrm{Aut}}(C, \{p_i\})$ is an algebraic group.

(c) $\underline{\mathrm{Aut}}(C, \{p_i\})$ is naturally a closed scheme of $\underline{\mathrm{Aut}}(\widetilde{C}, \{\widetilde{p}_i\}, \{\widetilde{q}_j\})$ with the same connected component of identity.

Proof. (a) Easy to see that we just need to verify that every smooth rational subcurve $\mathbb{P}^1 \subset C$ contains at least 3 special points if and only if every connected component of $(\widetilde{C}, \{\widetilde{p}_i\}, \{\widetilde{q}_j\})$ have the same property. This is also trivial as we just need to consider the rational component of $(\widetilde{C}, \{\widetilde{p}_i\}, \{\widetilde{q}_j\})$ and using the genus formula.

We omit the proof of (b),(c). □

Proposition 4.27. *Let $(C, p_1, ..., p_n)$ be an n-pointed prestable curve. The following are equivalent*

(i) $(C, p_1, ..., p_n)$ is stable;

(ii) $\mathrm{Aut}(C, p_1, ..., p_n)$ is finite; and

(iii) $\omega_C(p_1 + ... + p_n)$ is ample.

Proof. (i)⇔(ii). By the results in Section 2.1.2 we get for smooth connected projective curve, its automorphism group if finite if and only if ⇔ $2g-2+n > 0$ for (g, n). Now consider the normalization $f : (\widetilde{C}, \{\widetilde{p}_i\}_{i=1}^n, \{\widetilde{q}_j\}_{j=1}^{2s}) \to (C, p_1, ..., p_n)$ with $\widetilde{C} = \coprod_{j=1}^t \widetilde{C}_j$. By Proposition 4.26 (a), we have (i)⇔ for all j, $(\widetilde{C}_j, \{\widetilde{p}_i \in \widetilde{C}_j\}_{i=1}^n, \{\widetilde{q}_k \in \widetilde{C}_j\}_{k=1}^{2s})$ is stable. As all $\widetilde{C}_j$ have marked points and use Proposition 4.26 (c), (ii)$\Leftrightarrow$ for all j, $\mathrm{Aut}(\widetilde{C}_j, \{\widetilde{p}_i \in \widetilde{C}_j\}_{i=1}^n, \{\widetilde{q}_k \in \widetilde{C}_j\}_{k=1}^{2s})$ are finite. Hence by the case of smooth case, we win.

(i)$\Leftrightarrow$(iii). By Proposition A.14, 4.18 and consider the normalization $\pi : \widetilde{C} \to C$, we get $\omega_C(p_1 + ... + p_n)$ is ample if and only if $\pi^*\omega_C(p_1 + ... + p_n)$ is ample if and only if for any irreducible components $T \subset \widetilde{C}$, $\omega_C(p_1 + ... + p_n)|_T = \omega_T(\sum_{p_i \in T} p_i + D_T)$ is ample. This latter condition holds precisely if each $\mathbb{P}^1 \subset \widetilde{C}$ contains at least three points that lie over nodes or marked points (using Theorem 2.2) and we win. $\square$

4.3.2 Positivity of the dualizing sheaf

Theorem 4.28. *For any n-pointed stable curve $(C, p_1, ..., p_n)$, the bundle $(\omega_C(p_1 + ... + p_n))^{\otimes k}$ is very ample for $k \geq 3$.*

Proof. We refer Tag 0E8X. In this proof we will use a very general fact about globally generated bundles: Tag 0E3F. Note that we can let k be an algebraically closed since closed immersion is locally over fpqc topology. $\square$

Remark 4.29. *(a) From this we can get that $\pi_*\omega_{C/S}^{\otimes n}$ is locally free of rank $(2n-1)(g-1)$;*

(b)For another proof, we refer [29] Theorem (I.2) and [74] Proposition 13.2.17. Here we give a sketch as follows and omit some details:

We just prove the case of k is algebraically closed and no marked points. In this case we just need to show that its sections separates points and tangent vectors. As for $x, y \in C(k)$ (maybe the same points) and their ideal I_x, I_y, we have

$$0 \to \omega_C^{\otimes k} \otimes I_x I_y \to \omega_C^{\otimes k} \to \omega_C^{\otimes k} \otimes \mathscr{O}_C / I_x I_y \to 0.$$

So we just need to show that $H^1(C, \omega_C^{\otimes k} \otimes I_x I_y) = 0$. By Serre duality, we need to show

$$\begin{aligned} &H^1(C, \omega_C^{\otimes k} \otimes I_x I_y) = H^0(C, (\omega_C^{\otimes k} \otimes I_x I_y)^\vee \otimes \omega_C) \\ &= H^0(C, \mathscr{H}om(\omega_C^{\otimes k} \otimes I_x I_y, \omega_C)) = \mathrm{Hom}(I_x I_y, \omega_C^{\otimes(1-k)}) = 0. \end{aligned}$$

We need a case analysis on whether x, y are smooth or nodal.

If $x \in C$ is smooth, then $I_x = \mathscr{O}_C(-x)$ is invertible. If $x \in C$ is a node, consider the blowing up $\pi : C' \to C$ along x with $\pi^{-1}(x) = \{x_1, x_2\}$. Then for any line bundle L on C we claim that

$$\mathrm{Hom}(I_x, L) \cong H^0(C', \pi^* L), \mathrm{Hom}(I_x^2, L) \cong H^0(C', \pi^* L(x_1 + x_2)).$$

We just prove the first statement, the second is similar. First we have

$$0 \to \mathscr{O}_C \to \pi_* \mathscr{O}_{C'} \to \kappa(x) \to 0,$$

tensoring L we get

$$0 \to L \to \pi_* \mathscr{O}_{C'} \otimes L = \pi_* \pi^* L \to L(x) \to 0.$$

Hence we have

$$0 \to \mathrm{Hom}(I_x, L) \to \mathrm{Hom}(\pi^* I_x, \pi^* L) \xrightarrow{f} \mathrm{Hom}(I_x, L(x)) = \mathrm{Hom}(I_x/I_x^2, L(x)).$$

On the other hand, we have a short exact sequence

$$0 \to \pi^* L \to \pi^* L(x_1 + x_2) \to \pi^* L(x_1) \oplus \pi^* L(x_2) \to 0$$

inducing

$$0 \to H^0(C', \pi^* L) \to H^0(C', \pi^* L(x_1 + x_2)) \xrightarrow{g} \pi^* L(x_1) \oplus \pi^* L(x_2).$$

Let $J = \mathscr{O}_{C'}(-x_1 - x_2) \subset \mathscr{O}_{C'}$*, we get*

$$0 \to K \to \pi^* I_x \to J \to 0$$

where $\mathrm{supp}(K) = \{x_1, x_2\}$ *by checking locally. Since* $\pi^* L$ *is torsion free at* x_1, x_2*, we have* $\mathrm{Hom}(K, \pi^* L) = 0$*, so this defines an isomorphism*

$$\mathrm{Hom}(\pi^* I_x, \pi^* L) \cong \mathrm{Hom}(J, \pi^* L) \cong H^0(C', \pi^* L(x_1 + x_2)).$$

We also have $I_x/I_x^2 \cong \pi_*(J/J^2)$ *and* $\mathrm{Hom}(I_x/I_x^2, L(x)) \cong \mathrm{Hom}(\pi_*(J/J^2), L(x))$*. This isomorphism identifies*

$$\ker(\mathrm{Hom}(\pi^* I_x, \pi^* L) \to \mathrm{Hom}(I_x/I_x^2, L(x)))$$

and

$$\ker(H^0(C', \pi^* L(x_1 + x_2)) \to H^0(\pi^* L(x_1) \oplus \pi^* L(x_2))).$$

Hence we get $\mathrm{Hom}(I_x, L) = H^0(C', \pi^* L)$*, hence the claim is right.*

Case (I). *If* x, y *are all smooth points, then* $\deg(\omega_C^{\otimes(1-k)}(x+y)) = (1-k)(2g-2)+2 < 0$ *for* $k \geq 3$*. Hence*

$$\mathrm{Hom}(I_x I_y, \omega_C^{\otimes(1-k)}) = H^0(C, \omega_C^{\otimes(1-k)}(x+y)) = 0.$$

Case (II). *If* x *is a node and* y *is a smooth point, then by the claim, we win.*

Case (III.1). *If* $x = y$ *is a node, then by the claim we get*

$$\mathrm{Hom}(I_x^2, \omega_C^{\otimes(1-k)}) \cong H^0(C', \pi^* \omega_C^{\otimes(1-k)}(x_1 + x_2)).$$

Consider the normalization $\widetilde{C}$ *of* C *(and* C'*, also), we consider an irreducible component* $E \subset \widetilde{C}$*. Then* $\pi^* \omega_C^{\otimes(1-k)}(x_1 + x_2)$ *restrict to* E *has degree*

$$(1-k)(2g_E - 2 + \#\{E \cap \widetilde{\Sigma}\}) + \#(\{x_1, x_2\} \cap E)$$

is negative unless $k = 3$*,* $\{x_1, x_2\} \subset E$*,* E *is a rational curve meeting the other components of* C *in exactly one other point. In this case the degree on* E *is zero. So this global section is determined by its value at the point of* E *meeting the other components of* C*. Since not every component of* $\widetilde{C}$*, we win.*

Case (III.2). *If* $x \neq y$ *are all nodes, the blowing up* $\varpi : C'' \to C$ *along* $\{x, y\}$*. We can get similar conclusion*

$$\mathrm{Hom}(I_x I_y, \omega_C^{\otimes(1-k)}) \cong H^0(C', \varpi^* \omega_C^{\otimes(1-k)}).$$

This is zero since in any irreducible of normalization has negative degree.

4.3.3 Families of stable curves

Definition 4.30. *(1) A family of n-pointed nodal curves is a flat, proper and finitely presented morphism $C \to S$ of schemes with n sections $\sigma_1, ..., \sigma_n : S \to C$ such that every geometric fiber C_s is a (reduced) connected nodal curve.*

(2) A family of n-pointed stable curves (resp. semistable curves, prestable curves) is a family $C \to S$ of n-pointed nodal curves such that every geometric fiber $(C_s, \sigma_1(s), ..., \sigma_n(s))$ is stable (resp. semistable, prestable).

Remark 4.31. *(1) We can define the fibered category of groupoid $\overline{\mathscr{M}}_{g,n}$ as for any scheme S, define $\overline{\mathscr{M}}_{g,n}(S) = \{(C, \sigma_1, ..., \sigma_n) \to S : \text{is a family of stable curves of genus } g\}$. Note also that since the geometric fibers are stable curves, the image of each σ_i is a divisor contained in the smooth locus and we can form the line bundle $\omega_{C/S}(\sum_i \sigma_i)$.*

(2) We can define relative dualizing line bundle $\omega_{C/S}$ as $C \to S$ is l.c.i. By [57], we can get the following properties: (2.a) $\omega_{C/S}|_{C_s} = \omega_{C_s}$; (2.b) for any $f : T \to S$ we have $f^*\omega_{C/S} = \omega_{C\times_S T/T}$.

Proposition 4.32. *Let $\pi : (C, \sigma_1, ..., \sigma_n) \to S$ be a family of n-pointed stable curves of genus g. Let $L = \omega_{C/S}(\sum_i \sigma_i)$. If $k \geq 3$, then $L^{\otimes k}$ is relatively very ample and $\pi_* L^{\otimes k}$ is a vector bundle of rank $(2k-1)(g-1) + kn$.*

Proof. Similar as the smooth case by using Riemann Roch and cohomology and base change. Omitted here. □

Proposition 4.33 (Openness of stability)**.** *Let $\pi : (C, \sigma_1, ..., \sigma_n) \to S$ be a family of n-pointed nodal curves. The locus of S such that $(C_s, \sigma_i(s))$ is stable is open.*

Proof. As the locus such that $\sigma_i(s)$ is smooth is open, we just need to let this family is prestable. Using 4.27, we have two arguments:
Argument 1. Group scheme $\underline{\mathrm{Aut}}(C/S, \sigma_i) \to S$ has upper semicontinuous dimension of fibers, then as stable locus is the locus such that it is dimension 0 locus. Hence open.
Argument 2. Using the openness of ample locus. □

Proposition 4.34 (Openness of being nodal)**.** *Let $f : X \to S$ be a flat proper morphism of $\mathbb{C}$-schemes. Then the set of all $s \in S$ such that $X_s = f^{-1}(s)$ is not a connected nodal curve is closed in S. If, in addition, n sections σ_i of f are given, then the set of all $s \in S$ such that $(X_s; \sigma_i(s))$ is not a connected n-pointed nodal curve is closed in S.*

Sketch. We will give a sketch and for the detailed proof see [8] Proposition XI.5.1. First we need to let the fibers of f has dimension 1 by flatness and properness.
• **Step 1. Reduce to the case that fibers are connected and have no embedded components.** Easy to see that $\dim H^0(X_s, \mathscr{O}_{X_s}) = 1$ for all $s \in S$ if X_s is connected and reduced. As this is the stalks of $f_*\mathscr{O}_X$, we consider the free resolution $K^0 \xrightarrow{\alpha} K^1 \to \cdots$ at some open subset. Hence the locus of $\dim H^0(X_s, \mathscr{O}_{X_s}) > 1$ is the locus of $\mathrm{rank}(\alpha) \leq \mathrm{rank}(K^0) - 2$. Hence is closed.

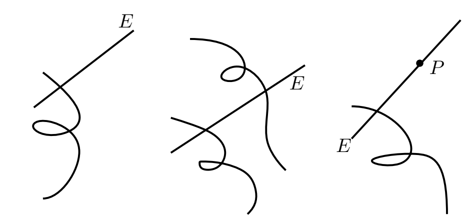
• **Step 2. Show that being neither nodal nor smooth is closed.** Here we need to represent nodes by some functions. Then we use some equivalent conditions (see [8] Lemma X.2.3) that if f be a function over 0 and $f(0) = 0$, then f defines the smooth point 0 if and only if the first-order partials of f not vanish at the origin; f defines the node 0 if and only if the first-order partials of f vanish and the Hessian not vanish. □

4.3.4 Rational tails and bridges

Definition 4.35. *Let $(C, p_1, ..., p_n)$ be a n-pointed prestable curve. We say a smooth rational subcurve $E \cong \mathbb{P}^1 \subset C$ is*

(i) a rational tail if E meets other irreducible components at exactly 1 time, and E contains no marked points;

(ii) a rational bridge if either E meets other irreducible components at exactly 2 time and contains no marked points, or E meets other irreducible components at exactly 1 time and contains exactly 1 marked point.

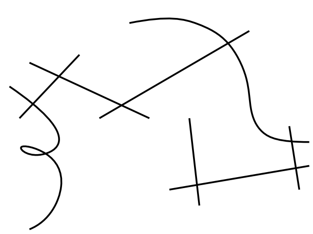


Rational Tail Rational bridges

Remark 4.36. *(1) C is stable if and only if it is prestable and has no rational tails and bridges;*

(2) C is semistable if and only if it is prestable and has no rational tails.

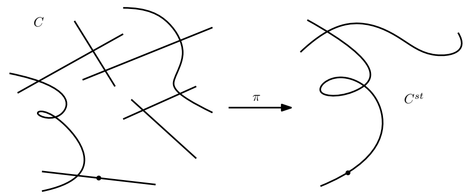
Or we can also have chain of rational tails and bridges like:



4.3.5 The stable model

• **The stable model of a single curve.**

Let $(C, p_1, ..., p_n)$ be a n-pointed prestable curve. Let $E_i \subset C$ are its rational tails and bridges. We let $C^{st} := \overline{C \backslash \bigcup_i E_i}$ and let $\pi : C \to C^{st}$ be the induced map. Let $p_i' = \pi(p_i)$, then $(C^{st}, \{p_i'\})$ is a stable curve, which we call the stable model of $(C, \{p_i\})$ and $\pi : C \to C^{st}$ the stabilization morphism. Like this:



For the serious argument of contraction to the stable curves, we refer [11]: contracting rational tails (St 0E3G), contracting rational bridges (St 0E7M), contracting to a stable curve (St 0E7N). We omitted here.

• **The stable model of a family of curves.**

For a family of nodal curves, we also have the following conclusion.

Proposition 4.37. *Let $(C \to S, \sigma_i)$ be a family of n-pointed prestable curves. Then there exists a unique (up to isomorphism) morphism $\pi : C \to C^{st}$ such that*

(a) $(C^{st} \to S, \{\sigma_i'\})$ is a n-pointed family of stable curves where $\sigma_i' = \pi \circ \sigma_i$;

(b) for each $s \in S$, $(C_s, \{\sigma_i(s)\}) \to (C_s^{st}, \{\sigma_i'(s)\})$ is the stable model;

(c) $\mathscr{O}_{C^{st}} = \pi_* \mathscr{O}_C$ and $R^1\pi_*\mathscr{O}_C = 0$ and this remains true after base change by a morphism $S' \to S$ of schemes;

(d) If $C \to S$ is a family of semistable curves, then $\omega_{C/S}(\sum_i \sigma_i)$ is the pullback of the relatively ample line bundle $\omega_{C^{st}/S}(\sum_i \sigma_i')$.

Proof. See St 0E7B for the detailed proof. □

4.4 Deformation theory of nodal and stable curves

After some basic results over arbitrary fields, we will focus on the curves over $\mathbb{C}$. We mainly follows [8] chapter XI (all results over $\mathbb{C}$) and some results over arbitrary fields we follows [1]. Some basic result and proofs we follows [77]. Here we let $k[\varepsilon] := k[x]/(x^2)$.

4.4.1 Elementary deformation theory and smooth objects

Definition 4.38. *Let X be a scheme over k. A first order deformation of X is a scheme $\mathcal{X}$ flat over $k[\varepsilon] = k[\varepsilon]/(\varepsilon^2)$ with $X \cong \mathcal{X} \times_{k[\varepsilon]} k$.*

We say $\mathcal{X}$ is trivial if $\mathcal{X}$ is isomorphic as first deformations to $\mathcal{X} \times_k k[\varepsilon]$, and locally trivial if there exists a Zariski-cover $X = \bigcup_i U_i$ such that $\mathcal{X}|_{U_i}$ is a trivial first order deformation of U_i, that is, $U_i \times_k k[\varepsilon] \cong \mathcal{X}|_{U_i}$ where $\mathcal{X}|_{U_i} \subset \mathcal{X}$ be a open subscheme with the same topology of U_i.

We let $\mathrm{Def}(X)$ be the isomorphism classes of first order deformations of X and $\mathrm{Def}^{lt}(X)$ be the isomorphism classes of locally trivial first order deformations of X.

Proposition 4.39 (See [1] D.1.11)**.** *For a scheme X of finite type over k with affine diagonal, there is a bijection*

$$\mathrm{Def}^{lt}(X) \leftrightarrow H^1(X, T_X).$$

In particular, if X_0 is smooth, then we have bijection

$$\mathrm{Def}(X) \leftrightarrow H^1(X, T_X),$$

as every first order deformations of smooth affine schemes is trivial.

Sketch. For a locally trivial first order deformation

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ \mathrm{Spec} k \arrow[r, hook] & \mathrm{Spec} k[\varepsilon] \end{tikzcd}$$

let affine covering $\{U_i\}$ of X such that $\mathcal{X}|_{U_i}$ be a trivial first order deformation. Hence we get isomorphisms $\phi_i : U_i \times_k k[\varepsilon] \cong \mathcal{X}|_{U_i}$. Let $\phi_{ij} := \phi_j^{-1}|_{U_{ij} \times_k k[\varepsilon]} \circ \phi_i|_{U_{ij} \times_k k[\varepsilon]}$ are automorphisms of first order defs, hence we get $\phi_{ij} \in \mathrm{Hom}_{\mathscr{O}_{U_{ij}}}(\Omega_{U_{ij}/k}, \mathscr{O}_{U_{ij}})$. As they satiefies cocycle condition, we get $\{\phi_{ij}\} \in H^1(X, T_X)$ by Čech theory (this is independent on the choice of covering, see [77] Proposition 1.2.9). Converse is trivial. $\square$

Remark 4.40. *For a locally trivial first order deformations ξ of X, we gives a class $\kappa(\xi) \in H^1(X, T_X)$ is called the Kodaira-Spencer class of ξ.*

Definition 4.41. *Consider a family of deformation of a smooth algebraic variety X over k*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X}_f \arrow[l] \arrow[d] \\ \mathrm{Spec} k \arrow[r, hook, "s"] \arrow[rr, bend right] & S & \mathrm{Spec} k[\varepsilon] \arrow[l, "f"'] \end{tikzcd}$$

hence we get

$$\kappa_{\mathcal{X}/S,s} : T_{S,s} \to H^1(X, T_X),$$

we called it Kodaira-Spencer map.

Definition 4.42. *Let $A' \twoheadrightarrow A$ has square-free kernel and $X \to \mathrm{Spec}(A)$ is flat. A deformation of $X \to \mathrm{Spec}(A)$ over A' is $X' \to \mathrm{Spec}(A')$ with $X' \times_{A'} A \cong X$. A morphism of deformations over A' is a morphism of schemes over A' restricting to the identity on X.*

Proposition 4.43. *Let $A' \twoheadrightarrow A$ has square-free kernel J. If $X \to \mathrm{Spec}(A)$ is a smooth morphism of schemes where X has affine diagonal, then*

(a) the group of automorphisms of a deformation $X' \to \mathrm{Spec}(A')$ of $X \to \mathrm{Spec}(A)$ over A' is bijective to $H^0(X, T_{X/A} \otimes_A J)$;

(b) If there exists a deformation of $X \to \mathrm{Spec}(A)$ over A', then the set of isomorphism classes of all such deformations is a torsor under $H^1(X, T_{X/A} \otimes_A J)$;

(c) There is an element $ob_X \in H^2(X, T_{X/A} \otimes_A J)$ with the property that there exists a deformation of $X \to \mathrm{Spec}(A)$ over A' if and only if $ob_X = 0$.

Proof. See [1] Proposition D.2.6. $\square$

Back to smooth curves over $\mathbb{C}$ pf genus g.

Theorem 4.44. *Let $(C; q_1, ..., q_n)$ be a n-pointed smooth n-pointed genus g curve over $\mathbb{C}$.*

(i) We have

$$\mathrm{Def}(C; q_1, ..., q_n) \leftrightarrow H^1(C, T_C(-\sum_{i=1}^{n} q_i));$$

(ii) There exists a deformation

$$\phi : \mathcal{C} \to (B, b_0), \sigma_i : B \to \mathcal{C} \text{ such that } \chi : (C; q_1, ..., q_n) \cong (\phi^{-1}(b_0), \sigma_i(b_0))$$

of $(C; q_1, ..., q_n)$ such that the Kodaira-Spencer map

$$\kappa : T_{b_0} B \to H^1(C, T_C(-\sum_{i=1}^{n} q_i))$$

is an isomorphism and B is a polydisc of dimension $3g - 3 + n + h^0(C, T_C(-\sum_{i=1}^{n} q_i))$.

Proof. See [8] Theorem XI.2.12. $\square$

4.4.2 Elementary deformations of nodal and stable curves

Lemma 4.45. *Let $(C, p_1, ..., p_n)$ be an n-pointed nodal, connected and projective curve over k with each $p_i \in C$ smooth. Let $\{q_1, ..., q_s\}$ be the nodes of C. Let $(\widetilde{C}, p_i, q'_j, q''_j)$ be the pointed normalization $\pi : \widetilde{C} \to C$ and $\pi^{-1}(q_j) = \{q'_j, q''_j\}$. Then we have the spectral sequence*

$$E_2^{p,q} = H^p(C, \mathscr{E}xt^q_{\mathscr{O}_C}(\Omega_C(p_1 + ... + p_n), \mathscr{O}_C)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_C}(\Omega_C(p_1 + ... + p_n), \mathscr{O}_C)$$

such that induce the following exact sequence

$$\begin{array}{ccc} & 0 \longrightarrow & H^1(C, \mathscr{H}om_{\mathscr{O}_C}(\Omega_C(p_1 + ... + p_n), \mathscr{O}_C)) \\ & & \downarrow \\ 0 \longleftarrow \bigoplus_j \mathrm{Ext}^1_{\widehat{\mathscr{O}}_{C,q_j}}(\Omega_{\widehat{\mathscr{O}}_{C,q_j}}, \widehat{\mathscr{O}}_{C,q_j}) & \longleftarrow & \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C(p_1 + ... + p_n), \mathscr{O}_C) \end{array}$$

Moreover, $\forall j$ we have $\mathrm{Ext}^1_{\widehat{\mathscr{O}}_{C,q_j}}(\Omega_{\widehat{\mathscr{O}}_{C,q_j}}, \widehat{\mathscr{O}}_{C,q_j}) = k$ and $\mathrm{Ext}^2_{\mathscr{O}_C}(\Omega_C(p_1 + ... + p_n), \mathscr{O}_C) = 0$.

Proof. By Grothendieck spectral sequence, we have

$$E_2^{p,q} = H^p(C, \mathscr{E}xt^q_{\mathscr{O}_C}(\Omega_C(p_1 + ... + p_n), \mathscr{O}_C)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_C}(\Omega_C(p_1 + ... + p_n), \mathscr{O}_C).$$

As C is a curve, $E_2^{p,q} = 0$ for $p \geq 2$.

By [39] Propostion 2.3, we get an exact sequence

$$0 \to E_2^{1,0} \to \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C(p_1+...+p_n), \mathscr{O}_C) \to E_2^{0,1} \to E_2^{2,0} \to 0.$$

As Ω_C is locally free away from nodes, $\mathscr{E}xt^1_{\mathscr{O}_C}(\Omega_C(p_1+...+p_n), \mathscr{O}_C)$ is zero-dimensional sheaf supported only at nodes. Hence $E_2^{1,1} = 0$ and

$$\begin{aligned} E_2^{0,1} &= H^0(C, \mathscr{E}xt^1_{\mathscr{O}_C}(\Omega_C(p_1+...+p_n), \mathscr{O}_C)) \\ &= \bigoplus_j \mathrm{Ext}^1_{\mathscr{O}_{C,q_j}}(\Omega_{C,q_j}, \mathscr{O}_{C,q_j}) = \bigoplus_j \mathrm{Ext}^1_{\widehat{\mathscr{O}}_{C,q_j}}(\Omega_{\widehat{\mathscr{O}}_{C,q_j}}, \widehat{\mathscr{O}}_{C,q_j}). \end{aligned}$$

where $\widehat{\Omega}_{C,q_j} = \Omega_{\widehat{\mathscr{O}}_{C,q_j}}$. Hence we get that exact sequence.

Similarly $\mathscr{E}xt^2_{\mathscr{O}_C}(\Omega_C(p_1+...+p_n), \mathscr{O}_C)$ is zero-dimensional sheaf supported only at nodes, then

$$E_2^{0,2} = H^0(C, \mathscr{E}xt^2_{\mathscr{O}_C}(\Omega_C(p_1+...+p_n), \mathscr{O}_C)) = \bigoplus_j \mathrm{Ext}^2_{\widehat{\mathscr{O}}_{C,q_j}}(\Omega_{\widehat{\mathscr{O}}_{C,q_j}}, \widehat{\mathscr{O}}_{C,q_j}).$$

Write $\widehat{\mathscr{O}}_{C,q_j} = k[[x,y]]/(xy)$ and consider the locally free resolution

$$0 \longrightarrow \widehat{\mathscr{O}}_{C,q_j} \xrightarrow{\binom{y}{x}} \widehat{\mathscr{O}}^{\oplus 2}_{C,q_j} \xrightarrow{(dx,dy)} \Omega_{\widehat{\mathscr{O}}_{C,q_j}} \longrightarrow 0$$

Hence we get $\mathrm{Ext}^1_{\widehat{\mathscr{O}}_{C,q_j}}(\Omega_{\widehat{\mathscr{O}}_{C,q_j}}, \widehat{\mathscr{O}}_{C,q_j}) = k$ and $\mathrm{Ext}^2_{\widehat{\mathscr{O}}_{C,q_j}}(\Omega_{\widehat{\mathscr{O}}_{C,q_j}}, \widehat{\mathscr{O}}_{C,q_j}) = 0$. Hence $E_2^{0,2} = E_2^{1,1} = E_2^{2,0} = 0$ and $\mathrm{Ext}^2_{\mathscr{O}_C}(\Omega_C(p_1+...+p_n), \mathscr{O}_C) = 0$. $\square$

Proposition 4.46. *Let $(C, p_1, ..., p_n)$ be an n-pointed nodal, connected and projective curve over k with each $p_i \in C$ smooth. Let $\{q_1, ..., q_s\}$ be the nodes of C. Let $(\widetilde{C}, p_i, q_j', q_j'')$ be the pointed normalization $\pi : \widetilde{C} \to C$ and $\pi^{-1}(q_j) = \{q_j', q_j''\}$. Then we have the following exact sequence*

$$0 \to \mathrm{Def}^{lt}(C) \to \mathrm{Def}(C) \to \bigoplus_j \mathrm{Def}(\widehat{\mathscr{O}}_{C,q_j}) \to 0$$

and

$$\mathrm{Def}^{lt}(C) \cong \mathrm{Def}(\widetilde{C}, p_i, q_j', q_j'') \cong H^1(\widetilde{C}, T_{\widetilde{C}}(-\sum_i p_i - \sum_j (q_j' + q_j''))),$$

$$\mathrm{Def}(C) \cong \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C(p_1+...+p_n), \mathscr{O}_C),$$

$$\mathrm{Def}(\widehat{\mathscr{O}}_{C,q_j}) \cong \mathrm{Ext}^1_{\widehat{\mathscr{O}}_{C,q_j}}(\Omega^1_{\widehat{\mathscr{O}}_{C,q_j}}, \widehat{\mathscr{O}}_{C,q_j}) \cong k.$$

Under these identifications, this exact sequence corresponds to the exact sequence in the Lemma.

Sketch. WLOG again we let $n = 0$. If $\mathcal{C} \to \mathrm{Spec}k[\varepsilon]$ is a locally trivial first order deformation of C, each node q_j extend to a section $\tilde{q}_j : \mathrm{Spec}k[\varepsilon] \to \mathcal{C}$. The pointed normalization of $\mathcal{C}$ along the sections $\tilde{q}_j$ is a first order deformation of the (possible disconnected) pointed normalization $(\widetilde{C}, p_i, q_j', q_j'')$. This gives a map $\mathrm{Def}^{lt}(C) \to \mathrm{Def}(\widetilde{C}, p_i, q_j', q_j'')$. The inverse is provided by gluing the sections of a first order deformation of $(\widetilde{C}, p_i, q_j', q_j'')$ along nodes.

If $\mathcal{C} \to \mathrm{Spec}k[\varepsilon]$ is a first order deformation of C, then ideal sheaf I of $C \to \mathcal{C}$ is $I = I/I^2 \cong \mathscr{O}_C$. The right exact sequence

$$I/I^2 \to \Omega_{\mathcal{C}/k} \to \Omega_{C/k} \to 0$$

is left exact at every smooth point of C. As $C \to \mathrm{Spec}k$ is generically smooth and it follows that $\mathscr{O}_C \cong I/I^2 \to \Omega_{\mathcal{C}/k}$ is generically injective, hence injective. Hence this defines $\mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C, \mathscr{O}_C)$. This is bijective (one can see [8] section XI.3). $\square$

Remark 4.47. *Hence we also have the Kodaira-Spencer map for some $\mathcal{C} \to (S, s)$ as*

$$\kappa_{S,s} : T_{S,s} \to \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C(p_1 + ... + p_n), \mathscr{O}_C).$$

Remark 4.48. *Let C be a nodal curve over $\mathbb{C}$ and let $p \in C$ be a node with normalization N and preimages $\{p_1, p_2\}$.*
$\bullet$***Claim 1.*** $\mathrm{Ext}^1(\Omega_{C,p}, \mathscr{O}_{C,p}) \cong \bigwedge^2(\mathfrak{m}_p/\mathfrak{m}_p^2) \otimes_{\mu_2} \tau$ *where $\mu_2 = \{\pm 1\}$ and τ be the set consisting of the two possible orderings of the branches of C at p.*

I omit this, see [8] page 180.
$\bullet$***Claim 2.*** $\mathrm{Ext}^1(\Omega_{C,p}, \mathscr{O}_{C,p}) \cong T_{N,p_1} \otimes T_{N,p_2}$.

Trivial by Claim 1 and $\mathfrak{m}_p/\mathfrak{m}_p^2 = T_{C,p} = T_{N,p_1} \oplus T_{N,p_2}$ and $\bigwedge^2 T_{N,p_1} \oplus T_{N,p_2}$ identify with $T_{N,p_1} \otimes T_{N,p_2}$ depends on the choice of an ordering of the two summands, we win.

Remark 4.49. *Let $(C; p_1, ..., p_n)$ be an n-pointed nodal curve over $\mathbb{C}$ for simplicity, and let $W = \{w_1, ..., w_l\}$ be some set of nodes of C. Let $f : N \to C$ be the partial normalization at these nodes with $f^{-1}(w_i) = \{r_i, q_i\}$. Let $D = \sum_i p_i$ with inverse $\widetilde{D}$ and $E = \sum(r_i + q_i)$.*
$\bullet$***Claim 1.*** $\mathscr{H}om(\Omega^1_C, \mathscr{O}_C(-D)) \cong f_*\mathscr{H}om(\Omega^1_N, \mathscr{O}_N(-\widetilde{D} - E))$.

This is trivially true at points away form W, so we just need to consider the points in W. Pick any $w_i \in W$, we get $\mathrm{Hom}(\Omega^1_{C,w_i}, \mathscr{O}_{C,w_i}) = \mathrm{Hom}(\mathscr{I}_{w_i}\omega_{C,w_i}, \mathscr{O}_{C,w_i})$ by Corollary 4.23. As $\mathscr{I}_{w_i}\omega_{C,w_i} = \omega_{N,r_i} \oplus \omega_{N,q_i}$ and $\mathscr{I}_{w_i} = \mathscr{O}_{N,r_i}(-r_i) \oplus \mathscr{O}_{N,q_i}(-q_i)$, we get

$$\mathrm{Hom}(\Omega^1_{C,w_i}, \mathscr{O}_{C,w_i}) = \bigoplus_{p=r_i,q_i} \mathrm{Hom}(\omega_{N,p}, \mathscr{O}_{N,p}(-p)),$$

hence $\mathscr{H}om(\Omega^1_C, \mathscr{O}_C(-D)) \cong f_*\mathscr{H}om(\Omega^1_N, \mathscr{O}_N(-\widetilde{D} - E))$.
$\bullet$***Claim 2.*** *We have*

$$\begin{aligned} 0 \to &\mathrm{Ext}^1(\Omega^1_N, \mathscr{O}_N(-\widetilde{D} - E)) \to \mathrm{Ext}^1(\Omega^1_C, \mathscr{O}_C(-D)) \to \\ &\bigoplus_{w_i \in W} \mathrm{Ext}^1(\Omega^1_{C,w_i}, \mathscr{O}_{C,w_i}) \to 0. \end{aligned}$$

By Claim 1, we get $H^1(N, \mathscr{H}om(\Omega^1_C, \mathscr{O}_C(-D))) \cong H^1(N, \mathscr{H}om(\Omega^1_N, \mathscr{O}_N(-\widetilde{D} - E)))$. *Hence by Lemma 4.45, we get*

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(N, \mathscr{H}om(\Omega^1_N, \mathscr{O}_N(-\widetilde{D} - E))) & \longrightarrow & \mathrm{Ext}^1(\Omega^1_N, \mathscr{O}_N(-\widetilde{D} - E)) \\ & & \| & & \downarrow \\ 0 & \longrightarrow & H^1(N, \mathscr{H}om(\Omega^1_C, \mathscr{O}_C(-D))) & \longrightarrow & \mathrm{Ext}^1(\Omega^1_C, \mathscr{O}_C(-D)) \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & \bigoplus_{w \in \mathrm{Sing}(C), w \notin W} \mathrm{Ext}^1(\Omega^1_{C,w}, \mathscr{O}_{C,w}) & \longrightarrow & 0 \\ & \downarrow & & \\ \longrightarrow & \bigoplus_{w \in \mathrm{Sing}(C)} \mathrm{Ext}^1(\Omega^1_{C,w}, \mathscr{O}_{C,w}) & \longrightarrow & 0 \end{array}$$

hence we get

$$\begin{aligned} 0 \to & \mathrm{Ext}^1(\Omega^1_N, \mathscr{O}_N(-\widetilde{D} - E)) \to \mathrm{Ext}^1(\Omega^1_C, \mathscr{O}_C(-D)) \to \\ & \bigoplus_{w_i \in W} \mathrm{Ext}^1(\Omega^1_{C,w_i}, \mathscr{O}_{C,w_i}) \to 0. \end{aligned}$$

Actually the left term in it classifies first-order deformations which are locally trivial at the nodes belonging to W, and the right term classifies first-order smoothings of these nodes.

• ***Claim 3.*** $\bigoplus_{w_i \in W} \mathrm{Ext}^1(\Omega^1_{C,w_i}, \mathscr{O}_{C,w_i}) = \bigoplus_{i=1}^{l} T_{N,r_i} \otimes T_{N,q_i}$.
By claims in Remark 4.48, this is trivial.

Here is a similar result as before over $\mathbb{C}$ via analytic GAGA.

Theorem 4.50. *Let $(C; p_1, ..., p_n)$ be an n-pointed nodal curve of genus g over $\mathbb{C}$. There exists a deformation*

$$\phi : \mathcal{C} \to (B, b_0), \sigma_i : B \to \mathcal{C} \text{ such that } \chi : (C; p_1, ..., p_n) \cong (\phi^{-1}(b_0), \sigma_i(b_0))$$

of $(C; p_1, ..., p_n)$ such that the Kodaira-Spencer map

$$\kappa : T_{b_0} B \to \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C(p_1 + ... + p_n), \mathscr{O}_C)$$

is an isomorphism and B is a polydisc of dimension $3g - 3 + n + \dim \mathrm{Hom}(\Omega_C, \mathscr{O}_C)$.

Finally, let s is the number of nodes in C. We can choose coordinates $t_1, ..., t_s, ...$ on B near b_0 such that the locus parameterizing deformations which are locally trivial at the i-th node is $t_i = 0$. In particular, the locus parameterizing singular curves is $t_1 \cdots t_s = 0$.

Proof. See [8] Theorem XI.3.17. □

Back to the general case.

Proposition 4.51. *Let $(C, p_1, ..., p_n)$ be an n-pointed nodal, connected and projective curve over k with each $p_i \in C$ smooth. Let $A' \twoheadrightarrow A$ be a surjection of artinian local k-algebras with residue field k such that $J = \ker(A' \to A)$ satiefies $\mathfrak{m}_{A'}J = 0$. If $C_A \to \mathrm{Spec}(A)$ be a family of nodal curves such that $C \cong C_A \times_A k$, then*

(a) The group of automorphisms of a deformation $C_{A'} \to \mathrm{Spec}(A')$ of $C_A \to \mathrm{Spec}(A)$ over A' is bijective to $\mathrm{Ext}^0_{\mathscr{O}_C}(\Omega_C(p_1 + ... + p_n), \mathscr{O}_C \otimes_k J)$;

(b) If there exists a deformation of $C_A \to \mathrm{Spec}(A)$ over A', then the set of isomorphism classes of all such deformations is a torsor under $\mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C(p_1+...+p_n), \mathscr{O}_C \otimes_k J)$;

(c) There is an element $ob_{C_A} \in \mathrm{Ext}^2_{\mathscr{O}_C}(\Omega_C(p_1 + ... + p_n), \mathscr{O}_C \otimes_k J)$ with the property that there exists a deformation of $C_A \to \mathrm{Spec}(A)$ over A' if and only if $ob_{C_A} = 0$.

Proof. Since nodal curves are generically smooth and local complete intersections, the unpointed case follows from the general theorem as in [1] Proposition D.2.11. □

Lemma 4.52 (St 0E68)**.** *Let k be an algebraically closed field. Let X be a proper connected nodal scheme of dimensional 1 over k. Let $f : \widetilde{X} \to X$ be the normalization of X and let $S \subset \widetilde{X}$ be the locus where f is not a bijection. Then*

$$\mathrm{Der}_k(\mathscr{O}_X, \mathscr{O}_X) \cong \{D' \in \mathrm{Der}_k(\mathscr{O}_{\widetilde{X}}, \mathscr{O}_{\widetilde{X}}) : D' \textit{ fixed every } x' \in S\}.$$

Proof. Let $x \in X$ be a node with the preimage $x', x'' \in \widetilde{X}$. Pick two uniformizers u, v in $\mathscr{O}_{\widetilde{X},x'}$ and $\mathscr{O}_{\widetilde{X},x''}$, respectively. Hence we have

$$0 \to \mathscr{O}_{X,x} \to \mathscr{O}_{\widetilde{X},x'} \times \mathscr{O}_{\widetilde{X},x''} \to k \to 0,$$

thus u, v as elements in $\mathscr{O}_{X,x}$ with $uv = 0$.

Since (u) is annihilator of v in $\mathscr{O}_{C,x}$ and similar as (v), we have $D(u) \in (u)$ and $D(v) \in (v)$. Since $\mathscr{O}_{\widetilde{C},x'} = k + (u)$, we can extend D to $\mathscr{O}_{\widetilde{C},x'}$ and the extension fixes x'. This makes a D' in the right hand side of the equality. Conversely, given a D' fixing x' and x'' and then we can find that D' preserves $\mathscr{O}_{C,x} \subset \mathscr{O}_{\widetilde{C},x'} \times \mathscr{O}_{\widetilde{C},x''}$ and this is how we go from right to left in the equality. □

Proposition 4.53. *Let $(C, p_1, ..., p_n)$ be an n-pointed stable curve of genus g over k. Then*

$$\dim_k \mathrm{Ext}^i_{\mathscr{O}_C}\left(\Omega_C\left(\sum_i p_i\right), \mathscr{O}_C\right) = \begin{cases} 0, & i = 0, 2; \\ 3g - 3 + n, & i = 1. \end{cases}$$

Proof. We let k is algebraically closed and has no marked point. Let $\pi : \widetilde{C} \to C$ be a normalization and let $\Sigma \subset C$ be the set of nodes. Let $\widetilde{\Sigma} = \pi^{-1}(\Sigma) \subset \widetilde{C}$.

By Lemma 4.45, we get $\dim_k \mathrm{Ext}^2_{\mathscr{O}_C}(\Omega_C, \mathscr{O}_C) = 0$. For Ext^0, we first claim that

$$\mathrm{Hom}_{\mathscr{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathscr{O}_{\widetilde{C}}) \cong \mathrm{Hom}_{\mathscr{O}_C}(\Omega_C, \mathscr{O}_C).$$

This is equivalent to show

$$\mathrm{Der}_k(\mathscr{O}_C, \mathscr{O}_C) \cong \{D' \in \mathrm{Der}_k(\mathscr{O}_{\widetilde{C}}, \mathscr{O}_{\widetilde{C}}) : D' \text{ fixes every points in } \widetilde{\Sigma}\}.$$

Actually this is just Lemma 4.52. This finish the claim. Hence we get

$$\mathrm{Hom}_{\mathscr{O}_C}(\Omega_C, \mathscr{O}_C) \cong \mathrm{Hom}_{\mathscr{O}_{\widetilde{C}}}(\Omega_{\widetilde{C}}(\widetilde{\Sigma}), \mathscr{O}_{\widetilde{C}}) \cong H^0(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})) = 0.$$

For Ext^1, by Lemma 4.45 we have

$$\begin{tikzcd}
0 \arrow[r] & H^1(C, \mathscr{H}om_{\mathscr{O}_C}(\Omega_C, \mathscr{O}_C)) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C, \mathscr{O}_C) \arrow[r] & \bigoplus_j \mathrm{Ext}^1_{\widehat{\mathscr{O}}_{C,q_j}}(\Omega_{\widehat{\mathscr{O}}_{C,q_j}}, \widehat{\mathscr{O}}_{C,q_j}) \arrow[d] \\
& H^1(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma})) & & 0
\end{tikzcd}$$

and $\mathrm{Ext}^1_{\widehat{\mathscr{O}}_{C,q_j}}(\Omega_{\widehat{\mathscr{O}}_{C,q_j}}, \widehat{\mathscr{O}}_{C,q_j}) = k$. This equality in this exact sequence is because

$$H^1(C, \mathscr{H}om_{\mathscr{O}_C}(\Omega_C, \mathscr{O}_C))$$

be the set of locally trivial first order deformation of C preserving nodes and this is equivalent to the set of locally trivial first order deformation of $\widetilde{C}$ fixed $\widetilde{\Sigma}$, which is $H^1(\widetilde{C}, T_{\widetilde{C}}(-\widetilde{\Sigma}))$.

Now let $\widetilde{C} = \coprod_{i=1}^t \widetilde{C}_i$ are connected components and $\widetilde{\Sigma}_i = \widetilde{C}_i \cap \widetilde{\Sigma}$. First we have

$$h^1(\widetilde{C}_i, T_{\widetilde{C}_i}(-\widetilde{\Sigma}_i)) = h^0(\widetilde{C}_i, \Omega^{\otimes 2}_{\widetilde{C}_i}(\widetilde{\Sigma}_i)) = 3g(\widetilde{C}_i) - 3 + \#(\widetilde{\Sigma}_i),$$

hence

$$\begin{aligned}
&\dim_k \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C, \mathscr{O}_C) = h^1(\widetilde{C}_i, T_{\widetilde{C}_i}(-\widetilde{\Sigma}_i)) + \#(\Sigma) \\
&= \sum_{i=1}^t (3g(\widetilde{C}_i) - 3 + \#(\widetilde{\Sigma}_i)) + \#(\Sigma) = 3\sum_{i=1}^t g(\widetilde{C}_i) - 3t + 3\#(\Sigma) = 3g - 3
\end{aligned}$$

by the genus formula, we win. □

4.4.3 Basic concept of Kuranishi family

We will work on analytic category over $\mathbb{C}$ via Serre's GAGA.

Definition 4.54. *Let $(C; p_1, ..., p_n)$ be an n-pointed connected nodal curve of genus g over $\mathbb{C}$. A deformation $\phi : \mathcal{C} \to (B, b_0), \sigma_i : B \to \mathcal{C}$ such that $\chi : (C; p_1, ..., p_n) \cong (\phi^{-1}(b_0), \sigma_i(b_0))$ of $(C; p_1, ..., p_n)$ is said to be a Kuranishi family for $(C; p_1, ..., p_n)$ if it satisfies the following condition:*
(Condition K). *For any deformation $\psi : \mathcal{D} \to (E, e_0)$ of $(C; p_1, ..., p_n)$ and for any small enough connected neighborhood U of e_0, there is a unique morphism of deformations of n-pointed curves*

$$\begin{tikzcd}
\mathcal{D}|_U \arrow[r, "F"] \arrow[d, "\psi|_U"] & \mathcal{C} \arrow[d, "\phi"] \\
(U, e_0) \arrow[r, "f"] & (B, b_0)
\end{tikzcd}$$

Remark 4.55. *In the algebraic case, the neighborhood U of e_0 taken étale locally. Actually this is the same as all analytic local and étale local here.*

Remark 4.56 (Versal)**.** *If we just let the deformation satisfies (condition K) except for uniqueness and the Kodaira-Spencer map at the central fiber be an isomorphism, then we call it a versal deformation.*

We will show, the Kuranishi family for $(C; p_1, ..., p_n)$ exists if and only if $(C; p_1, ..., p_n)$ is stable, at next two sections.

Corollary 4.57. *At the base point of a Kuranishi family, the Kodaira-Spencer map is an isomorphism.*

Proof. This is trivial as the family $\mathcal{C}_\varepsilon \to \mathrm{Spec}\mathbb{C}[\varepsilon]$ just has and has unique map to $\phi : \mathcal{C} \to (B, b_0)$. This defines a bijection between T_{B,b_0} and $\mathrm{Def}(C) = \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C(p_1 + ... + p_n), \mathscr{O}_C)$ via κ_{B,b_0}. □

Corollary 4.58. *consider a deformation of a stable n-pointed curve $(C; p_1, ..., p_n)$ over the pointed analytic space (E, e_0) and let its Kodaira-Spencer map at e_0 is an isomorphism and E is smooth at e_0. Then this deformation is a Kuranishi family for $(C; p_1, ..., p_n)$.*

Proof. The proof is immediate. □

Corollary 4.59. *The base of the Kuranishi family of a stable n-pointed curve $(C; p_1, ..., p_n)$ of genus g is smooth of dimension $3g - 3 + n$.*

Proof. By the previous corollary and Theorem 4.50 and the uniqueness of the Kuranishi family by the universal property. □

Corollary 4.60. *Let $X \to S$ be a family of stable curves with n marked points and s_0 be a fixed point of S. If $X \to S$ is a Kuranishi family for X_{s_0}, then it is a Kuranishi family for X_s for all s in some small neighborhood U of s_0.*

Proof. From the previous results we know that $X \to S$ is Kuranishi for X_s if and only if s is smooth in S and the Kodaira-Spencer map at s is an isomorphism. Hence we just need to check these two things. The first of these conditions is clearly open. The second condition can be induced into a rank condition for a map between vector bundles ss the dimension of $\mathrm{Ext}^1(\Omega^1_{X_s}, \mathscr{O}_{X_s}(-\sum \sigma_i(s)))$ is independent of s. Hence it is open. □

4.4.4 The Hilbert scheme of ν-canonical curves

For any stable n-pointed genus g curve $(C; p_i)$, if we let $D = \sum_i p_i$, then by Proposition 4.32 that for all $\nu \geq 3$, the ν-log-canonical bundle $\omega_C(D)^{\otimes \nu}$ is very ample and embeds C into $\mathbb{P}^{N-1}$ where $N = (2\nu - 1)(g - 1) + \nu n$. Let $P_\nu(t) = (2\nu t - 1)(g - 1) + \nu n t$, we consider the Hilbert scheme $\underline{\mathrm{Hilb}}^{P_\nu}_{\mathbb{P}^{N-1}}$.

By Proposition 4.34, we get the nonempty subset $U \subset \underline{\mathrm{Hilb}}^{P_\nu}_{\mathbb{P}^{N-1}}$ parameterizing connected n-pointed nodal curves is open. Let the $(\pi : \mathcal{Y} \to U, \sigma_i)$ be the restriction of the universal family. As the general points of U does not correspond to an n-pointed curve embedded by the ν-fold log-canonical sheaf, we need to define a new subscheme.

Definition 4.61. *Let $F = (\pi^*\mathscr{O}_{\mathbb{P}^{N-1}}(1))^{-1} \otimes \omega_\pi(\sum_i \sigma_i)^{\otimes \nu}$. We define $H_{\nu,g,n} \subset U \subset \underline{\mathrm{Hilb}}^{P_\nu}_{\mathbb{P}^{N-1}}$ as a subscheme by*

$$H_{\nu,g,n}(X) := \left\{ \alpha : X \to U \;\middle|\; \begin{array}{l} \textit{correspond to} \quad \begin{array}{ccc} \mathcal{Y} \times_U X & \xrightarrow{\beta} & \mathcal{Y} \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\alpha} & U \end{array} \quad \textit{such} \\ \textit{that } \beta^*F \cong \eta^*G \textit{ for some } G \in \mathrm{Pic}(X) \end{array} \right\}.$$

We call $H_{\nu,g,n}$ as the Hilbert scheme of ν-log-canonically embedded, stable, n-pointed, genus g curves.

Lemma 4.62. *Let $h = (C \subset \mathbb{P}^m; p_1, ..., p_n)$ be a nodal curve where $p_1, ..., p_n$ be distinct smooth points of C and $D = \sum_i p_i$. Let H be the Hilbert scheme parameterizing the $(n+1)$-tuples $(Y; q_1, ..., q_n)$, where Y is a subscheme of $C \subset \mathbb{P}^m$ and $q_1, ..., q_n$ points on it, then we have the exact sequence*

$$\begin{aligned} 0 \to &\mathrm{Hom}_{\mathscr{O}_C}(\Omega^1_C, \mathscr{O}_C(-D)) \to \mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^m}}(\Omega^1_{\mathbb{P}^m}, \mathscr{O}_C) \\ &\to T_h H \xrightarrow{\beta} \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega^1_C, \mathscr{O}_C(-D)) \end{aligned}$$

where β is just the Kodaira-Spencer map at h associated to the universal family over H.

Proof. Consider

$$\begin{array}{ccccccccc} \mathcal{D}^* & & & \mathcal{C}^* & & \mathcal{B}^* & & \mathcal{A}^* & \\ \| & & & \| & & \| & & \| & \\ \mathscr{O}_C & & 0 \longrightarrow & \mathscr{I}_C/\mathscr{I}_C^2 & \longrightarrow & \Omega^1_{\mathbb{P}^m} \otimes \mathscr{O}_C & \longrightarrow & \Omega^1_C & \longrightarrow 0 \\ \downarrow & & & \downarrow & & \downarrow & & \downarrow & \\ \mathscr{O}_D & & 0 \longrightarrow & \mathscr{I}_D/\mathscr{I}_D^2 & \longrightarrow & \Omega^1_{\mathbb{P}^m} \otimes \mathscr{O}_D & \longrightarrow & 0 & \end{array}$$

Hence we get

$$0 \to \mathrm{Hom}_{\mathscr{O}_C}(\mathcal{A}^*, \mathcal{D}^*) \to \mathrm{Hom}_{\mathscr{O}_C}(\mathcal{B}^*, \mathcal{D}^*) \to \mathrm{Hom}_{\mathscr{O}_C}(\mathcal{C}^*, \mathcal{D}^*) \to \mathrm{Ext}_{\mathscr{O}_C}(\mathcal{A}^*, \mathcal{D}^*).$$

As $\mathcal{A}^* \to \mathcal{D}^*$ is equivalent to $\Omega^1_C \to \ker(\mathscr{O}_C \to \mathscr{O}_D) = \mathscr{O}_C(-D)$, hence $\mathrm{Hom}_{\mathscr{O}_C}(\mathcal{A}^*, \mathcal{D}^*) = \mathrm{Hom}(\Omega^1_C, \mathscr{O}_C(-D))$. As $\mathcal{B}^* \to \mathcal{D}^*$ determined by $\mathcal{B}^0 \to \mathcal{D}^0$, we get

$$\mathrm{Hom}_{\mathscr{O}_C}(\mathcal{B}^*, \mathcal{D}^*) = \mathrm{Hom}_{\mathscr{O}_C}(\Omega^1_{\mathbb{P}^m} \otimes \mathscr{O}_C, \mathscr{O}_C) = \mathrm{Hom}_{\mathscr{O}_{\mathbb{P}^m}}(\Omega^1_{\mathbb{P}^m}, \mathscr{O}_C).$$

It is trivial that $\mathrm{Hom}_{\mathscr{O}_C}(\mathcal{C}^*, \mathcal{D}^*) \cong T_h H$. The final term is actually the isomorphism classes of first-order deformations of h, hence is $\mathrm{Ext}^1_{\mathscr{O}_C}(\Omega^1_C, \mathscr{O}_C(-D))$ (see [8] XI.(3.11)). Hence we win. $\square$

Theorem 4.63. *Let $2g-2+n>0$ and $\nu \geq 3$ and $N=(2\nu-1)(g-1)+\nu n$. Then $H_{\nu,g,n}$ defined as above satisfied the following statements.*

(i) Let $h=(C;p_1,...,p_n)$ be a stable curve in $\mathbb{P}^{N-1}$ embedded by the ν-fold log-canonical system and $D=\sum_i p_i$. Then we have the exact sequence

$$0 \to H^0(C,\mathscr{O}_C(1))^{\oplus N}/H^0(C,\mathscr{O}_C) \to T_h(H_{\nu,g,n}) \xrightarrow{\lambda} \operatorname{Ext}^1(\Omega^1_C,\mathscr{O}_C(-D)) \to 0$$

where λ is the Kodaira-Spencer map at h of the universal family on $H_{\nu,g,n}$. In particular,

$$\dim T_h H_{\nu,g,n} = 3g-3+n+N^2-1;$$

(ii) $H_{\nu,g,n}$ is smooth and quasi-projective of dimension $3g-3+n+N^2-1$.

Sketch. (i) By Euler sequence and Lemma 4.62, we have

$$\begin{array}{ccccccccc}
 & & & & 0 & \longrightarrow & \operatorname{Hom}(\Omega^1_C,\mathscr{O}_C(-D)) & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & H^0(C,\mathscr{O}_C) & \longrightarrow & H^0(C,\mathscr{O}_C(1))^{\oplus N} & \longrightarrow & \operatorname{Hom}(\Omega^1_{\mathbb{P}^{N-1}},\mathscr{O}_C) & \xrightarrow{\delta} & H^1(C,\mathscr{O}_C) \\
 & & & & & & \downarrow{\scriptstyle\gamma} & & \\
 & & & & & & T_hH & \xrightarrow{\beta} & \operatorname{Ext}^1(\Omega^1_C,\mathscr{O}_C(-D))
\end{array}$$

Now we will analyze several groups and morphisms above.

•**The map β associates to every first-order embedded deformation of h.** (Trivial)

•**The elements of $\operatorname{Hom}(\Omega^1_{\mathbb{P}^{N-1}},\mathscr{O}_C)$ correspond to fiber space maps $j: C\times \operatorname{Spec}\mathbb{C}[\varepsilon] \to \mathbb{P}^{N-1}\times \operatorname{Spec}\mathbb{C}[\varepsilon]$.** (Omitted, see [8] page 200)

•**The map δ associates to any such object the infinitesimal deformation of line bundles on C given by $j^*(\mathscr{O}^1_{\mathbb{P}^{N-1}}(1)\otimes \mathscr{O}_{\operatorname{Spec}\mathbb{C}[\varepsilon]})\otimes(\mathscr{O}^1_C(1)\otimes\mathscr{O}_{\operatorname{Spec}\mathbb{C}[\varepsilon]})^{-1}$.** (Omitted, see [8] page 201)

•**The elements of $H^0(C,\mathscr{O}_C(1))^{\oplus N}/H^0(C,\mathscr{O}_C)$ is the tangent space to $\mathrm{PGL}(N)$.** (Omitted)

Let $v=\Gamma(\alpha)\in T_hH$ tangent to $H_{\nu,g,n}$ where $\alpha\in\operatorname{Hom}(\Omega^1_{\mathbb{P}^{N-1}},\mathscr{O}_C)$. Hence v from a fiber space map $j: C\times\operatorname{Spec}\mathbb{C}[\varepsilon]\to\mathbb{P}^{N-1}\times\operatorname{Spec}\mathbb{C}[\varepsilon]$ such that

$$j^*(\mathscr{O}^1_{\mathbb{P}^{N-1}}(1)\otimes\mathscr{O}_{\operatorname{Spec}\mathbb{C}[\varepsilon]}) = \omega_C(D)^{\otimes\nu}\otimes\mathscr{O}_{\operatorname{Spec}\mathbb{C}[\varepsilon]}.$$

Then $\delta(\alpha)=0$ and hence $\alpha\in H^0(C,\mathscr{O}_C(1))^{\oplus N}/H^0(C,\mathscr{O}_C)$. Conversely we find that the image of $H^0(C,\mathscr{O}_C(1))^{\oplus N}/H^0(C,\mathscr{O}_C)$ in T_hH contained in $T_hH_{\nu,g,n}$. By Proposition 4.53 we get $\operatorname{Hom}(\Omega^1_C,\mathscr{O}_C(-D))=0$, hence we have

$$0\to H^0(C,\mathscr{O}_C(1))^{\oplus N}/H^0(C,\mathscr{O}_C)\to T_h(H_{\nu,g,n})\xrightarrow{\lambda}\operatorname{Ext}^1(\Omega^1_C,\mathscr{O}_C(-D)).$$

Actually λ is surjective since any infinitesimal deformations of h can be embedded via the ν-fold log-canonical system. Hence we win.

(ii) By the basic theory of Hilbert schemes, $H_{\nu,g,n}$ is quasi-projective by the trivial reason. We now will show that $H_{\nu,g,n}$ is smooth of dimension $3g-3+n+N^2-1$. By (i) we get $\dim T_h H_{\nu,g,n} = 3g-3+n+N^2-1$, hence $\dim H_{\nu,g,n} \leq 3g-3+n+N^2-1$. If we have showed that $\dim H_{\nu,g,n} \geq 3g-3+n+N^2-1$, then well done.

Here we just give a sketch, details see [8] Proposition XI.5.12. By Theorem 4.50, we get a $(3g-3+n)$-dimensional deformation $\phi : \mathcal{C} \to (B, b_0)$. Let $\mathcal{C}_b = \phi^{-1}(b)$ and $D_b = \sum_i \sigma_i(b)$. Consider a principle $\mathrm{PGL}(N)$-bundle over B as

$$\mathcal{B} := \left\{ (b, F) \middle| \begin{array}{l} b \in B \text{ and } F \text{ a basis of } H^0(\mathcal{C}_b, \omega_{\mathcal{C}_b}(D_b)^{\otimes \nu}), \\ \text{modulo homotheties} \end{array} \right\}.$$

Take F_0 correspond to $C \subset \mathbb{P}^{N-1}$ and consider the family

$$\mathcal{X} := \mathcal{B} \times_B \mathcal{C} \xrightarrow{\psi} \mathcal{B}, \tau_i : \mathcal{B} \to \mathcal{X}.$$

Via some projective frame of $\psi_*(\omega_{\mathcal{X}/\mathcal{B}}(\sum \tau_i)^{\otimes \nu})$, we have $\mathcal{X} \to \mathbb{P}^{N-1} \times \mathcal{B}$, which induce $\xi : \mathcal{B} \to H_{\nu,g,n}$. Hence we have

$$\begin{tikzcd}
0 \arrow[r] & T_e(G) \arrow[r] \arrow[d, equal] & T_{(b_0,F_0)}\mathcal{B} \arrow[r] \arrow[d, "d\xi"] & T_{b_0}B \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & H^0(C, \mathscr{O}_C(1))^{\oplus N}/H^0(C, \mathscr{O}_C) \arrow[r] & T_h(H_{\nu,g,n}) \arrow[r] & \mathrm{Ext}^1(\Omega^1_C, \mathscr{O}_C(-D)) \arrow[r] & 0
\end{tikzcd}$$

where ρ is Kodaira-Spencer map. As ρ is an isomorphism, we have $d\xi$ is also an isomorphism. Hence locally ξ is a local isomorphism at (b_0, F_0). As $\dim \mathcal{B} = 3g-3+n+N^2-1$, well done. □

4.4.5 Construction of Kuranishi families

Let $\nu \geq 3$ and $(C; p_1, ..., p_n) \subset \mathbb{P}^{N-1}$ be a stable n-pointed genus g curve where $N = (2\nu-1)(g-1)+\nu n$, via ν-fold log-canonical system. We consider it as $x_0 \in H_{\nu,g,n}$. Fix the universal family $\mathcal{Y} \to H_{\nu,g,n}$ with sections $\sigma_i : H \to \mathcal{Y}$. Let

$$H_{\nu,g,n} \subset \underline{\mathrm{Hilb}}^{P_\nu}_{\mathbb{P}^{N-1}} \times (\mathbb{P}^{N-1})^n \subset \mathbb{P}^M \times (\mathbb{P}^{N-1})^n \subset \mathbb{P}^K$$

acted by $\mathbb{G} = \mathrm{PGL}(N) \subset \mathrm{PGL}(K+1)$ and let $\mathrm{Aut}(C; p_i) = \mathbb{G}_{x_0} \subset \mathbb{G} = \mathrm{PGL}(N)$ be the stabilizer of x_0.

Theorem 4.64. *There is a locally closed smooth subscheme $X \subset H_{\nu,g,n}$ of dimension $(3g-3+n)$ including x_0 such that the restriction of the universal family of $H_{\nu,g,n}$ over X is a Kuranishi family for all of its fibers and we have the following properties:*

(i) X is affine and $\mathbb{G}_{x_0}$-invariant;

(ii) For any $y \in X$, we have $\mathbb{G}_y \subset \mathbb{G}_{x_0}$;

(iii) For any $y \in X$, there is a $\mathbb{G}_y$-invariant neighborhood $U \subset X$ of y such that $\mathbb{G}_y = \{\gamma \in \mathbb{G} : \gamma(U) \cap U \neq \emptyset\}$ in the analytic (etale) topology.

Proof. See [8] Theorem XI.6.5. □

Hence we get a Kuranishi family $(\pi : \mathcal{C} \to (X, x_0), \sigma_i)$.

Definition 4.65 (Standard algebraic Kuranishi family)**.** *Let $(C; p_1, ..., p_n)$ be a stable n-pointed curve of genus g with $G = \mathrm{Aut}(C; p_i)$. Let $(\pi : \mathcal{C} \to (X, x_0), \sigma_i)$ be the Kuranishi family in Theorem 4.64 and it is called a standard algebraic Kuranishi family if the following conditions are satisfied:*

(a) The action of G_{x_0} on the central fiber can extend to compatible actions on $\mathcal{C}$ and X;

(b) For any $y \in X$ we have $G_y := \mathrm{Aut}(\mathcal{C}_y; \sigma_i(y)) \cong \mathrm{stab}_{G_{x_0}}(y)$;

(c) For any $y \in X$, there is a G_y-invariant analytic (etale) neighborhood U of y in X such that any isomorphism of n-pointed curves between fibers over U is induced by an element of G_y.

Definition 4.66 (Standard Kuranishi family)**.** *Let $(C; p_1, ..., p_n)$ be a stable n-pointed genus g curve with $G = \mathrm{Aut}(C; p_i)$. We will say a Kuranishi family $\mathcal{X} \to (B, b_0), \tau_i : B \to \mathcal{X}$ of $(C; p_1, ..., p_n)$ is called a standard Kuranishi family if the following conditions are satisfied.*

(a) B is a connected complex manifold and the family is a Kuranishi family at every points of B;

(b) The action of G on the central fiber extends to compatible actions on $\mathcal{X}$ and B;

(C) Any isomorphism (of n-pointed curves) between fibers is induced by an element of G.

Remark 4.67. *In fact, given any Kuranishi family, there is a neighborhood of the base point such that the restriction is standard. By the uniqueness of the Kuranishi family, it suffices to notice that this is true for a standard algebraic Kuranishi family. By Theorem 4.64 and we win.*

Corollary 4.68. *Any nodal curves $(C; p_1, ..., p_n)$ has a versal deformation (is unique up to an isomorphism, which however need not be unique).*

Proof. Adding some smooth marked points such that it becomes a stabel curve. Then taking the Kuranishi family of it and ignore the added marked points. $\square$

Corollary 4.69. *Any family of nodal curves can be locally embedded in a family of nodal curves with a reduced (even smooth) base.*

Proof. For any family of nodal curves $\eta : X \to S$ and let $s_0 \in S$. By the previous corollary we can get a versal deformation $\pi : \mathcal{X} \to (B, b_0)$ of $\eta^{-1}(s_0)$. After shrinking S (étale locally), we have a closed immersion $S \hookrightarrow T$ where T is smooth and a cartesian

$$\begin{array}{ccc} X & \xrightarrow{\beta} & \mathcal{X} \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\alpha} & B \end{array}$$

with $b_0 = \alpha(s_0)$. Hence we get cartesians

$$\begin{tikzcd}
X \arrow[r, "{(\eta,\beta)}"] \arrow[d, "\eta"] & S \times \mathscr{X} \arrow[r] \arrow[d, "{(\mathrm{id}_S,\pi)}"] & T \times \mathscr{X} \arrow[d, "{(\mathrm{id}_T,\pi)}"] \\
S \arrow[r, "{(\mathrm{id}_S,\alpha)}"] & S \times B \arrow[r] & T \times B
\end{tikzcd}$$

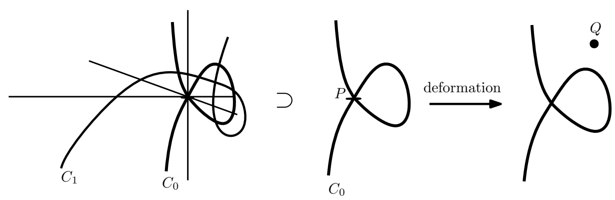
Clearly, $T \times B$ is smooth, and $S \to T \times B$ is a closed immersion. □

4.5 The stack of all curves

4.5.1 Families of all arbitrary curves

Definition 4.70. *Here we redefine a curve over k is a scheme C of finite type over k of dimension 1 (rather than pure dimension 1). The genus of C is defined as $g(C) = 1 - \chi(C, \mathscr{O}_C)$.*

Remark 4.71. *Why we not allow pure dimension 1? Since they may arise as deformations of connected pure one-dimensional curves; without this relaxation, the stack of all curves would fail to be algebraic. For example in [58] Example III.9.8.4, a flat family of rational curves defined by $\mathbb{P}^1 \to \mathbb{P}^3$ via $[x : y] \mapsto [x^3 : x^2y : xy^2 : ty^3]$ for any $t \neq 0$. As $t \to 0$, we may get a singular non-reduced curve C_0 with an embedded point at that node, but C_0 can deforms to the disjoint union of a plane nodal cuvre and a point in $\mathbb{P}^3$.*



Definition 4.72. *(i) A family of curves over a scheme S is a flat, proper and finitely presented morphism $C \to S$ of algebraic spaces such that every fiber is a curve.*

(ii) A family of n-pointed curves is a family of curves $C \to S$ together with n sections $\sigma_1, ..., \sigma_n : S \to C$ (with no condition on whether they are distinct or land in the relative smooth locus of C over S).

Remark 4.73. *(i) When we consider a family of stable curve, since the relative dualizing sheaf is ample, we can get it is projective, hence must be a scheme;*

(ii) There are some examples such that C are not a scheme.

Proposition 4.74. *If $C \to S$ is a family of curves over S, there exists an étale cover $S' \to S$ such that $C_{S'} \to S'$ is projective.*

Sketch-Local to global. Consider cartesians

$$\begin{tikzcd}
C_0 := C_s \arrow[r, hook] \arrow[d] & C_1 \arrow[r, hook] \arrow[d] & \cdots \arrow[r] & \widehat{C} \arrow[r] \arrow[d] & C \arrow[d] \\
S_0 := \mathrm{Spec}\kappa(s) \arrow[r, hook] & S_1 := \mathrm{Spec}\mathscr{O}_{S,s}/\mathfrak{m}_s^2 \arrow[r, hook] & \cdots \arrow[r] & \widehat{S} := \mathrm{Spec}\widehat{\mathscr{O}}_{S,s} \arrow[r] & S
\end{tikzcd}$$

Step 1. $C_0 \to \mathrm{Spec}\kappa(s)$. By St 0ADD, every separated algebraic space of dimension one is a scheme. Any one-dimensional proper $\kappa(s)$-scheme is projective by St 0A26. In particular we get a ample line bundle L_0 on C_0.
Step 2. $C_n \to S_n$. The obstruction to deforming a line bundle L_n on C_n to L_{n+1} on C_{n+1} lives in $H^2(C_0, \mathscr{O}_{C_0})$ and thus vanishes as $\dim C_0 = 1$. Thus there exists a compatible sequence of line bundles L_n on C_n. Since ampleness is an open condition in families and L_0 is ample, L_n is also ample.
Step 3. $\widehat{C} \to \widehat{S}$ with $\widehat{S}$ noetherian. Use Grothendieck's Existence Theorem we get an equivalence $\mathrm{Coh}(\widehat{C}) \to \varprojlim \mathrm{Coh}(C_n)$. As $\widehat{C} \to \widehat{S}$ is proper, then by Chow's lemma there exists a projective birational morphism $C' \to \widehat{C}$ of algebraic spaces such that $C' \to S$ is projective. This allows one to reduce Grothendieck's Existence Theorem for $\widehat{C} \to \widehat{S}$ to $C' \to \widehat{S}$ using devissage. As a result, using again that ampleness is an open condition in families we can extend the sequence of line bundle L_n to a line bundle $\widehat{L}$ on $\widehat{C}$ which is ample.
Step 4. S is of finite type over $\mathbb{Z}$. For every closed point $s \in S$, apply Artin Approximation to the functor

$$(Sch/S) \to (Sets), (T \to S) \mapsto \mathrm{Pic}(C_T)$$

to obtain an étale neighborhood $(S', s') \to (S, s)$ of s and a line bundle L' on $C_{S'}$ extending L_0. By openness of ampleness, we can replace S' with an open neighborhood of s' such that L' is relatively ample over S'.
Step 5. General S. Use noetherian approximation. □

4.5.2 Algebraicity of the stack of all curves

Definition 4.75. *Let $\mathscr{M}_{g,n}^{all}$ denote the category over Sch_{et} whose objects over S consists of families of curves $C \to S$ and n sections $\sigma_i : S \to C$. A morphism $(C' \to S', \sigma_i') \to (C \to S, \sigma_i)$ is the data of the cartesian*

$$\begin{tikzcd}
C' \arrow[r, "g"] \arrow[d] & C \arrow[d] \\
S' \arrow[r, "f"] \arrow[u, bend left, "\sigma_i'"] & S \arrow[u, bend left, "\sigma_i"]
\end{tikzcd}$$

with $g \circ \sigma_i' \to \sigma_i \circ f$.

Lemma 4.76. *$\mathscr{M}_{g,n}^{all}$ is a stack over Sch_{et}.*

Proof. Handle $n = 0$. Let $S' \to S$ be an étale cover with $C' \to S'$. And $\alpha : p_1^* C' \to p_2^* C'$ is an isomorphism over $S' \times_S S'$ satisfying the cocycle condition. The quotient of the étale equivalence relation

$$\begin{tikzcd} R := p_1^* C' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2 \circ \alpha"'] \arrow[d] & C' \arrow[r, dashed] \arrow[d] & C := C'/R \arrow[d, dashed] \\ S' \times_S S' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & S' \arrow[r] & S \end{tikzcd}$$

Well done. $\square$

Lemma 4.77. $\Delta : \mathscr{M}_{g,n}^{all} \to \mathscr{M}_{g,n}^{all} \times \mathscr{M}_{g,n}^{all}$ *is representable.*

Proof. Handle $n = 0$. Consider the cartesian

$$\begin{tikzcd} \underline{\mathrm{Isom}}_T(C_1, C_2) \arrow[r] \arrow[d] & T \arrow[d, "(C_1, C_2)"'] \\ \mathscr{M}_{g,n}^{all} \arrow[r, "\Delta"] & \mathscr{M}_{g,n}^{all} \times \mathscr{M}_{g,n}^{all} \end{tikzcd}$$

We need to show $\underline{\mathrm{Isom}}_T(C_1, C_2)$ is an algebraic space. By Proposition 4.74, there exists an étale cover $T' \to T$ such that $C_{i,T'} \to T'$ is projective. Hence we may let C_1, C_2 are projective over T. Indeed, as

$$\underline{\mathrm{Isom}}_T(C_1, C_2) \times_T T' = \underline{\mathrm{Isom}}_{T'}(C_{1,T'}, C_{2,T'}),$$

we get $\underline{\mathrm{Isom}}_{T'}(C_{1,T'}, C_{2,T'}) \to \underline{\mathrm{Isom}}_T(C_1, C_2)$ is representable, surjective and étale. Hence if $\underline{\mathrm{Isom}}_{T'}(C_{1,T'}, C_{2,T'})$ is an algebraic space, so is $\underline{\mathrm{Isom}}_T(C_1, C_2)$.
Fact. (St 05XD) If $f : X \to Y$ are T-morphism such that X, Y are proper, flat and locally of finite presention over T, then for any $U \to T$ such that $X_U \cong Y_U$ if and only if $U \to T$ factor through an open subscheme $T_0 \subset T$.

Now we get the inclutions

$$\underline{\mathrm{Isom}}_T(C_1, C_2) \subset \underline{\mathrm{Mor}}_T(C_1, C_2) \subset \underline{\mathrm{Hilb}}(C_1 \times_T C_2 / T)$$

where the second inclusion is $(g : C_1 \to C_2) \mapsto (\Gamma_g : C_1 \to C_1 \times_T C_2)$. The first inclusion is representable open immersion by the above fact. The second inclusion, we find that a subspace $[Z \subset C_1 \times_T C_2] \in \underline{\mathrm{Hilb}}(C_1 \times_T C_2/T)$ is in the image of the inclusion if and only if $Z \to C_1 \times_T C_2 \to C_1$ is an isomorphism (and similarly for other valued points). Therefore by the above fact we win. $\square$

Theorem 4.78. $\mathscr{M}_{g,n}^{all}$ *is an algebraic stack locally of finite type over* $\mathbb{Z}$.

Sketch. **Step 1. Reduce to $n = 0$.** Since $\mathscr{M}_{g,n+1}^{all}$ is the universal family over $\mathscr{M}_{g,n}^{all}$, we can prove the conclusion at the case $\mathscr{M}_g^{all}$.
Step 2. Look for possible smooth cover H' of $\mathscr{M}_g^{all}$. Let C_0 be any projective curves C_0 over k. Choosing an embedding $C_0 \subset \mathbb{P}_k^N$ such that $h^1(C_0, \mathscr{O}(1)) = 0$ by

Serre's vanishing theorem. Let $P(t)$ be its Hilbert polynomial. Let $H := \underline{\mathrm{Hilb}}^P_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}}$ be the Hilbert scheme which is projective over $\mathbb{Z}$. Consider the universal family

$$\begin{tikzcd}
\mathcal{C} \arrow[r, hook] \arrow[d] & \mathbb{P}^N_H \arrow[dl] \\
H &
\end{tikzcd}$$

there is a point $h_0 \in H(k)$ such that $\mathcal{C}_{h_0} = C_0$. By Review A.1 we can find an open neighborhood $H' \subset H$ of h_0 such that for any $s \in H'$ we have $h^1(\mathcal{C}_s, \mathscr{O}_{\mathcal{C}_s}(1)) = 0$. Now consider

$$H' \to \mathscr{M}_g^{all}, [C \hookrightarrow \mathbb{P}^N] \mapsto [C],$$

and by the representability of the diagonal, this map is representable as H' is a scheme.
Step 3. Show that $H' \to \mathscr{M}_g^{all}$ is smooth. Using Infinitesimal Lifting Criterion such that for all surjections $A \to A_0$ of artinian local rings with residue field k such that $k = \ker(A \to A_0)$ and for all diagrams

$$\begin{tikzcd}
& \mathrm{Spec} k \arrow[dl] \arrow[dr, "{[C \subset \mathbb{P}^N_k]}"] & \\
\mathrm{Spec} A_0 \arrow[rr, "{[\mathcal{C}_0 \subset \mathbb{P}^N_{A_0}]}"] \arrow[d, hook] & & H' \arrow[d] \\
\mathrm{Spec} A \arrow[rr, "\mathcal{C}"] \arrow[urr, dashed, "{[\mathcal{C} \subset \mathbb{P}^N_A]}"] & & \mathscr{M}_g^{all}
\end{tikzcd}$$

We need to find that dotted arrow. This diagram is equivalent to

$$\begin{tikzcd}
& \mathbb{P}^N_k & & \mathbb{P}^N_{A_0} & & \mathbb{P}^N_A \arrow[dddl] \\
& & & & & \\
C \arrow[uur, hook] \arrow[rr, hook] \arrow[d] & & \mathcal{C}_0 \arrow[uur, hook] \arrow[rr, hook] \arrow[d] & & \mathcal{C} \arrow[uur, hook, dashed] \arrow[d] & \\
\mathrm{Spec} k \arrow[rr, hook] & & \mathrm{Spec} A_0 \arrow[rr, hook] & & \mathrm{Spec} A &
\end{tikzcd}$$

For simplifying, we let C is of locally complete intersection (general case see [51] and [50]). Let $\mathcal{J}$ be the ideal sheaf of $C \to \mathbb{P}^N_k$ generated by regular sequence locally and that $\mathcal{J}/\mathcal{J}^2$ is a vector bundle on C with

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega_{\mathbb{P}^N_k}|_C \to \Omega_C \to 0.$$

By long exact sequence we get

$$\mathrm{Hom}_{\mathscr{O}_C}(\mathcal{J}/\mathcal{J}^2, \mathscr{O}_C) \to \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C, \mathscr{O}_C) \to \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_{\mathbb{P}^N_k}|_C, \mathscr{O}_C) = H^1(C, T_{\mathbb{P}^N_k}|_C).$$

Consider the canonical sequence

$$0 \to \mathscr{O}_C \to \mathscr{O}_C(1)^{\oplus N+1} \to T_{\mathbb{P}^N}|_C \to 0.$$

Since $H^2(C, \mathscr{O}_C) = 0$ and $H^1(C, \mathscr{O}_C(1)) = 0$ by $[C] \in H'$ we get $H^1(C, T_{\mathbb{P}^N_k}|_C) = 0$. Hence we get a surjection

$$\mathrm{Hom}_{\mathscr{O}_C}(\mathcal{J}/\mathcal{J}^2, \mathscr{O}_C) \twoheadrightarrow \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C, \mathscr{O}_C).$$

Use some deformation theory we get $\mathrm{Hom}_{\mathscr{O}_C}(\mathcal{J}/\mathcal{J}^2, \mathscr{O}_C)$ classifies embedded deformations of $\mathcal{C}_0 \to \mathbb{P}^N_{A_0}$ to $\mathcal{C}' \to \mathbb{P}^N_A$ and $\mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C, \mathscr{O}_C)$ classifies deformations of $\mathcal{C}_0$ over A_0 to $\mathcal{C}'$ over A. As the map is $[\mathcal{C}' \to \mathbb{P}^N_A] \mapsto \mathcal{C}'$ and is surjective, we win. □

4.5.3 Algebraicity of several stacks and boundedness of stable curves

Proposition 4.79 (Several stacks)**.** *We have inclusions*

$$\mathscr{M}_{g,n} \subset \overline{\mathscr{M}}_{g,n} \subset \mathscr{M}^{ss}_{g,n} \subset \mathscr{M}^{pre}_{g,n} \subset \mathscr{M}^{\leq nodal}_{g,n} \subset \mathscr{M}^{all}_{g,n}$$

of prestacks. Then all of these are open substacks, hence all of these are algebraic stacks locally of finite type over $\mathbb{Z}$.

Proof. • By Theorem 4.78, $\mathscr{M}^{all}_{g,n}$ is an algebraic stack locally of finite type over $\mathbb{Z}$.
• $\mathscr{M}^{\leq nodal}_{g,n} \subset \mathscr{M}^{all}_{g,n}$ is an open substack. Actually by Corollary 4.21 we get the nodal locus is open when C is a scheme. In general for an étale cover $g : C' \to C$ by a scheme, we find that a point $p \in C'$ is a node in its fiber if and only if $g(p)$ is a node in its fiber. We win.
• $\mathscr{M}^{pre}_{g,n} \subset \mathscr{M}^{\leq nodal}_{g,n}$ is an open substack. This is because for a family $(C \to S, \{\sigma_i\})$ of nodal curves, the locus $\{s \in S : \sigma_i(s) \text{ are disjoint and smooth}\}$ is open.
• $\mathscr{M}^{ss}_{g,n} \subset \mathscr{M}^{pre}_{g,n}$ is an open substack. This is because the nef locus is open (as the log canonical divisor is nef if and only if semistable).
• $\overline{\mathscr{M}}_{g,n} \subset \mathscr{M}^{ss}_{g,n}$ is an open substack. Indeed the stable locus is open by Proposition 4.33.
• $\mathscr{M}_{g,n} \subset \overline{\mathscr{M}}_{g,n}$ is an open substack. Indeed this is by the fact that smooth locus is open. □

Proposition 4.80. *$\overline{\mathscr{M}}_{g,n}$ is a quasi-compact smooth Deligne-Mumford stack of dimension $3g - 3 + n$ over $\mathbb{Z}$.*

Proof. • $\overline{\mathscr{M}}_{g,n}$ is quasi-compact. Let $(C, p_1, ..., p_n)$ be a n-pointed stable curve. By Theorem 4.28, we get $(\omega_C(p_1 + ... + p_n))^{\otimes 3}$ is very ample, we get $C \hookrightarrow \mathbb{P}^N$ with Hilbert polynomial $P(t)$. This is independent of C. Hence consider closed subscheme $H \subset \underline{\mathrm{Hilb}}^P_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}} \times (\mathbb{P}^N)^n$ of embedded curve and n points $(C \hookrightarrow \mathbb{P}^N, p_i \in C)$. Consider a forgetful functor

$$H \to \mathscr{M}^{all}_{g,n}, (C \hookrightarrow \mathbb{P}^N, p_i \in C) \mapsto (C, \{p_i\}).$$

Then the image of $|H| \to |\mathscr{M}^{all}_{g,n}|$ contains $\overline{\mathscr{M}}_{g,n}$. As $\underline{\mathrm{Hilb}}^P_{\mathbb{P}^N_{\mathbb{Z}}/\mathbb{Z}}$ is projective, then H is quasi-compact. Hence $\overline{\mathscr{M}}_{g,n}$ is quasi-compact.
• $\overline{\mathscr{M}}_{g,n}$ is Deligne-Mumford stack. By Proposition 4.53 for $i = 0$ and Proposition 4.51 (a), we get a n-pointed stable curve $(C, p_1, ..., p_n)$ has no infinitesimal automorphisms,

i.e. that the Lie algebra $T_e\mathrm{Aut}(C, p_1, ..., p_n)$ is trivial. Since the automorphism group scheme $\mathrm{Aut}(C, p_1, ..., p_n)$ is of finite type, this implies that $\mathrm{Aut}(C, p_1, ..., p_n)$ is finite and discrete, hence $\overline{\mathscr{M}}_{g,n}$ is a quasi-compact Deligne-Mumford stack.

• $\overline{\mathscr{M}}_{g,n}$ is smooth over $\mathrm{Spec}\mathbb{Z}$. Proposition 4.53 for $i = 2$ and Proposition 4.51 (c) implies that there are no obstructions to deforming C. As the algebraicity of $\overline{\mathscr{M}}_{g,n}$, this will allow us to invoke the Infinitesimal Lifting Criterion to establish that $\overline{\mathscr{M}}_{g,n}$ is smooth over $\mathrm{Spec}\mathbb{Z}$.

• $\overline{\mathscr{M}}_{g,n}$ has relative dimension $3g - 3 + n$ over $\mathrm{Spec}\mathbb{Z}$. Proposition 4.53 for $i = 1$ and Proposition 4.51 (b) implies that isomorphism classes of deformations of $(C, p_1, ..., p_n)$, it is identified with the Zariski tangent space of $\overline{\mathscr{M}}_{g,n}$ at the point corresponding to $(C, p_1, ..., p_n)$. This will allow us to conclude that $\overline{\mathscr{M}}_{g,n}$ has relative dimension $3g-3+n$ over $\mathbb{Z}$. □

4.6 Stable reduction: why $\overline{\mathscr{M}}_{g,n}$ is proper?

In this section we will using the Valuative Criterion (Theorem 3.49 (1)) to show that $\overline{\mathscr{M}}_{g,n}$ is proper. The existence of extention is called stable reduction, which is our main theorem:

Lemma 4.81. *The diagonal of the stack $\mathscr{M}_{g,n}^{all}$ is separated. In particular, $\mathscr{M}_{g,n}^{all} \to \mathrm{Spec}(\mathbb{Z})$ is quasi-separated.*

Proof. Omitted. See St 0DSQ. □

Theorem 4.82 (Stable reduction)**.** *Let R be a DVR with fraction field K and $\Delta = \mathrm{Spec}(R), \Delta^* = \mathrm{Spec}(K)$. If $(\mathcal{C}^* \to \Delta^*, s_1^*, ..., s_n^*)$ is a family of n-pointed stable curves of genus g, then there exists a finite cover $\Delta' \to \Delta$ of spectrums of DVRs and a family $(\mathcal{C}' \to \Delta', s_1', ..., s_n')$ of stable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$. As*

$$\begin{tikzcd}
 & & \mathcal{C}^* \arrow[d] & & \\
\mathcal{C}^* \times_{\Delta^*} \Delta'^* \arrow[urr] \arrow[rrr, dashed] \arrow[dd] & & & \mathcal{C}' \arrow[dd, dashed] & \\
 & & \Delta^* \arrow[uu, bend left, "s_i^*"] \arrow[rr] & & \Delta \\
\Delta'^* \arrow[uu, bend left] \arrow[urr] \arrow[rrr] & & & \Delta' \arrow[uu, dashed, bend left, "s_i'"] \arrow[ur] &
\end{tikzcd}$$

given by

$$\begin{tikzcd}
\Delta'^* \arrow[r] \arrow[d] & \Delta^* \arrow[r, "{(\mathcal{C}^* \to \Delta^*, \{s_i^*\})}"] \arrow[d, "{(\mathcal{C}' \to \Delta', \{s_1'\})}"] & \overline{\mathscr{M}}_{g,n} \\
\Delta' \arrow[r] \arrow[urr, dashed] & \Delta &
\end{tikzcd}$$

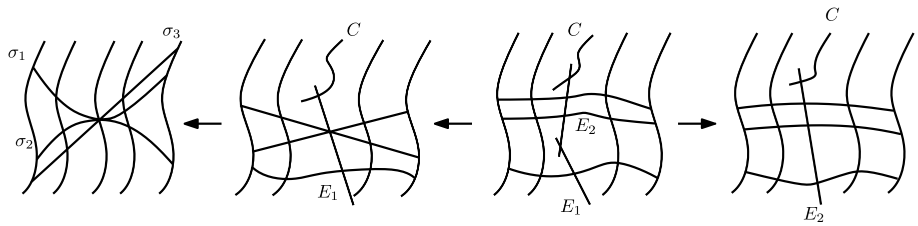
After proving this and the uniqueness, we can get the following conclusion:

Theorem 4.83. *If $2g - 2 + n > 0$, then $\overline{\mathscr{M}}_{g,n}$ is a proper smooth Deligne-Mumford stack of dimension $3g - 3 + n$ over $\mathbb{Z}$.*

By using the Keel-Mori Theorem, we get

Corollary 4.84. *If $2g-2+n>0$, there exists a coarse moduli space $\overline{\mathscr{M}}_{g,n} \to \overline{M}_{g,n}$ where $\overline{M}_{g,n}$ is a proper algebraic space over $\mathbb{Z}$.*

Example 4.85. *Let $\Delta = \mathrm{Spec}(R)$ where R be a DVR with uniformizer t. Let C be a smooth curve and consider $\mathcal{C} = C \times \Delta$ with sections $(\sigma_1, \sigma_2, \sigma_3) = (t^2, -t^2, 4t)$ as following diagram. The first two arrows are blowing up and the third is contracting*



E_1.

Remark 4.86. *Actually there are several methods to prove this. The first proof due to the original paper [29] by consider the Jacobians of curves and reduce the case into the semistable reduction of abelian varieties. Our method follows [55] by using some birational geometry of surfaces to prove the case of characteristic 0. There is another method can deal the positive or mixed characteristic case by [10], for this we refer St 0E8C.*

4.6.1 Proof of stable reduction in characteristic 0

Lemma 4.87. *Let R be a DVR with uniforming t and $0 := (t)$. Let $\mathcal{C} \to \Delta = \mathrm{Spec}(R)$ be a flat, proper and finitely presented morphisms such that each geometric fiber is a curve. Assume that $\mathcal{C}$ is regular. Let $p \in \mathcal{C}_0$.*

(a) If p is a smooth point in the reduced fiber $(\mathcal{C}_0)_{red}$. Show that after possibly an extension of DVRs, there exists an étale neighborhood of p (defined over R)

$$\mathrm{Spec} R[x,y]/(x^a - t) \to \mathcal{C}.$$

(b) If p is a node in the reduced fiber $(\mathcal{C}_0)_{red}$. Show that there exists an étale neighborhood of p (defined over R)

$$\mathrm{Spec} R[x,y]/(x^a y^b - t) \to \mathcal{C}.$$

Proof. This is easy to see. Or see [8] X.4. □

Lemma 4.88. *Let a, b, m be positive integers such that both a and b divide m.*

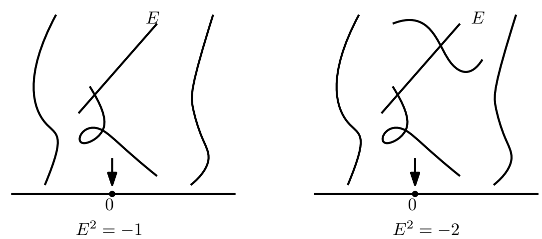
(a) Let $X = \mathrm{Spec} k[x,t]/(t^m - x^a)$ and normalization $\widetilde{X} \to X$. Then each preimage of the origin is locally defined by $x = t^k$ for some k.

(b) Let $X = \mathrm{Spec} k[x,y,t]/(t^m - x^a y^b)$ and normalization $\widetilde{X} \to X$. Then each preimage of the origin is locally defined by $t^k = xy$. In particular is a reduced and nodal point in the fiber over $t = 0$.

Proof. (a) We have $x^a - t^m = \prod_{i=0}^{a-1}(x - \zeta^i t^{m/a})$ where ζ be a primitive a-th root of unity. Hence the origin has a preimages in $\widetilde{X}$ locally defined by $x - \zeta^i t^{m/a}$, respectively.

(b) Similarly, see [8] Page 107. $\square$

Lemma 4.89. *Let $\mathcal{C} \to \Delta = \mathrm{Spec}(R)$ be a family of nodal curves where R be a DVR such that the general fiber $\mathcal{C}^*$ is smooth. Then if E is a rational tail (rational bridge with out marked points) of $\mathcal{C}_0$, then $E^2 = -1$ ($E^2 = -2$). As*



Proof. For any $E \cong \mathbb{P}^1 \subset \mathcal{C}_0$, then $0 = E \cdot \mathcal{C}_0 = E^2 + E \cdot E^c$. We win. $\square$

For simplicity of notation, we assume that there are no marked points, i.e. $n = 0$. Fix a spectrum of DVR $\Delta = \mathrm{Spec}(R), \Delta^* = \mathrm{Spec}(K)$ and $t \in R$ is the uniformizer, and $0 = (t) \in \mathrm{Spec} R$ the unique closed point. Consider $\mathcal{C}^* \to \Delta^*$ be a family of stable curve.

STEP 1. Reduce to the case where $\mathcal{C}^* \to \Delta^*$ is smooth. If $\mathcal{C}^*$ has k nodes, then after a finite extension of K we can arrange that each node is given by K-points $p_i \in \mathcal{C}^*(K)$. Let the pointed normalization $(\widetilde{\mathcal{C}}^*, q_1, ..., q_{2k})$ of it. By induction on the genus g, we perform stable reduction on each connected component and then take the nodal union along sections. After possibly an extension of K (and R), this produces a family of curves $\mathcal{C} \to \Delta$ extending $\mathcal{C}^* \to \Delta^*$.

STEP 2. Find some flat extension $\mathcal{C} \to \Delta$. As $\omega_{\mathcal{C}^*/\Delta^*}^{\otimes 3}$ is very ample, we can get an embedding as follows

$$\begin{tikzcd}
\mathbb{P}^{5g-6} \times \Delta^* \arrow[r, hook] & \mathbb{P}^{5g-6} \times \Delta \\
\mathcal{C}^* \arrow[u, hook, "|\omega_{\mathcal{C}^*/\Delta^*}^{\otimes 3}|"] \arrow[r, hook] \arrow[d] & \mathcal{C} := \overline{\mathcal{C}^*} \arrow[u, hook] \arrow[d, "f"'] \\
\Delta^* \arrow[r, hook] & \Delta
\end{tikzcd}$$

where $\mathcal{C} := \overline{\mathcal{C}}$ be the scheme-theoretic image of $\mathcal{C}^* \hookrightarrow \mathbb{P}^{5g-6} \times \Delta$. Now we focus on f. Actually the scheme-theoretic closure does not bring more embedded points. Hence by Proposition 2.16 we get f is flat.

STEP 3. Use embedded resolutions to find a resolution of singularities $\widetilde{\mathcal{C}} \to \mathcal{C}$ so that the reduced central fiber $(\widetilde{\mathcal{C}}_0)_{red}$ is nodal. By Theorem A.7, there exists a finite sequence of blow-ups at closed points of $\mathcal{C}_0$ yielding a projective birational morphism

$$\begin{tikzcd} \widetilde{\mathcal{C}}_0 \subset \widetilde{\mathcal{C}} \arrow[r] \arrow[drr] & \cdots \arrow[r] & \mathcal{C} \supset \mathcal{C}_0 \arrow[d] \\ & & \Delta \end{tikzcd}$$

such that $\widetilde{\mathcal{C}}$ is regular flat family of curves and such that the reduced central fiber $(\widetilde{\mathcal{C}}_0)_{red}$ is nodal. Now replace $\mathcal{C}$ by $\widetilde{\mathcal{C}}$.

STEP 4. Perform a base change $\Delta' \to \Delta$ such that the normalization of the total family $\mathcal{C} \times_\Delta \Delta'$ has a reduced nodal central fiber with many rational tails and bridges. By Lemma 4.87, we choose local coordinates x, y around $p \in \mathcal{C}_0$ (étale locally and formally locally) such that $\mathcal{C} \to \Delta$ can be described as follows:

(i) If $p \in (\mathcal{C}_0)_{red}$ is a smooth point, then $(x, y) \mapsto x^a$ and the multiplicity of the irreducible component of $\mathcal{C}_0$ containing p is a;

(ii) If $p \in (\mathcal{C}_0)_{red}$ is a (separated) node, then $(x, y) \mapsto x^a y^b$ and the two components of $\mathcal{C}_0$ containing p have multiplicities a and b.

Let m be the least common multiple of the multiplicities of the irreducible components of $\mathcal{C}_0$. Let the ramified morphism $\Delta' = \mathrm{Spec}(R) \to \Delta$ given by $t \mapsto t^m$. Hence we get

$$\begin{tikzcd} \widetilde{\mathcal{C}'} \arrow[r] & \mathcal{C}' \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ & \Delta' \arrow[r] & \Delta \end{tikzcd}$$

where $\mathcal{C}' = \mathcal{C} \times_\Delta \Delta'$ and $\widetilde{\mathcal{C}'} \to \mathcal{C}'$ be the normalization. Consider $p \in (\mathcal{C}_0)_{red}$.

(a) If p is a smooth point, then the unique preimage of p in $\mathcal{C}'$ defined locally by $x^a - t^m$. By Lemma 4.88 (a), we get each preimage of p in $\widetilde{\mathcal{C}'}$ is locally defined by $x = t^k$ which are the smooth points in $\widetilde{\mathcal{C}'}_0$;

(b) If p is a node, then the unique preimage of p in $\mathcal{C}'$ defined locally by $x^a y^b - t^m$. By Lemma 4.88 (a), we get each preimage of p in $\widetilde{\mathcal{C}'}$ is locally defined by $xy = t^k$ which are reduced and nodal points in $\widetilde{\mathcal{C}'}_0$. If $k > 1$, $\widetilde{\mathcal{C}'}$ have A_{k-1}-singularity.

Hence now we replace $\mathcal{C}$ by $\widetilde{\mathcal{C}'}$, which has a reduced central fiber with many rational tails and bridges.

STEP 5. After taking the minimal model, contract all rational tails and bridges in the central fiber. Using Theorem A.6 we get a minimal resolution $\mathcal{C}' \to \mathcal{C}$ and we get a family of prestable curves $\mathcal{C}' \to \Delta$ where $\mathcal{C}'$ is regular. By Lemma 4.89 and Corollary A.9, we can get a projective birational map $\mathcal{C}' \to \mathcal{C}'_{\min}$ where $\mathcal{C}'_{\min}$ is semistable. So we replace $\mathcal{C}$ by $\mathcal{C}'_{\min}$. (This is the semistable reduction!) Using Proposition 4.37, we can get a relative canonical stabel model $\mathcal{C}' \to \mathcal{C}^{st}$.

4.6.2 Explicit stable reduction

Proposition 4.90. *Let $\mathcal{C} \to \Delta$ be a generically smooth, proper and flat family such that $(\mathcal{C}_0)_{red}$ is nodal. Let $\mathcal{C}_0 = \sum_i a_i D_i$ where a_i is the multiplicity of D_i. Let $\Delta' \to \Delta$ defined by $t \mapsto t^p$ where p prime and set $\mathcal{C}' = \mathcal{C} \times_\Delta \Delta'$. Then after taking normalization $\widetilde{\mathcal{C}'} \to \mathcal{C}$ is branched cover ramified over $\sum_i (a_i \pmod p) D_i$.*

Example 4.91 (Stable reduction of A_{2k+1}-singularity)**.** *Let $\mathcal{C} \to \Delta = \operatorname{Spec}(R)$ be a generically smooth family degenerating to a A_{2k+1}-singularity in the central fiber where have local equation around the singular point is $y^2 = x^{2k+1} + t$. Now we will work through the steps in the proof of stable reduction. The first two steps have already finished, now we start at step 3.*

▶***STEP 3. Use embedded resolutions to find a resolution of singularities $\widetilde{\mathcal{C}} \to \mathcal{C}$ so that the reduced central fiber $(\widetilde{\mathcal{C}}_0)_{red}$ is nodal.*** *We consider two charts in blowing up with coordinates x', y' where the original coordinates are x, y, as:*

$$\begin{tikzcd} E|_{U_1} = V(x') \arrow[r, hook] & U_1 \arrow[r, hook] \arrow[d] & \widetilde{\mathcal{C}} = \mathrm{Bl}_p\mathcal{C} \arrow[dl] & (x', y') \arrow[d, mapsto] \\ & \mathcal{C} & & (x', x'y') \end{tikzcd} \qquad \begin{tikzcd} E|_{U_2} = V(y') \arrow[r, hook] & U_2 \arrow[r, hook] \arrow[d] & \widetilde{\mathcal{C}} = \mathrm{Bl}_p\mathcal{C} \arrow[dl] & (x', y') \arrow[d, mapsto] \\ & \mathcal{C} & & (x'y', y') \end{tikzcd}$$

•***The first blowing up.*** *In the first chart, the preimage of $y^2 - x^{2k+1}$ is $x'^2y'^2 - x'^{2k+1} = x'^2(y'^2 - x'^{2k-1})$; in the second chart it is $y'^2 - (x'y')^{2k+1} = y'^2(1 - x'^{2k+1}y'^{2k-1})$. Hence the exceptional divisor E_1 has multiplicity 2.*

•***The second blowing up.*** *In the first chart, the preimage of $x^2(y^2 - x^{2k-1})$ is $x'^4(y'^2 - x'^{2k-3})$; in the second chart it is $x'^2y'^4(1 - x'^{2k-1}y'^{2k-3})$. Hence the exceptional divisor E_2 has multiplicity 4.*

•***After k blowing ups.*** *We get $x^{2k}(y^2 - x)$ with the exceptional divisors E_i has multiplicity $2i$.*

•***One more blowing up.*** *We get the preimage of $x^{2k}(y^2 - x)$ in the second chart is*

$$x'^{2k}y'^{2k+1}(y' - x')$$

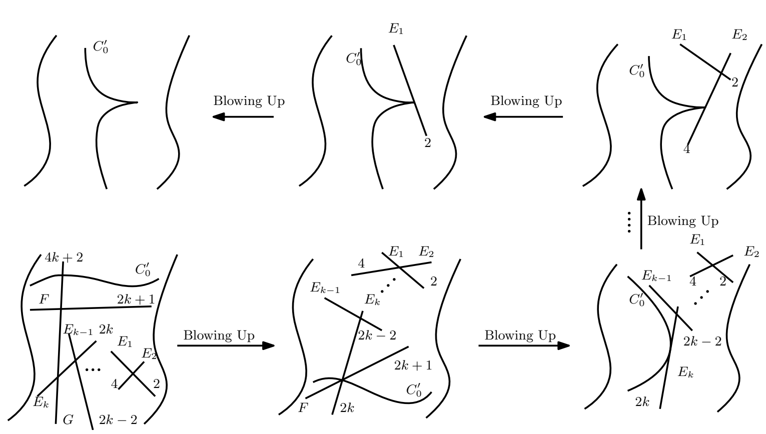
with the exceptional divisor F has multiplicity $2k + 1$.

•***The final blowing up.*** *We get the preimage of $x^{2k}y^{2k+1}(y - x)$ in the first chart is*

$$x'^{4k+2}y'^{2k+1}(y' - 1)$$

with the exceptional divisor G has multiplicity $4k + 2$.

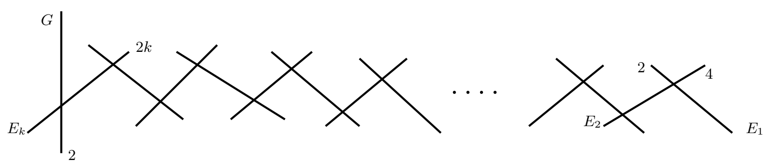
The process as follows:



▶STEP 4. Perform a base change $\Delta' \to \Delta$ such that the normalization of the total family $\mathcal{C} \times_\Delta \Delta'$ has a reduced nodal central fiber with many rational tails and bridges.

*• **The first base change.** First consider $\Delta' \to \Delta, t \mapsto t^{2k+1}$ and normalizing. After inductively apply to the prime factorization $2k+1$ and normalization, we will use the proposition to analyze the preimage of these irreducible component. Actually we get this $2k+1$-degree cover ramified over $C_0' + \sum_i E_i$ and we just need to consider F, G. For G, its preimage G' ramified at two points (intersects E_k, C_0') with index $2k$. By Riemann-Hurwitz Theorem, we get $2g(G')-2 = (2k+1)(2g(G)-2)+4k = -2$. Hence $g(G') = 0$ and $G' \cong \mathbb{P}^1$. For F, its preimage F' is unramified at all points, hence $F' = \coprod_{j=1}^{2k+1} F_j$ are copies of F. Hence replace Δ by Δ', we get the central fiber as $\mathcal{C}_0 = C_0' + 2G' + \sum_j F_j + \sum_i 2iE_i$.*

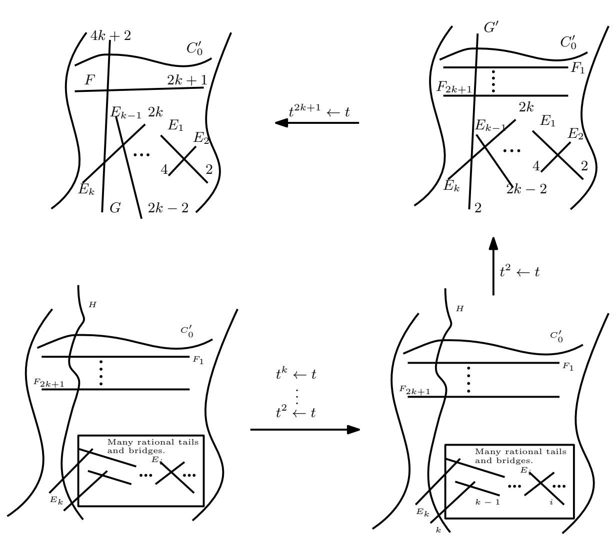
*• **The second base change.** Consider $\Delta' \to \Delta, t \mapsto t^2$ and normalizing. Actually we get this 2-degree cover ramified over $C_0' + \sum_j F_j$ and we just need to consider G', E_i. For G', the preimage H ramified at $2k+2$ points (C_0', F_j). Hence we get $g(H) = k$ by Riemann-Hurwitz Theorem. For E_i, the things become more complicated as follows:*



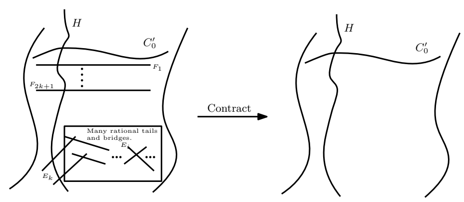
But we can easy to see that after this process, these things are just plenty of rational bridges and rational tails.

*• **The final base changes.** Here we just need to consider E_i and these have multiplicity i. Consider $t \mapsto t^k$, then (many) E_k has two ramified points, hence by Riemann-Hurwitz Theorem $g(E_k') = 0$, hence rational. Then consider $t \mapsto t^{k-1}$,...,$t \mapsto t^2$, we have the same results. Hence we also get plenty of rational tails and bridges, which are all multiplicity 1.*

The whole process as follows:



►***STEP 5. Contract all rational tails and bridges in the central fiber.*** *Now we kill all -1-curves (many E_1 and all F_j), and then (every) E_2 become -1-curves. Inductively, we kill all E_i and F_j and get a stable central fiber as follows and we win.*



4.6.3 Separatedness of $\overline{\mathscr{M}}_{g,n}$

Proposition 4.92. *Let R be a DVR with fraction field K with $\Delta = \operatorname{Spec}(R), \Delta^* = \operatorname{Spec}(K)$. Let $(\mathcal{C} \to \Delta, \sigma_1^*, ..., \sigma_n^*)$ and $(\mathcal{D} \to \Delta, \tau_1^*, ..., \tau_n^*)$ are n-pointed stable curves, then for any $\alpha^* : \mathcal{C}^* \to \mathcal{D}^*$ with $\tau_i^* = \alpha^* \circ \sigma_i^*$ over generic fiber can extends to a unique isomorphism $\alpha : \mathcal{C} \to \mathcal{D}$ with $\tau_i = \alpha \circ \sigma_i$.*

$$\begin{tikzcd}
\mathcal{C}^* \arrow[r, "\alpha^*"] \arrow[dr] & \mathcal{D}^* \arrow[d] & \mathcal{C} \arrow[r, dashed, "\alpha"] \arrow[dr] & \mathcal{D} \arrow[d] \\
 & \Delta^* \arrow[rr, hook] & & \Delta
\end{tikzcd}$$

Proof. We only prove the case of $n = 0$ generically smooth curves. Let $\mathcal{C}' \to \mathcal{C}, \mathcal{D}' \to \mathcal{D}$ be the minimal resolutions and let $\Gamma \subset \mathcal{C}' \times_\Delta \mathcal{D}'$ be the clpsure of the graph of $\mathrm{id} \times \alpha^* : \mathcal{C}^* \to \mathcal{C}^* \times_{\Delta^*} \mathcal{D}^*$. Let $\Gamma' \to \Gamma$ be the minimal resolution. Hence we get birational projective maps $\Gamma' \to \mathcal{C}'$ and $\Gamma' \to \mathcal{D}'$. By the same proof of [58] Theorem II.8.19, we get

$$\Gamma(\mathcal{C}', \omega_{\mathcal{C}'/\Delta}^{\otimes k}) \cong \Gamma(\Gamma', \omega_{\Gamma'/\Delta}^{\otimes k}) \cong \Gamma(\mathcal{D}', \omega_{\mathcal{D}'/\Delta}^{\otimes k})$$

for all $k \geq 0$. As the canonical bundle are ample, we get

$$\mathcal{C}' \cong \operatorname{Proj} \bigoplus_k \Gamma(\mathcal{C}', \omega_{\mathcal{C}'/\Delta}^{\otimes k}) \cong \operatorname{Proj} \bigoplus_k \Gamma(\mathcal{D}', \omega_{\mathcal{D}'/\Delta}^{\otimes k}) \cong \mathcal{D}'.$$

Furthermore, we know that $\mathcal{C}, \mathcal{D}$ are stable models of $\mathcal{C}', \mathcal{D}'$, respectively. By the uniqueness of stable models, we get $\alpha : \mathcal{C} \cong \mathcal{D}$ extending α^*. □

4.6.4 A general version of stable reduction

For simplicity, we consider the case without marked points.

Recall that we have proved the stable reduction of stabel curves over a spectrum of DVR:

Theorem 4.93 (Stable reduction)**.** *Let R be a DVR with fraction field K and $\Delta = \operatorname{Spec}(R), \Delta^* = \operatorname{Spec}(K)$. If $(\mathcal{C}^* \to \Delta^*)$ is a family of stable curves of genus g, then* *there exists a finite cover $\Delta' \to \Delta$ of spectrums of DVRs and a family $(\mathcal{C}' \to \Delta')$ of stable curves extending $\mathcal{C}^* \times_{\Delta^*} \Delta'^* \to \Delta'^*$. As*

$$\begin{tikzcd}
\Delta'^* \arrow[r] \arrow[d] & \Delta^* \arrow[r, "(\mathcal{C}^*\to\Delta^*)"] \arrow[d] & \overline{\mathscr{M}}_{g,n} \\
\Delta' \arrow[r] \arrow[urr, dashed, "(\mathcal{C}'\to\Delta')"] & \Delta &
\end{tikzcd}$$

Now how can we let the DVR to be an integral scheme S? We will use a technical lemma (le lemme de Gabber) to show the case when S is a normal integral scheme directly. Then we will give a canonical way to construct like in le lemme de Gabber. These are all have some same methods with de Jong's alteration.

Theorem 4.94. *Let S be a normal integral scheme with a generic point $\eta = \mathrm{Spec}K$ and C_η be a stable curve over η. Then there exists a generically étale, generically finite, proper and dominant morphism of integral schemes $S' \to S$ with generic point $\eta' = \mathrm{Spec}K'$ such that the stable curve $C_\eta \times_\eta \eta'$ extends to a stable curve over S'.*

First we give some comments. Actually we find that C is already extended to the family of stable curves in $\overline{\mathscr{M}}_g$! So if we consider some compact neighborhood S' of C in $\overline{\mathscr{M}}_g$, then we have such S' extended C.

$$\begin{tikzcd}
\eta \arrow[r] \arrow[dr] & S' \arrow[r, hook] \arrow[d, dashed, "?"] & \overline{\mathscr{M}}_g \\
& S &
\end{tikzcd}$$

But now $S' \to S$ need not be proper surjective and even S' may not be a scheme! Hence we need to find some finite cover of scheme over this stack to deal with this problem.

Proof. Consider $\eta \to \overline{\mathscr{M}}_g$ induced by C_η. By Gabber's Lemma 3.54, we find a finite surjective generically étale morphism from a scheme $M \to \overline{\mathscr{M}}_g$. Now M is proper as the stack $\overline{\mathscr{M}}_g$ is proper. Hence we obtain a finite separable K-scheme $\eta \times_{\overline{\mathscr{M}}_g} M$ and there exists a finite separable extension K'/K such that $\eta' := \mathrm{Spec}K' \to \eta$ factor through $\eta \times_{\overline{\mathscr{M}}_g} M$. Hence the curve $C'_\eta := C_\eta \times_\eta \eta'$ induced from a family of curve $C'_M \to M$ by $M \to \overline{\mathscr{M}}_g$.

Hence we claim that there exists an proper,dominant and generically finite morphism of integral schemes $S' \to S$ with generic point η' such that the map $\eta' \to M$ factor through $S' \to M$.

Now consider $\eta \to S_1 = \mathrm{Norm}_{\eta'}(S) \to S$ be the normalization. Consider S' be the closure of the image of diaganol $\Delta : \eta' \to S_1 \times_{\mathrm{Spec}\mathbb{Z}} M$. Hence we have:

$$\begin{tikzcd}
\eta' \arrow[dd] \arrow[dr, "\Delta"] \arrow[ddr] \arrow[rr, bend left=10] & & \\
& S' := \overline{\mathrm{Im}(\Delta)} \arrow[r] \arrow[d] & M \arrow[d] \\
S & S_1 = \mathrm{Norm}_{\eta'}(S) \arrow[l] & \overline{\mathscr{M}}_g
\end{tikzcd}$$

Now we find that $S' \to S$ is generically étale, generically finite and dominant. Hence we just need to show $S' \to S$ is proper. This comes from the fact that $S_1 \to S$ is finite (this follows from St 032L) and $S' \to S_1$ is proper as the composition of the base change of $M \to \mathrm{Spec}\mathbb{Z}$ and a closed immersion. This gives us the claim. Now let C' induced by C'_M along $S' \to M$, then well done. $\square$

Now we will give a canonical construction of M due to P. Deligne. Here we will follows [28]. We all knew that the obstruction of $\overline{\mathscr{M}}_g$ to be a scheme (or algebraic space) is that it has non-trivial automorphisms. So we need to get a ℓ-level structure to kill these automorphisms.

Let $\ell \geq 3$ is invertible on T, then we will call a family of genus g smooth curves $X \to T$ with ℓ-level structure if $\mathrm{Pic}^0_S(X)[\ell] \cong (\mathbb{Z}/\ell\mathbb{Z})^{\oplus 2g}$ (when over some algebraically closed field, one can let $H^1_{\text{ét}}(C, \mathbb{Z}/\ell\mathbb{Z}) \cong (\mathbb{Z}/\ell\mathbb{Z})^{\oplus 2g}$ or $R^1_{\text{ét}} f_*(\mathbb{Z}/\ell\mathbb{Z}) \cong (\mathbb{Z}/\ell\mathbb{Z})^{\oplus 2g}$ for families). Then we consider the stack ${}_\ell\mathscr{M}_g$ over $\mathrm{Spec}\mathbb{Z}[1/\ell]$ parameterizing smooth curves of genus g with ℓ-level structure. Hence we have:

$$\begin{tikzcd}
{}_\ell\mathscr{M}_g \arrow[d] & \\
\mathscr{M}_g \arrow[r, hook] & \overline{\mathscr{M}}_g
\end{tikzcd}$$

By Lemma 3.5.1 in [28], we get ${}_\ell\mathscr{M}_g$ is an algebraic space. since Deligne proved that all automorphisms are killed after adding the ℓ-level structure.

Let ${}_\ell\overline{\mathscr{M}}_g = \mathrm{Norm}_{\overline{\mathscr{M}}_g}({}_\ell\mathscr{M}_g)$, Deligne also proved that ${}_\ell\overline{\mathscr{M}}_g$ is a proper algebraic space in Corollary 3.6 in [28]. By the algebraic-space version of Chow's lemma, one get a projective scheme M over $\mathbb{Z}$ as:

$$\begin{tikzcd}
{}_\ell\mathscr{M}_g \arrow[r] \arrow[d] & {}_\ell\overline{\mathscr{M}}_g \arrow[d] & M \arrow[l] \\
\mathscr{M}_g \arrow[r] & \overline{\mathscr{M}}_g &
\end{tikzcd}$$

Hence we give a canonical construction of M now!

4.6.5 The moduli stack of elliptic curves $\mathscr{M}_{1,1}$ and $\overline{\mathscr{M}}_{1,1}$

Proposition 4.95. *$\mathscr{M}_1$ is not a stack.*

Proof. See [74] Remark 8.4.15 for the References. □

Remark 4.96. *If we let $\mathscr{M}_1'$ as morphisms of algebraic spaces, then this will be a stack. This follows the Picard functor and the stack $\mathscr{M}_{1,1}$. In fact if we consider the universal elliptic curve $\mathscr{E} \to \mathscr{M}_{1,1}$, then Picard functor gives $\mathscr{M}_1' \to \mathscr{M}_{1,1}$ which induce $\mathscr{M}_1' \cong B\mathscr{E}$. We omitted here.*

Actually we hae the following explicit expression of the moduli stacks of elliptic curves which rely on the discussion of the families of elliptic curves before.

Proposition 4.97. *(i) Let $W := \mathrm{Spec}\mathbb{Z}[a_1, a_2, a_3, a_4, a_6, 1/\Delta]$ and*

$$H = \left\{ \begin{pmatrix} u^2 & s & 0 \\ 0 & u^3 & 0 \\ r & t & 1 \end{pmatrix} \middle| \begin{matrix} u \text{ unit} \\ s, r, t \text{ arbitrary} \end{matrix} \right\} \subset \mathrm{GL}_{3,\mathbb{Z}}$$

be a subgroup act on W naturally, then we have

$$\mathscr{M}_{1,1} \cong [W/H];$$

(ii) we have

$$\mathscr{M}_{1,1} \times_{\mathbb{Z}} \mathbb{Z}[1/6] \cong [(\mathbb{A}^2 \backslash V(\delta))/\mathbb{G}_m],$$

with action $t \cdot (a, b) = (t^4 a, t^6 b)$ and $\delta = 4a^3 + 27b^2$;

(iii) we have

$$\overline{\mathscr{M}}_{1,1} \times_{\mathbb{Z}} \mathbb{Z}[1/6] \cong [(\mathbb{A}^2 \backslash 0)/\mathbb{G}_m] \cong \mathscr{P}(4, 6),$$

with action $t \cdot (a, b) = (t^4 a, t^6 b)$ and $\mathscr{P}(m_1, ..., m_k)$ be the weighted projective stack.

Proof. This is very simple:

For (i), as any family of elliptic curves have the Weierstrass form, we find that $W \to \mathscr{M}_{1,1}$ is surjective. By Theorem 2.20 and the observation that $S \times_{\mathscr{M}_{1,1}} W = \mathrm{Isom}_{S \times W}(E \times W, S \times E_W)$ we find that $W \to \mathscr{M}_{1,1}$ is an H-principal bundle. Hence $\mathscr{M}_{1,1} \cong [W/H]$.

For (ii), we know that the family version of Weierstrass form we discribed before is very complicated. But here if we let 6 is invertible, we can easy to see that there exist unique x, y such that $a_1 = a_2 = a_3 = 0$ at before. Hence we have a much easier Weierstrass form $y^2 = x^3 + a_4 x + a_6$. In this case $\Delta = -16\delta = -16(4a_4^3 + 27a_6^2)$ Hence by (i) we have $\mathscr{M}_{1,1} \times_{\mathbb{Z}} \mathbb{Z}[1/6] \cong [(\mathbb{A}^2 \backslash V(\delta))/\mathbb{G}_m]$ with action $t \cdot (a, b) = (t^4 a, t^6 b)$ and $\delta = 4a^3 + 27b^2$.

For (iii), this is trivial by the argument of (ii). □

Proposition 4.98. *$\mathscr{M}_{1,1}$ $\overline{\mathscr{M}}_{1,1}$ has coarse moduli spaces $M_{1,1} \cong \mathbb{A}^1$ and $\overline{M}_{1,1} \cong \mathbb{P}^1$.*

Proof. Now over $\mathbb{Z}[1/6]$ these are trivial by the theorem above. But we will give a general argument.

By Keel-Mori theorem, stack $\mathscr{M}_{1,1}$ has a coarse moduli space $M_{1,1}$. By Theorem 2.20 we have the j-invariant map $j : \mathscr{M}_{1,1} \to \mathbb{A}^1$. By the universal property of coarse moduli space we have the unique map:

$$\begin{tikzcd} \mathscr{M}_{1,1} \arrow[d] \arrow[dr, "j"] & \\ M_{1,1} \arrow[r, dashed, "\exists ! j'"'] & \mathbb{A}^1 \end{tikzcd}$$

We just need to claim that j' is an isomorphism.

Now by valuative criterion, we can see that j is proper by [49]. Then j' is also proper. As j' is also quasi-finite, we know that j' is finite. Hence to prove j' is an isomorphism, we just need to prove j' is birational by Zariski's main theorem. For any geometric point $t : \mathrm{Spec} k \to \mathbb{A}^1$ the $M_{1,1} \times_{\mathbb{A}^1} \mathrm{Spec} k$ is a single point as elliptic curves over algebraically closed fields are classified by j-invariant. Hence we conclude the proof. $\square$

4.7 Gluing and forgetful morphisms

We follows [34].

4.7.1 Gluing morphisms

Proposition 4.99. *There are finite morphisms of algebraic stacks*

$$F : \overline{\mathscr{M}}_{i,n} \times \overline{\mathscr{M}}_{g-i,m} \to \overline{\mathscr{M}}_{g,n+m-2}$$

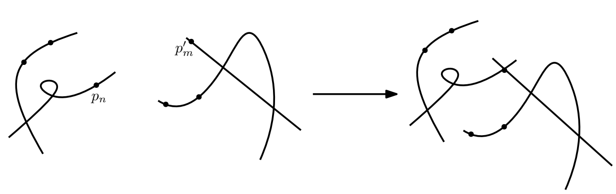
$$((C, p_1, ..., p_n), (C', p'_1, ..., p'_m)) \mapsto (C \cap C', p_1, ..., p_{n-1}, p'_1, ..., p'_m),$$

and

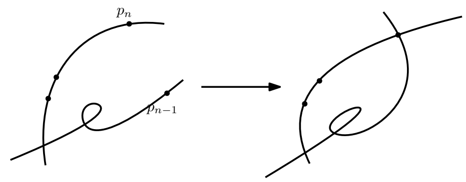
$$G : \overline{\mathscr{M}}_{g-1,n} \to \overline{\mathscr{M}}_{g,n-2}$$

$$(C, p_1, ..., p_n) \mapsto (C/(p_{n-1} \sim p_n), p_1, ..., p_{n-2}).$$

As follows:



Sketch-using pushout. By the stable reduction, these maps are of course representable and proper. As they have the finite fibers, these maps are now finite. Now for F we let $n = m = 1$ and for G we let $n = 2$.



For F: Let $(\pi : \mathcal{C} \to S, \sigma), (\pi' : \mathcal{C}' \to S, \sigma')$ are stable curves. As σ, σ' are closed immersions, we get the pushout exists by the theory of Ferrand (St 0ECH) and as we have the finite cover $\mathcal{C} \sqcup \mathcal{C}' \to \mathcal{C}$, we get this pushout is proper and flat (omitted):

$$\begin{tikzcd}
& \operatorname{Spec}(A) \arrow[rr] \arrow[dl] \arrow[dd] & & \operatorname{Spec}A[y] \arrow[dl] \arrow[dd] \\
\operatorname{Spec}A[x] \arrow[rr] \arrow[dd] & & \operatorname{Spec}A[x,y]/(xy) \arrow[dd] & \\
& S \arrow[rr, "\sigma"] \arrow[dl, "\sigma'"'] & & \mathcal{C} \arrow[dl] \\
\mathcal{C}' \arrow[rr] & & \widetilde{\mathcal{C}} &
\end{tikzcd}$$

where $\operatorname{Spec}A[x]$ is an étale neighborhood of $\sigma(s)$ which is the pulback of étale neighborhood $\operatorname{Spec}(A)$ of any $s \in S$. Since an étale morphism from an affine scheme extend over closed immersions, there is an étale neighborhood $\operatorname{Spec}A[y]$ is an étale neighborhood of $\sigma'(s)$. Then the pushout can be easy to compute as $\operatorname{Spec}(A[x] \times_A A[x]) \cong \operatorname{Spec}A[x,y]/(xy)$. By some results of pushout (in St 0D2G), we get $\operatorname{Spec}A[x,y]/(xy) \to \widetilde{\mathcal{C}}$ is an étale neighborhood of s. Hence $\widetilde{\mathcal{C}} \to S$ is nodal along S. Checking fibers we get $\widetilde{\mathcal{C}}_s$ is stable.

For G: Let $(\mathcal{C} \to S, \sigma_1, \sigma_2)$ are stable curve. Here we consider the pushout:

$$\begin{tikzcd}
S \sqcup S \arrow[r, "\sigma_1 \sqcup \sigma_2"] \arrow[d] & \mathcal{C} \arrow[d] \\
S \arrow[r] & \widetilde{\mathcal{C}}
\end{tikzcd}$$

which is étale locally like

$$\begin{tikzcd}
\operatorname{Spec}(A \times A) \arrow[r, "(0,1)"'] \arrow[d] & \operatorname{Spec}A[t] \arrow[d] \\
\operatorname{Spec}A \arrow[r] & \operatorname{Spec}A[x,y]/(y^2 - x^2(x+1))
\end{tikzcd}$$

where we find that $x := t^2 - 1, y = t^3 - t$ generate $A \times_{A \times A} A[t]$, then well done. $\square$

4.7.2 Boundary divisors of $\overline{\mathscr{M}}_g$

Consider the closed substacks

$$\delta_0 = \operatorname{Im}(\overline{\mathscr{M}}_{g-1,2} \to \overline{\mathscr{M}}_g)$$

$$\delta_i = \operatorname{Im}(\overline{\mathscr{M}}_{i,1} \times \overline{\mathscr{M}}_{g-i,1} \to \overline{\mathscr{M}}_g)$$

where $i = 1, ..., \lfloor g/2 \rfloor$.

As these maps are finite, we get $\dim \delta_0 = \dim \overline{\mathscr{M}}_{g-1,2} = 3(g-1) - 3 + 2 = 3g - 4$ and similar $\dim \delta_i = 3g - 4$. Hence these are divisors of $\overline{\mathscr{M}}_g$.

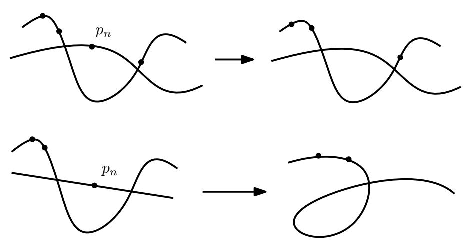
(By analyzing the formal deformation space of a stable curve, one can show that $\delta = \bigcup_{j=0}^{\lfloor g/2 \rfloor} \delta_j$ is a normal crossings divisor.)

4.7.3 Forgetful morphisms

Proposition 4.100. *By Proposition 4.37, there is a morphism of algebraic stacks*

$$\overline{\mathscr{M}}_{g,n} \to \overline{\mathscr{M}}_{g,n-1}, (C, p_1, ..., p_n) \mapsto (C^{st}, p_1, ..., p_{n-1}).$$

As



4.7.4 Universal family $\overline{\mathscr{M}}_{g,n+1} \to \overline{\mathscr{M}}_{g,n}$

This section we follows [34] and [35]. We consider the universal family $\mathscr{U}_{g,n} \to \overline{\mathscr{M}}_{g,n}$ of $\overline{\mathscr{M}}_{g,n}$. Actually the definition of universal family as for any family of stable curves $(\mathcal{C} \to S, \{\sigma_i\})$, we have the following universal property of cartesian

$$\begin{tikzcd}
\mathcal{C} \arrow[r, dashed] \arrow[d] & \mathscr{U}_{g,n} \arrow[d] \\
S \arrow[r, dashed, "\exists!"] & \overline{\mathscr{M}}_{g,n}
\end{tikzcd}$$

The existence given by 2-Yoneda's Lemma and some descent theory (omitted). Here we express this family as follows:

Lemma 4.101 (See [35])**.** *$\mathscr{U}_{g,n}(S)$ to be the set of families of curves $(\mathcal{C} \to S, \sigma_1, ..., \sigma_n, \sigma)$ where $(\mathcal{C} \to S, \sigma_1, ..., \sigma_n) \in \overline{\mathscr{M}}_{g,n}(S)$ and σ is an extra section without smooth condition.*

Proof. Fix $\pi : \mathcal{C} \to S$. We first let $\Sigma(S) : \mathscr{U}_{g,n}(S) \to \overline{\mathscr{M}}_{g,n}(S)$ as $(\pi, \sigma_i, \sigma) \mapsto (\pi, \sigma_i)$ be the canonical map and $\Sigma_i(S) : \overline{\mathscr{M}}_{g,n}(S) \to \mathscr{U}_{g,n}(S)$ as $(\pi, \sigma_1, ..., \sigma_n) \mapsto (\pi, \sigma_1, ..., \sigma_n; \sigma_i)$. Finally we need to define $\mathcal{C} \to \mathscr{U}_{g,n}$ as $(\mathrm{pr}_2 : \mathcal{C} \times_S \mathcal{C} \to \mathcal{C}, s_i, \Delta)$ where $s_i = (\sigma_i \circ \pi, \mathrm{id}_{\mathcal{C}})$ and $\Delta = (\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$. Hence we get the following cartesian diagram of fibered categories

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, "\pi"] & \mathscr{U}_{g,n} \arrow[d, "\Sigma"'] \\
S \arrow[r] \arrow[u, bend left, "\sigma_1,...,\sigma_n"] & \overline{\mathscr{M}}_{g,n} \arrow[u, bend right, "\Sigma_1,...,\Sigma_n"']
\end{tikzcd}$$

Well done. $\square$

Now we consider

$$\overline{\mathscr{M}}_{g,n+1} \to \mathscr{U}_{g,n}, (\mathcal{C} \to S) \mapsto (\mathcal{C}^{st} \to S, \sigma_1', ..., \sigma_n', \sigma')$$

where this stabilization aiming to make $(\mathcal{C}^{st} \to S, \sigma_1', ..., \sigma_n')$ in $\overline{\mathscr{M}}_{g,n}(S)$.

$$\begin{tikzcd} \overline{\mathscr{M}}_{g,n+1} \arrow[r] \arrow[dr] & \mathscr{U}_{g,n} \arrow[d] \\ & \overline{\mathscr{M}}_{g,n} \end{tikzcd}$$

Remark 4.102 (More explicit construction)**.** *Fix $f : X \to S$ in $\overline{\mathscr{M}}_{g,n+1}(S)$ and hence we get*

$$X = \underline{\mathrm{Proj}}_S \left(\bigoplus_{m \geq 0} f_* \omega_{X/S} \left(\sum_{i=1}^{n+1} \sigma_i \right)^{\otimes m} \right).$$

Now we let

$$c(X) := \underline{\mathrm{Proj}}_S \left(\bigoplus_{m \geq 0} f_* \omega_{X/S} \left(\sum_{i=1}^{n} \sigma_i \right)^{\otimes m} \right).$$

with $\sigma_i' : S \xrightarrow{\sigma_i} X \to \underline{c}(X)$. Hence $(c(X); \sigma_1', ..., \sigma_n')$ be a family of n-pointed stable curves. Hence we get $\overline{\mathscr{M}}_{g,n+1} \to \mathscr{U}_{g,n}$. Here we follows the proof before chapter 8 in [23].

Proposition 4.103. *The morphism $\overline{\mathscr{M}}_{g,n+1} \to \mathscr{U}_{g,n}$ is an isomorphism over $\overline{\mathscr{M}}_{g,n}$.*

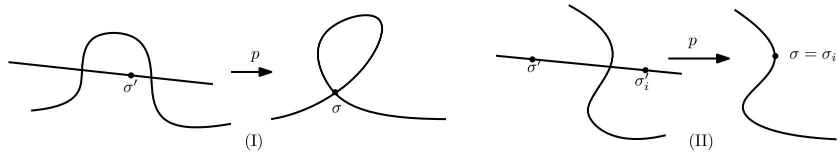
Sketch. Now we construct an inverse map $\mathscr{U}_{g,n} \to \overline{\mathscr{M}}_{g,n+1}$.
Step 1. Construct that family of curves. Let $(\mathcal{C} \to S, \sigma_1, ..., \sigma_n, \sigma)$ be an element in $\mathscr{U}_{g,n}(S)$. As σ is a closed immersion, it defined by an ideal sheaf $i : \mathscr{J}_\sigma \hookrightarrow \mathscr{O}_{\mathcal{C}}$. Define the coherent sheaf K by the exact sequence

$$0 \to \mathscr{O}_{\mathcal{C}} \xrightarrow{\delta} \mathscr{J}_\sigma^\vee \oplus \mathscr{O}_{\mathcal{C}}(\sigma_1 + ... + \sigma_n) \to K \to 0$$

where $\delta = (i^\vee, j)$ where j is also an embedding. Now consider $\mathcal{C}' = \underline{\mathrm{Proj}}\mathrm{Sym}(K) \xrightarrow{p} \mathcal{C} \to S$.
Step 2. Construct the section. In [34], Knudsen introduce a notion called stably reflexive module. Knudsen separate the two cases of σ as this is local on S: (I) σ meets a non-smooth point in the fiber; (II) σ is a divisor meets one of these sections σ_i.

In both cases we find the surjections form as $\sigma^* K \to \sigma^*(-)$ or $\sigma_i^* K \to \sigma_i^*(-)$ to getting lifts where showed that all $\sigma^*(-)$ are line bundles (may using stably reflexive module). The picture when $S = \mathrm{Spec}(k)$ as follows:

Hence we omitted all details and get $(\mathcal{C}' \to S, \sigma_1', ..., \sigma_n', \sigma') \in \overline{\mathscr{M}}_{g,n+1}(S)$. For this detailed proof, we refer the original paper [34] Theorem 2.4 or the new paper [35]. One can also see [8] X.8 for more detailed proof over $\mathbb{C}$. $\square$



4.8 Irreducibility

As $\overline{\mathscr{M}}_{g,n}$ is a smooth Deligne-Mumford stack, its irreducibility if and only if connectedness. As $\overline{\mathscr{M}}_{g,n+1} \to \overline{\mathscr{M}}_{g,n}$ be a universal family, it has connected fibers. Hence by induction, we can reduce the case of $\overline{\mathscr{M}}_g$. Moreover, by Keel-Mori theorem we get the coarse moduli space $\overline{\mathscr{M}}_g \to \overline{M}_g$ which induce the homeomorphism $|\overline{\mathscr{M}}_g| \cong |\overline{M}_g|$. Hence we can reduce the case of $\overline{M}_g$. Hence we have the following relations:

$$\begin{aligned}\overline{\mathscr{M}}_{g,n} \text{ irreducible} \Leftrightarrow \overline{\mathscr{M}}_{g,n} \text{ connected} \Leftrightarrow \overline{\mathscr{M}}_g \text{ connected (or irreducible)}\\ (\Leftrightarrow \mathscr{M}_g \text{ connected and dense in} \overline{\mathscr{M}}_g) \Leftrightarrow \overline{M}_g \text{ connected.}\end{aligned}$$

Here the denceness of $\mathscr{M}_g$ in the proper Deligne-Mumford stack $\overline{\mathscr{M}}_g$ is called **Deligne-Mumford compactification**.

Remark 4.104 (Some historical remarks)**.** *(i) In 19th century, Clebsch and Hurwitz establishing irreducibility of M_g in characteristic 0 by using the classical topological argument;*

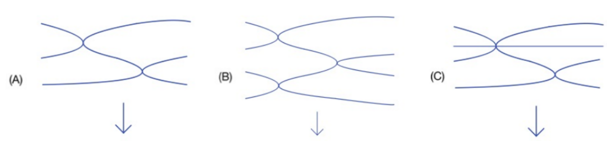
(ii) In the appendix of the paper [56] by Fulton in [41] gives a completely algebraic proof for this in characteristic 0 in 1982;

(iii) In the paper [29], Deligne and Mumford give two arguments of irreducibility of $\overline{M}_{g,n}$ in characteristic p (by reduction to characteristic 0) in 1969;

(iv) In paper [40], Fulton established the irreducibility of $\overline{M}_{g,n}$ in characteristic p where $p > g+1$ in 1969.

4.8.1 Preliminaries–Branched coverings

Definition 4.105. *Let C be a connected smooth curve on k. A branched covering of $\mathbb{P}^1_k$ is a separable finite morphism $f : C \to \mathbb{P}^1_k$. We say f is simply branched if for any branched point $x \in \mathbb{P}^1_k$, there is at most one ramification point in the fiber $f^{-1}(x)$ and such a point has index 2.*



Here (A) is simply branched but (B),(C) are not.

Lemma 4.106. *Let C be a smooth, connected and projective curve of genus g over an algebraically closed field k of characteristic 0. If L is a line bundle of degree $d \geq g+1$, then for a subspace $V \subset H^0(C, L)$ of dimension 2 we get $C \to \mathbb{P}^1$ a simply branched.*

Proof. As $h^0(C, L) = d + 1 - g$, we get $\dim \mathrm{Gr}(2, H^0(C, L)) = 2(d - g - 1)$. Here $\mathrm{char}(k) = 0$, the map $C \to \mathbb{P}^1$ is finite separable. So $C \to \mathbb{P}^1$ is not a simply branched covering if and only if one of the following conditions holds

(a) V has a base point;

(b) there exists a ramification point with index > 2;

(c) there exists 2 ramification points in the same fiber.

This is easy to see that both of these cases have dimensions smaller than dimension of $\mathrm{Gr}(2, H^0(C, L))$. □

Lemma 4.107. *If $C \to \mathbb{P}^1$ is a simply branched cover of degree $d > 2$ in characteristic 0, then $\mathrm{Aut}(C/\mathbb{P}^1)$ is trivial.*

Proof. Any $\alpha \in \mathrm{Aut}(C/\mathbb{P}^1)$ must fix the $2g + 2d - 2$ branched points by Riemann-Hurwtiz Theorem and simplyness. By Proposition 2.13, there are no non-trivial automorphisms of a smooth curve fixing more than $2g + 2$ points. Hence as $d > 2$, $\mathrm{Aut}(C/\mathbb{P}^1)$ is trivial. □

Remark 4.108. *Here we give some notes for the proof of Clebsch and Hurwitz in 19th to show that M_g is connected over $\mathbb{C}$. We define*

$$H_{d,b} = \{C \to \mathbb{P}^1 \text{ simply branched covering of degree } d \text{ over } b \text{ points}\}$$

where $b = 2g+2d-2$. By the previous lemma, $H_{d,b}$ is an algebraic space or a topological space (if $k = \mathbb{C}$, as there are no stacky issues here). Let $\mathrm{Sym}^b\mathbb{P}^1\backslash\Delta$ as the variety of b unordered distinct points in $\mathbb{P}^1$ (which can also be written as the complement $\mathbb{P}^b\backslash\Delta$ of the discriminant hypersurface), we have a diagram

$$\begin{tikzcd} & H_{d,b} \arrow[dl] \arrow[dr] & \\ \mathscr{M}_g & & \mathrm{Sym}^b\mathbb{P}^1\backslash\Delta \end{tikzcd}$$

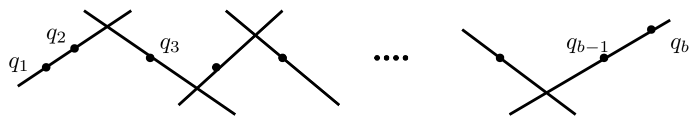
with the canonical maps. Then they showed that $H_{d,b} \to \mathrm{Sym}^b\mathbb{P}^1\backslash\Delta$ is finite étale (actually this can be showed by using deformation theory pure algebraically, see[1] Lemma 5.7.9. We omitted here). By Lemma 4.106, we get $H_{d,b} \to \mathscr{M}_g$ is surjective. Hence we need to show that $H_{d,b}$ is connected. Combining these and some properties of monodromy theory, they proved this. For more detail, see [1] subsection 5.7.2.

4.8.2 Irreducibility over characteristic 0 using admissible covers

In this section we will use the method of admissible covers to gives a sketch algebraic proof for the irreducibility in characteristic 0, which appears in the appendix of the paper [56] by Fulton in [41].

Proposition 4.109. *Let C be a smooth, connected and projective curve of genus g over an algebraically closed field k of characteristic 0. There exists a connected curve T with points $t_1, t_2 \in T$ and a family $\mathcal{C} \to T$ of stable curves such that $\mathcal{C}_{t_1} \cong C$ and $\mathcal{C}_{t_2}$ is a singular stable curve.*

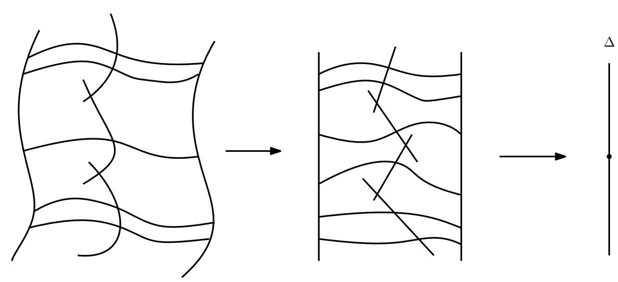
Sketch of proof. By Lemma 4.106 we get for $d \gg 0$ there exists a finite cover $C \to \mathbb{P}^1$ of degree d simply branched over $b = 2g+2d-2$ distinct points $p_1, ..., p_b$ in $\mathbb{P}^1$. This gives a b-pointed stable curve $G = [\mathbb{P}^1, \{p_i\}] \in M_{0,b}$. By Remark 4.108 ($H_{d,b} \to \mathrm{Sym}^b\mathbb{P}^1 \backslash \Delta$ is finite étale), we get $G \in M_{0,b}$ in general. Then G can degenerates to $(D_0, q_1, ..., q_b)$ as the following picture



In the other words, there is a DVR R and fraction field K with $\Delta = \mathrm{Spec}(R) \to \overline{M}_{0,n}$ be a stable curve $(\mathcal{D} \to \Delta, \sigma_i)$ with generic fiber $(\mathbb{P}^1, p_i)$ and special fiber $(D_0, q_1, ..., q_b)$. Hence we have a simply branched covering $\mathcal{C}^* \to \Delta^*$ and extend to $\mathcal{C} \to \mathcal{D}$ by taking $\mathcal{C}$ as the integral closure of $\mathscr{O}_{\mathcal{D}}$ in $K(\mathcal{C}^*)$ as

$$\begin{tikzcd}
\mathcal{C} \arrow[r, dashed] & \mathcal{D} \arrow[r] & \Delta \\
\mathcal{C}^* \arrow[r] \arrow[u, hook, dashed] & \mathcal{D}^* \arrow[r] \arrow[u, hook] & \Delta^* \arrow[u, hook]
\end{tikzcd}$$

Hence we get this diagram:



Now we just need to make $\mathcal{C}$ be a singular stable curve. Purity of the branch locus implies that the ramification of $\mathcal{C} \to \mathcal{D}$ is a divisor when restricted to the relative smooth locus of $\mathcal{C} \to \mathcal{D}$ and the central fiber $\mathcal{C}_0 \to \mathcal{D}_0$ is ramified at $\sigma_i(0)$. By $\Delta' \to \Delta, t \mapsto t^m$ we can replace $\mathcal{C}$ such that $\mathcal{C}_0 \to \mathcal{D}_0$ is ramified only over $\sigma_i(0)$ and possibly over nodes of $\mathcal{D}_0$. By an analysis of possible extensions $\mathcal{C} \to \mathcal{D}$, one can show that $\mathcal{C}_0$ is a nodal curve. Therefore $\mathcal{C} \to \Delta$ is a family of nodal curves.

Now we take $\mathcal{C} \to \mathcal{C}^{st}$ and just need to check $\mathcal{C}_0^{st}$ is singular. For any irreducible component $T \subset \mathcal{C}_0^{st}$, apply Riemann-Hurwitz to $T \to \mathbb{P}^1 \subset \mathcal{D}_0$ we get $2g(T) - 2 = -2d + R$. If $\mathbb{P}^1$ is the middle one, we get $R \leq 2 + d - 1$; if $\mathbb{P}^1$ is the boundary one, we get $R \leq 1 + 2d - 2$. Hence $R \leq 2d - 1$ and $g(T) = 0$. Hence T is rational. Hence $\mathcal{C}_0^{st}$ is singular. $\square$

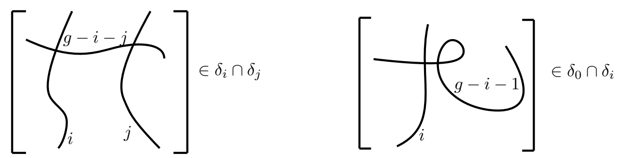
Proposition 4.110. *If $\overline{\mathscr{M}}_{g',n'}$ is irreducible for all $g' < g$, then $\delta = \overline{\mathscr{M}}_{g,2} \backslash \mathscr{M}_{g,2}$ is connected.*

Proof. Let $\delta = \delta \cup \delta_1 \cup \cdots \cup \delta_{\lfloor g/2 \rfloor}$ where

$$\delta_0 = \mathrm{Im}(\overline{\mathscr{M}}_{g-1,2} \to \overline{\mathscr{M}}_g)$$

$$\delta_i = \mathrm{Im}(\overline{\mathscr{M}}_{i,1} \times \overline{\mathscr{M}}_{g-i,1} \to \overline{\mathscr{M}}_g)$$

where $i = 1, ..., \lfloor g/2 \rfloor$. Hence δ_0, δ_i are connected by hypotheses. Easy to see that these divisors intersect as the points of $|\overline{\mathscr{M}}_g|$:



where when $i = j = g/2$ we may remove the middle line. □

Theorem 4.111. *$\overline{\mathscr{M}}_{g,n}$ is irreducible.*

Proof. By the argument at begining, we just need to show $\overline{\mathscr{M}}_g$ is connected. By Proposition 4.109 every smooth curve degenerates to a stable singular curve in the boundary $\delta = \overline{\mathscr{M}}_g \backslash \mathscr{M}_g$. By induction on g and Proposition 4.110 we get δ is connected, so is $\overline{\mathscr{M}}_g$. □

Remark 4.112. *For the irreducibility in positive characteristic, we omitted and we refer the original [29], [40]. For the sketch, we refer subsection 5.7.4 in [1].*

4.9 Projectivity

We will prove the coarse moduli space $\overline{M}_{g,n}$ is projective follows [65] and [80].

Remark 4.113. *Some generalizations of the projectivities:*

(a) In [64] shows the moduli of stable varieties in any dimension is projective;
(b) In [22] and [82] shows the moduli of K-polystable Fano varieties is projective.

Let the universal family $\pi : \mathscr{U}_g \to \overline{\mathscr{M}}_g$ and we define k-th pluri-canonical bundle as the vector bundle $\pi_*(\omega^{\otimes k}_{\mathscr{U}_g/\overline{\mathscr{M}}_g})$. Indeed, $\pi_*(\omega^{\otimes k}_{\mathscr{U}_g/\overline{\mathscr{M}}_g})$ is a coherent sheaf on the stack $\overline{\mathscr{M}}_g$ by the coherence theorem. We need to check that it is a vector bundle. By definition of the vector bundle over Deligne-Mumford stack, we need to show for any $S \to \overline{\mathscr{M}}_g$ the sheaf $(\pi_*(\omega^{\otimes k}_{\mathscr{U}_g/\overline{\mathscr{M}}_g}))|_S$ is a vector bundle over S. As $S \to \overline{\mathscr{M}}_g$ correspond to $\pi_S : \mathcal{C} \to S$, we get

$$(\pi_*(\omega^{\otimes k}_{\mathscr{U}_g/\overline{\mathscr{M}}_g}))|_S \cong \pi_{S,*}(\omega^{\otimes k}_{\mathcal{C}/S}).$$

By some argument with Review A.1 we can show that $\pi_{S,*}(\omega^{\otimes k}_{\mathcal{C}/S})$ is a vector bundle.

Moreover, we get use the Riemann-Roch Theorem to deduce that

$$\mathrm{rank}(\pi_*(\omega^{\otimes k}_{\mathscr{U}_g/\overline{\mathscr{M}}_g})) = \begin{cases} g, & k = 1; \\ (2k-1)(g-1), & k > 1. \end{cases}$$

Now we consider the line bundle over $\overline{\mathscr{M}}_g$

$$\lambda_k := \det \pi_*(\omega^{\otimes k}_{\mathscr{U}_g/\overline{\mathscr{M}}_g}).$$

We will show that for $k \gg 0$, the line bundle λ_k descends to an ample line bundle on $\overline{M}_g$, then we get $\overline{M}_g$ is a projective scheme.

4.9.1 Kollár's Criteria

Lemma 4.114. *Let $\mathscr{X}$ be a proper Deligne-Mumford stack with coarse moduli space $\mathscr{X} \to X$. Suppose L line bundle over $\mathscr{X}$ with*

(a) L is semiample (i.e. L^N is basepoint-free for some $N > 0$);

(b) for every proper integral curve T and map $f : T \to \mathscr{X}$ such that $f(T) \subset |\mathscr{X}|$ is not a single point, $\deg L|T > 0$.

Then for some $N > 0$, $L^{\otimes N}$ descends to an ample line bundle over X.

Proof. This is the stack-version of the Corollary 1.2.15 in [68]. Actually we consider the following diagram which come from (a) and the universal property of coarse moduli space:

$$\begin{tikzcd} \mathscr{X} \arrow[d, "\pi"] \arrow[dr, "f"] & \\ X \arrow[r, "g"] & \mathbb{P}(H^0(\mathscr{X}, L^{\otimes N})) \end{tikzcd}$$

By (b), f doesn't contract curves, so is g. Hence g is quasi-finite and proper, hence finite by Zariski main theorem. Hence $M := g^*\mathscr{O}(1)$ is ample; moreover, $\pi^*M = L^{\otimes N}$, we win. □

Theorem 4.115 (Nakai-Moishezon Criterion)**.** *If X is a proper algebraic space, a line bundle L is ample if and only if for all irreducible closed subvarieties $Z \subset X$,*

$$L^{\dim Z} \cdot Z > 0.$$

Proof. This is the algebraic space-version of the Theorem 1.21 in [27]. By Le Lemme de Gabber (Theorem 3.54), there exists a finite surjection $f : X' \to X$ and by the algebraic space version St 0GFB we get L is ample if and only if f^*L is ample. Hence by the scheme-version of Nakai-Moishezon Criterion ([27] Theorem 1.21), we win. □

Let X be a proper algebraic space over k. Let $W \to Q$ be a surjection of vector bundles of rank w and q. Suppose that W has structure group $G \to \mathrm{GL}_w$. There is a classifying map

$$X \to [\mathrm{Gr}(q, w)/G], x \mapsto [W \otimes \kappa(x) \twoheadrightarrow Q \otimes \kappa(x)]$$

which is well defined because these killed by G.

Here we state our main theorem in this section. For simplicity, we only state it in characteristic 0. The criteria first appears in [65] and more general case we refer [64].

Theorem 4.116 (Kollár's Criterion)**.** *Let X be a proper algebraic space over a field k of characteristic* 0*. Let $W \twoheadrightarrow Q$ be a surjection of vector bundles of rank w and q, where W has structure group $G \to \mathrm{GL}_w$. Suppose that*

(a) The classifying map $X(k) \to \mathrm{Gr}(q, w)(k)/G(k)$ has finite fibers;

(b) W is nef.

Then $\det Q$ *is ample.*

Proof. By Nakai-Moishezon criterion, for any irreducible subvariety $Z \subset X$ we need to verify $\det(Q)|_Z$ is big. As (a),(b) can restrict to Z, we can let X is an integral scheme and show that $\det Q$ is big.

By Le Lemme de Gabber (Theorem 3.54), there exists a finite projective surjection $f : Y \to X$ of schemes. Hence we have $\det(f^*Q)^{\dim Y} = \deg(f)\det(Q)^{\dim X}$ and $\det Q$ is big if and only if $\det(f^*Q)$ is big. By taking the normalization, we can assume Y is normal and integral. So by Lemma 4.117 we win. □

Lemma 4.117. *Let Y be a normal projective integral scheme over a field k of characteristic 0. Let $W \twoheadrightarrow Q$ be a surjection of vector bundles of rank w and q, where W has structure group $G \to \mathrm{GL}_w$. Suppose that*

(a) The classifying map $Y(k) \to \mathrm{Gr}(q, w)(k)/G(k)$ generically has finite fibers;

(b) W is nef.

Then $\det Q$ is big.

Proof. See Proposition 5.8.9 in [1]. □

4.9.2 Nefness of pluri-canonical bundles

Theorem 4.118. *Let $\pi : \mathcal{C} \to T$ be a family of stable curves over a smooth curve T over k, then $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is nef for $k \geq 2$.*

Proof. We following several steps:

•**Step 1. Reduction to characteristic p.** Now we let k is of characteristic 0. Since $\mathcal{C}$ and T are finite type over k, their defining equations only involve finitely many coefficients of k. Thus there exists a finitely generated $\mathbb{Z}$-subalgebra $A \subset k$ and a cartesian diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \widetilde{\mathcal{C}} \arrow[d] \\
T \arrow[r] \arrow[d] & \widetilde{T} \arrow[d] \\
\mathrm{Spec} k \arrow[r] & \mathrm{Spec} A
\end{tikzcd}$$

where $\widetilde{\mathcal{C}}, \widetilde{T}$ are schemes of finite type over A. By possibly enlarging A, we can arrange that $\widetilde{T} \to \mathrm{Spec}(A)$ is smooth and projective family and $\widetilde{\mathcal{C}} \to \widetilde{T}$ is a family of stable curves. After restricting along a morphism $\Delta \to \mathrm{Spec} A$ from a DVR such that the images of the closed and generic points have characteristic p and 0, respectively, we may assume that A is a DVR. As the nef locus is open for proper flat families, it suffices to prove the theorem when $\mathrm{char}(k) = p > 0$ by Proposition A.12.

•**Step 2. Second reductions.** We can reduce to the case that

(a) $\mathcal{C}$ is a smooth and minimal surface;

(b) $\mathcal{C} \to T$ is generically smooth;

(c) The genus of T is at least 2.

These implies $\mathcal{C}$ is of general type.

•**Step 3. Positive characteristic case.** Let $p = \mathrm{char}(k)$. If $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$ is not nef, then there exists a quotient line bundle $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow M^\vee$ where $d = \deg(M) > 0$. Consider the absolute Frobenius

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{Frob}_{\mathcal{C}}"] \arrow[d] & \mathcal{C} \arrow[d] \\ T \arrow[r, "\mathrm{Frob}_T"] & T \end{tikzcd}$$

By the property of the dualizing sheaf, we get $\mathrm{Frob}_T^*\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \cong \pi_*(\omega_{\mathcal{C}/T}^{\otimes k})$. And $\deg \mathrm{Frob}_T^* M = pd$, we can let $d \gg 0$. Hence we can let $M = \omega_T^{\otimes k} \otimes L$ where L is very ample.

The surjection $\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \twoheadrightarrow (\omega_T^{\otimes k} \otimes L)^\vee$ induce

$$\pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L \twoheadrightarrow \mathscr{O}_T.$$

As $h^1(T, \mathscr{O}_T) \geq 2$, we have $h^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L) \geq 2$. Use the Leray spectral sequence

$$H^1(T, \pi_*(\omega_{\mathcal{C}/T}^{\otimes k}) \otimes \omega_T^{\otimes k} \otimes L) \Rightarrow H^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L),$$

hence $h^1(\mathcal{C}, \omega_{\mathcal{C}}^{\otimes k} \otimes \pi^* L) \geq 2$ by some calculation. By Lemma 4.119, we win. □

Lemma 4.119 (Bombieri-Ekedahl)**.** *Let S be a smooth projective surface over an algebraically closed field k which is minimal and of general type. Let D be an effective divisor with $D^2 = 0$. If $\mathrm{char}(k) \neq 2$, then $H^1(S, \omega_S^{\otimes n}(D)) = 0$ for all $n \geq 2$. If $\mathrm{char}(k) = 2$, then $h^1(S, \omega_S^{\otimes n}(D)) \leq 1$ for all $n \geq 2$.*

4.9.3 Positivity via positivity theory

For a morphism $S \to \overline{\mathscr{M}}_g$ correspond to $\mathcal{C} \to S$. Consider an integral d, we have

$$\mathrm{Sym}^d \pi_*(\omega_{\mathcal{C}/S}^{\otimes k}) \to \pi_*(\omega_{\mathcal{C}/S}^{\otimes dk}).$$

When $S = \mathrm{Spec}(K), \mathcal{C} = C$, we get

$$\mathrm{Sym}^d H^0(C, \omega_C^{\otimes k}) \to H^0(C, \omega_C^{\otimes dk})$$

with kernel consists of degree d equations cutting out the image of $|\omega_C^{\otimes k}| : C \to \mathbb{P}^{r(k)-1}$. If $k \geq 3$, $\omega_{\mathcal{C}/S}^{\otimes k}$ is very ample and thus $\mathcal{C} \to S$ can be recovered from the kernel of the multiplication map.

Proposition 4.120. *For $k \gg 0$ and N sufficiently divisible, then $\lambda_k = \det \pi_*(\omega_{\mathscr{U}_g/\overline{\mathscr{M}}_g}^{\otimes k})$ descends to an ample line bundle on $\overline{\mathscr{M}}_g$.*

Proof. Consider $\mathcal{C} = \mathscr{U}_g, S = \overline{M}_g$. Choose k, d such that

(a) $\omega_{\mathcal{C}/S}^{\otimes k}$ is relatively very ample and $R^1\pi_*\omega_{\mathcal{C}/S}^{\otimes k} = 0$;

(b) Every curve $|\omega_C^{\otimes k}| : C \hookrightarrow \mathbb{P}^{r(k)-1}$ is cut out by equations degree d;

(c) $\pi_*(\omega_{\mathcal{C}/S}^{\otimes k})$ is nef (by Theorem 4.118).
These implies surjection

$$W := \mathrm{Sym}^d \pi_*(\omega_{\mathcal{C}/S}^{\otimes k}) \twoheadrightarrow \pi_*(\omega_{\mathcal{C}/S}^{\otimes dk}) =: Q.$$

Let w, q be the rank of W, Q, respectively. Let W has structure group $G \to \mathrm{GL}_w$. Consider the classifying map

$$\overline{\mathscr{M}}_g \to [\mathrm{Gr}(q,w)/G], x \mapsto [\underbrace{\mathrm{Sym}^d H^0(C, \omega_C^{\otimes k})}_{\Gamma(\mathbb{P}^{r(k)-1}, \mathscr{O}(d))} \twoheadrightarrow \underbrace{H^0(C, \omega_C^{\otimes dk})}_{\Gamma(C, \mathscr{O}(d))}]$$

is injective as the conditions on d and k imply that the kernel of the multiplication map uniquely determines C.

By Le Lemme de Gabber we get a finite cover $X \to \overline{\mathscr{M}}_g$. By Kollár's Criterion (Theorem 4.116), we get the pullback of λ_k to X is ample for $k \gg 0$. By Proposition 3.55, we get for N sufficiently divisible, $\lambda_k^{\otimes N}$ descends to a line bundle L on $\overline{M}_g$. Since the pullback of L under the finite morphism $X \to \overline{\mathscr{M}}_g \to \overline{M}_g$, by St 0GFB we get the conclusion that L is ample. □

Theorem 4.121. *If $2g - 2 + n > 0$, then $\overline{M}_{g,n}$ is projective.*

Proof. The universal family $\overline{\mathscr{M}}_{g,n+1} \to \overline{\mathscr{M}}_{g,n}$ is projective by Proposition 4.32. Hence we just consider $n = 0$. This is right directly by the previous proposition. □

Remark 4.122. *If we consider $\omega_{\mathscr{U}_{g,n}/\overline{\mathscr{M}}_{g,n}}^{\otimes k}(\Sigma_1 + ... + \Sigma_n)$ from begining, we can prove the projectivity of $\overline{M}_{g,n}$ directly.*

4.9.4 Projectivity via GIT, a sketch

By our old way, we have $\overline{\mathscr{M}}_g \cong [H'/\mathrm{PGL}_{r(k)}]$ for some locally closed $\mathrm{PGL}_{r(k)}$-invariant subscheme of $\underline{\mathrm{Hilb}}_{\mathbb{P}^{r(k)-1}}^P$ where $P(t) = \chi(C, \omega_C^{\otimes kt})$ and $r(k) = (2k-1)(g-1)$.

Remark 4.123. *In fact we have $\overline{\mathscr{M}}_{g,n} \cong [H_{\nu,g,n}/\mathrm{PGL}(N)]$ where $N = (2\nu - 1)(g - 1) + \nu n$ and $H_{\nu,g,n} \subset \underline{\mathrm{Hilb}}_{\mathbb{P}^{N-1}}^{P_\nu}$ be the Hilbert scheme of ν-log-canonically embedded n-pointed stable curves of genus g where $P_\nu(t) = (2\nu t - 1)(g-1) + \nu n t$ for $\nu \geq 3$. See [8] Theorem XII.5.6 for the proof.*

Let H be the closure of H' in $\underline{\mathrm{Hilb}}_{\mathbb{P}^{r(k)-1}}^P$, By the proof of the representability of the quotient scheme, we get a closed immersion for $d \gg 0$:

$$\begin{array}{ccccc} H & \hookrightarrow & \underline{\mathrm{Hilb}}_{\mathbb{P}^{r(k)-1}}^P & \hookrightarrow & \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathscr{O}(d))) \\ [C \hookrightarrow \mathbb{P}^{r(k)-1}] & & \longmapsto & & [\Gamma(\mathbb{P}^{r(k)-1}, \mathscr{O}(d)) \twoheadrightarrow \Gamma(C, \mathscr{O}(d))] \end{array}$$

Next consider the Plücker embedding

$$\begin{array}{ccc} \mathrm{Gr}(P(d), \Gamma(\mathbb{P}^{r(k)-1}, \mathscr{O}(d))) & \hookrightarrow & \mathbb{P}\left(\bigwedge^{P(d)} \Gamma(\mathbb{P}^{r(k)-1}, \mathscr{O}(d))\right) \\ [\Gamma(\mathbb{P}^{r(k)-1}, \mathscr{O}(d)) \twoheadrightarrow \Gamma(C, \mathscr{O}(d))] & \longmapsto & [\bigwedge^{P(d)} \Gamma(\mathbb{P}^{r(k)-1}, \mathscr{O}(d)) \twoheadrightarrow \bigwedge^{P(d)} \Gamma(C, \mathscr{O}(d))] \end{array}$$

we can get $L_d := \mathscr{O}_{\mathrm{Gr}(P(d),\Gamma(\mathbb{P}^{r(k)-1},\mathscr{O}(d)))}(1)|_H$ be the very ample line bundle over H. All these morphisms are $\mathrm{PGL}_{r(k)}$-equivariant, hence L_d inherits a $\mathrm{PGL}_{r(k)}$-linearization. Hence L_d can defined on $[H/\mathrm{PGL}_{r(k)}]$.

Using the theory of Hilbert-Mumford criteria, we can prove the following difficult result.

Theorem 4.124. *Let $k \geq 5$ and $d \gg 0$. For $h = [C \hookrightarrow \mathbb{P}^{r(k)-1}] \in H$, the curve C is stable if and only if $h \in H$ is GIT semistable with respect to L_d, that is, there exists an equivariant section $s \in \Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}$ with $N > 0$ such that $s(h) \neq 0$. Moreover, we have*

$$\overline{M}_g \cong \mathrm{Proj}\left(\Gamma(H, L_d^{\otimes N})^{\mathrm{PGL}_{r(k)}}\right),$$

hence projective.

5 More theory of the moduli of curves

5.1 Preliminaries

We now consider $\mathscr{M}_{g,n}$ and $\overline{\mathscr{M}}_{g,n}$ as the groupoid over the category $(Sch/\mathrm{Spec}\mathbb{C})$. Then by the same arguments in the previous part, we can get $\overline{\mathscr{M}}_{g,n}$ is also a proper smooth Deligne-Mumford stack of dimension $3g-3+n$ over $\mathbb{C}$ with a coarse moduli space $\overline{M}_{g,n}$ which is a projective variety over $\mathbb{C}$. Similarly for $\mathscr{M}_{g,n}$ and $M_{g,n}$. We will refer [8].

5.1.1 Boundary geometry I. Graphs and dual graphs

We can associate a graph to a nodal curve with marked points.

Definition 5.1. *A graph Γ is the datum of:*

(a) a finite nonempty set $V = V(\Gamma)$ (the set of vertices);

(b) a finite set $L = L(\Gamma)$ (the set of half-edges);

(c) an involution ι of L;

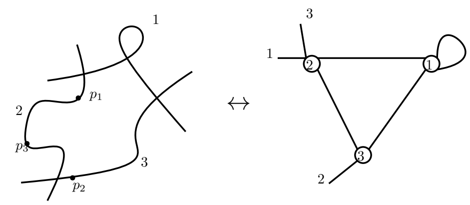
(d) a partition of L indexed by V, that is, each $v \in V$ has a (possibly empty) subset L_v of L such that $L = \bigcup_{v \in V} L_v$ with $L_v \cap L_w = \emptyset$ when $v \neq w$.

A pair of distinct elements in L which can be interchanged by ι is called an edge of the graph. The fixed points of ι is called a leg of the graph. The set of edges of Γ is denoted by $E(\Gamma)$. Define the dual graph is a graph together with a nonnegative integer g_v to each vertex v. The genus of a dual graph Γ is defined to be

$$g_\Gamma = \sum_{v \in V(\Gamma)} g_v + 1 - \chi(\Gamma).$$

A (dual) graph with a one-to-one correspondence between a finite set P and its legs will said to be P-marked.

Let C be a nodal curve and D be a finite set of its smooth points. Let the vertex be the connected components of the normalization of C, and its g_v is the genus of the components. The half-edges from a vertex are the points of the corresponding component which are nodes of C or marked points. Easy to see that the edges of the graph are node sof C; the legs are the marked points. This graph we denote it $\mathrm{Graph}(C; D)$. (Easy to see by Theorem 4.14, we get the genus of the dual graph associated to $(C; D)$ is equal to the genus of C!) For example:

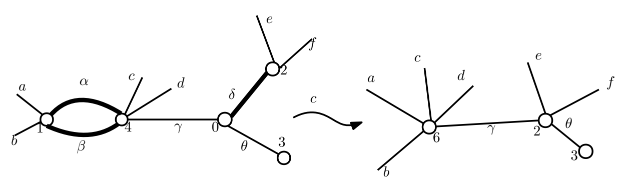


Definition 5.2. *A curve is tree-like if, after deleting edges leading from a node to itself, the dual graph becomes a tree; it is of compact type if the dual graph actually is a tree.*

Remark 5.3. *Moreover, we also have some kind of localization. For such $(C; D)$, we fix some set S of nodes of C. We let a graph $\mathrm{Graph}^S(C; D)$: (a) the vertices be the connected components of the partial normalization C^S of C at S; (b) g_v is the genus of corresponding component of C^S; (c) the edges correspond to the nodes in S; (d) the half-edges are the marked points or the points of C^S mapping to nodes in S.*

Definition 5.4. *Let Γ be a P-marked graph and let I be a subgraph without legs with all the vertices of Γ. We define Γ_I to be the graph that contracting each connected component of I to a point (see [8] page 313). Hence we have a caonincal map $c_I : \Gamma \to \Gamma_I$. There is a bijection between vertices of Γ_I and the connected components of I, and we can let $g_w(\Gamma_I) = g(I_w)$ where w be a vertice of Γ_I and I_w be its connected component.*

A P-marked dual graph Γ' is defined to be a specialization of Γ if Γ is isomorphic to Γ'_I for some $I \subset \Gamma'$. We call $c : \Gamma' \to \Gamma \cong \Gamma'_I$ an I-contraction For example:

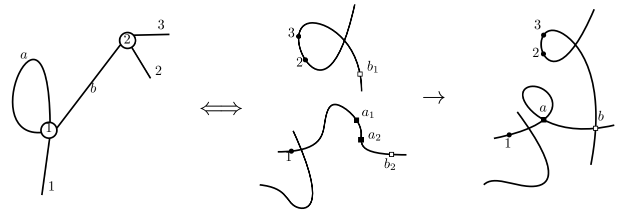


5.1.2 Boundary geometry II. More on gluing morphisms

• Gluing via graphs.

•• Gluing of curves.

Fix a P-marked dual graph Γ and for any $v \in V$ we give a L_v-pointed nodal curve C_v of genus g_v. Let $C' = \coprod_{v \in V} C_v$ and let $C = C'/\sim$ where $\sim$ means two points need to gluing together if and only if they are marked points labeled by the two halves of an edge of Γ. Hence $C' \to C$ is actually a partial normalization. For example:



Here we need to note that this graph here is kind of partial diagram.

•• Gluing of families of curves.

Fix a P-marked dual graph Γ and for any $v \in V$ we give a family of stable L_v-pointed genus g_v curves $F_v = (f_v : X_v \to S, \sigma_l$ where $l \in L_v)$. Let $X' = \coprod_v X_v$ and we get $F' = (f' : X \to S, \sigma_i)$ a family of L-pointed nodal curves.

For any $m \in L$, by taking residue along σ_m we get a surjection

$$\omega_{f'}^k(k \sum \sigma_l) \to \mathscr{O}_{\sigma_m(S)}.$$

Hence we get

$$R_l^{(k)} : f_* \left(\omega_{f'}^k(k \sum \sigma_l) \right) \to \mathscr{O}_S$$

by some kind of positivity (see [8] Lemma X.6.1(i)) it is surjective for all $k > 1$. Consider

$$R^{(k)} : f_* \left(\omega_{f'}^k(k \sum \sigma_l) \right) \to \mathscr{O}_S^E$$

indexed by pairs of edges $\{l, l'\}$ with components $R_l^{(k)} + (-1)^{k-1} R_{l'}^{(k)}$. The kernel of its fiber at $s \in S$ is $H^0(X_s, \omega_{f'}^k(k \sum_{p \in P} \sigma_{l_p}(s)))$ where X_s be the gluing of X'_s via Γ, hence its dimension is independent of s. Hence the kernel of $R^{(k)}$, which we denote by $\mathscr{S}_k$, is locally free. It is locally finitely generated (see [8] Corollary X.6.4). Let

$$X = \underline{\mathrm{Proj}}_S \bigoplus_{k \geq 0} \mathscr{S}_k$$

and hence the fibers of $X \to S$ is gluing via Γ. Let $\sigma'_p : S \to X$ is the composition of σ_{l_p} and $X' \to X$. Hence we get $F = (X \to S, \sigma'_p)_{p \in P}$.

• Gluing functors.

Fix a P-pointed dual genus g dual graph Γ and consider a Deligne-Mumford stack

$$\overline{\mathscr{M}}_\Gamma = \prod_{v \in V} \overline{\mathscr{M}}_{g_v, L_v}.$$

Fixed S and we let $\eta = (\eta_v)_{v \in V} \in \overline{\mathscr{M}}_\Gamma(S)$ where $\eta_v : X_v \to S$ be a family of stable L_v-pointed curves of genus g_v. The morphisms are isomorphisms between these families Hence we get a gluing map via Γ to get $\xi_\Gamma(\eta) : X \to S$, a family of stable P-pointed genus g curves. Hence we get the gluing morphism of stacks

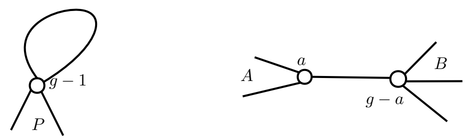
$$\xi_\Gamma : \overline{\mathscr{M}}_\Gamma \to \overline{\mathscr{M}}_{g,P}.$$

Let $\mathscr{D}_\Gamma \subset \overline{\mathscr{M}}_{g,P}$ be a closed substack as

$$\mathscr{D}_\Gamma(S) = \begin{Bmatrix} \sigma : X \to S \text{ families of } P\text{-pointed stable curves of genus } g : \\ \text{fibers have dual graphs which are specializations of } \Gamma \end{Bmatrix}$$

(as the image of ξ_Γ). It is also a Deligne-Mumford stack with a coarse moduli space $\Delta_\Gamma \subset \overline{M}_{g,P}$ as a closed subvariety. We often refer to the $\mathscr{D}_\Gamma$ (or the Δ_Γ) as the boundary strata of $\overline{\mathscr{M}}_{g,P}$ (or of $\overline{M}_{g,P}$).

The simplest boundary strata are those of codimension 1 (as a divisor as before), which correspond to the stable graphs with a single edge. If we consider the following graphs:



then for the first we let Γ_{irr} and the second $\Gamma_{a,A}$ (or $\Gamma_{\mathcal{P}}$ if $\mathcal{P} = \{(a, A), (b = g - a, B)\}$ be a stable bipartition). Hence we can also define $\mathscr{D}_{irr} := \mathscr{D}_{\Gamma_{irr}}$ and $\mathscr{D}_{a,A} := \mathscr{D}_{\Gamma_{a,A}}$ (or $\mathscr{D}_{\mathcal{P}}$). The coarse case are the same $\Delta_{irr}, \Delta_{a,A}, \Delta_{\mathcal{P}}$. Moreover, in the case we get the old gluing way:

$$\xi_{irr} : \overline{\mathscr{M}}_{g-1,P\cup\{x,y\}} \to \overline{\mathscr{M}}_{g,P}, \xi_{a,A} : \overline{\mathscr{M}}_{a,A\cup\{x\}} \times \overline{\mathscr{M}}_{g-a,A^c\cup\{y\}} \to \overline{\mathscr{M}}_{g,P}.$$

Definition 5.5 (Weak Γ-marking)**.** *Consider a family of stable P-pointed genus g curves $(\pi : \mathcal{C} \to S, \tau_p)$. Let subvariety $\Sigma \subset \mathrm{Sing}(\mathcal{C})$ proper and étale over S, then for any $s \in S$, the fiber Σ_s be a finite set of nodes. Hence we can consider* $\mathrm{Graph}^{\Sigma_s}(\mathcal{C}_s)$.

Fix a P-marked graph Γ of genus g, if $\mathrm{Graph}^{\Sigma_s}(\mathcal{C}_s) \cong \Gamma$ *for any s, then we call Σ is a weak Γ-marking. Hence we can define a stack $\mathscr{E}_\Gamma$ as*

$$\mathscr{E}_\Gamma(S) = \begin{Bmatrix} \pi : \mathcal{C} \to S \textit{ families of } P\textit{-pointed stable curves} \\ \textit{of genus } g : \textit{ endowed with a weak } \Gamma\textit{-marking} \end{Bmatrix}$$

Definition 5.6 (Γ-marking)**.** *If $\mathcal{C} \to S$ coming form $(X \to S) \in \overline{\mathscr{M}}_\Gamma(S)$ by gluing via Γ with Σ the locus of nodes produced by gluing. As Σ be a union of sections on $\mathcal{C}$, so is the preimage over X (partial normalization). Hence we can get* $\mathrm{Graph}^\Sigma(\mathcal{C})$ *with a family. Moreover* $\Gamma \cong \mathrm{Graph}^\Sigma(\mathcal{C})$*. If these data exist for $\mathcal{C} \to S$, we called it endowed a Γ-marking.*

Hence we can see that in this case we can do it conversely, hence we have $\overline{\mathscr{M}}'_\Gamma$ as

$$\overline{\mathscr{M}}_\Gamma(S) \Leftrightarrow \begin{Bmatrix} \pi : \mathcal{C} \to S \textit{ families of } P\textit{-pointed stable curves} \\ \textit{of genus } g : \textit{ endowed with a } \Gamma\textit{-marking} \end{Bmatrix} := \overline{\mathscr{M}}'_\Gamma(S)$$

Hence we can find that the gluing map can be composited as

$$\begin{tikzcd}
\overline{\mathscr{M}}_\Gamma \arrow[d, "\cong"] \arrow[drrr, "\xi_\Gamma"] & & & \\
\overline{\mathscr{M}}'_\Gamma \arrow[r, "F"] & \mathscr{E}_\Gamma \arrow[r, "F'"] & \mathscr{D}_\Gamma \arrow[r, hook] & \overline{\mathscr{M}}_{g,P}
\end{tikzcd}$$

where F, F' are forgetful maps.

Proposition 5.7. *(i) $\mathscr{E}_\Gamma$ be the normalization of substack $\mathscr{D}_\Gamma \subset \overline{\mathscr{M}}_{g,P}$;*
(ii) The morphism $\overline{\mathscr{M}}_\Gamma \to \mathscr{E}_\Gamma$ can be identified with $\overline{\mathscr{M}}_\Gamma \to [\overline{\mathscr{M}}_\Gamma/\mathrm{Aut}(\Gamma)]$.

Proof. See [8] Proposition XII.10.11. □

Corollary 5.8. *We can seen $\mathrm{Im}(\xi_\Gamma) = \mathscr{D}_\Gamma$ as before.*

Proof. Trivial by the Proposition. □

Corollary 5.9. *Let Γ be a stable P-marked dual graph of genus g. Assume that $\mathrm{Aut}(\Gamma) = \{\mathrm{id}_\Gamma\}$. Furthermore, assume that, for every graph Γ' which is a specialization of Γ, all the elements in $\mathrm{Aut}(\Gamma')$ are specializations of id_Γ. Then $\xi_\Gamma : \overline{\mathscr{M}}_\Gamma \to \overline{\mathscr{M}}_{g,P}$ is a closed immersion.*

Proof. See [8] Corollary XII.10.22. □

Theorem 5.10. *The map $\xi_\Gamma : \overline{\mathscr{M}}_\Gamma \to \overline{\mathscr{M}}_{g,P}$ is representable.*

Proof. ▶ **Step 1. Construct a new graph $\widehat{\Gamma}$ from Γ.**

Fix an edge $\ell = \{l, l'\} \in E(\Gamma)$, consider the following graph $\Gamma_l, \Gamma_{l'}$ and spliting ℓ into l, l' and joint $\Gamma_l, \Gamma_{l'}$:

Repeat this operation for all edge of Γ, we get $\widehat{\Gamma}$. Hence $\widehat{\Gamma}$ is $P \cup H$-marked where H the set of half-edges of Γ which are not legs.

▶ **Step 2. Decomposite ξ_Γ into closed immersion and projection.**

Consider maps

$$\begin{aligned}\iota_\Gamma : \overline{\mathscr{M}}_\Gamma = \prod_{v\in V} \overline{\mathscr{M}}_{g_v,L_v} \to \\ \overline{\mathscr{M}}_{\widehat{\Gamma}} = \prod_{v\in V} \overline{\mathscr{M}}_{g_v,L_v} \times \prod_{\{l,l'\}\in E} \left(\overline{\mathscr{M}}_{0,\{l_0,l_1,l_\infty\}} \times \overline{\mathscr{M}}_{0,\{l'_0,l'_1,l'_\infty\}}\right)\end{aligned}$$

and

$$\xi_{\widehat{\Gamma}} : \overline{\mathscr{M}}_{\widehat{\Gamma}} = \prod_{v\in V} \overline{\mathscr{M}}_{g_v,L_v} \times \prod_{\{l,l'\}\in E} \left(\overline{\mathscr{M}}_{0,\{l_0,l_1,l_\infty\}} \times \overline{\mathscr{M}}_{0,\{l'_0,l'_1,l'_\infty\}}\right) \to \overline{\mathscr{M}}_{g,P\cup H}$$

and $\pi_H : \overline{\mathscr{M}}_{g,P\cup H} \to \overline{\mathscr{M}}_{g,P}$ be the natural projection. Then

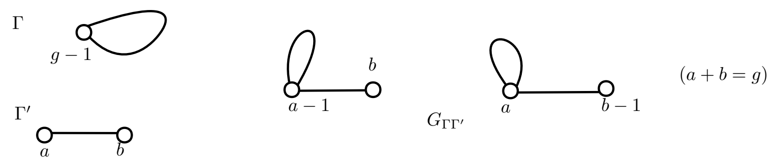
$$\xi_\Gamma = \pi_H \circ \xi_{\widehat{\Gamma}} \circ \iota_\Gamma.$$

As ι_Γ is isomorphism and π_H is representable (as a universal family) and $\xi_{\widehat{\Gamma}}$ is a closed immersion by Corollary 5.9 and $\mathrm{Aut}(\widehat{\Gamma}) = \{\mathrm{id}_{\widehat{\Gamma}}\}$. □

It is important to describe how the various boundary strata intersect. Let Γ, Γ' are two P-marked dual graph of genus g. Consider

$$G_{\Gamma\Gamma'} = \begin{Bmatrix} (\Lambda, c, c')/\cong: \Lambda \text{ be a } P\text{-marked dual graph of genus } g, \\ c : \Lambda \to \Gamma, c' : \Lambda \to \Gamma' \text{ are contractions with the} \\ \text{property that } E(\Lambda) = c^{-1}(E(\Gamma)) \cup c'^{-1}(E(\Gamma')) \end{Bmatrix}.$$

For example:



Proposition 5.11. *If we let* $\overline{\mathscr{M}}_{\Gamma\Gamma'} := \overline{\mathscr{M}}_{\Gamma} \times_{\overline{\mathscr{M}}_{g,P}} \overline{\mathscr{M}}_{\Gamma'}$, *then*

$$\overline{\mathscr{M}}_{\Gamma\Gamma'} = \coprod_{\Lambda \in G_{\Gamma\Gamma'}} \overline{\mathscr{M}}_{\Lambda}.$$

Proof. Fix a scheme T.

First we let $\xi : \mathcal{C} \to T$ in $\overline{\mathscr{M}}_{\Lambda} = \overline{\mathscr{M}}'_{\Lambda}$, then we are given a subvariety $\Sigma \subset \mathrm{Sing}(C)$, proper and étale over T, whose inverse image in the partial normalization along Σ itself is a union of sections, plus an isomorphism $\gamma : \mathrm{Graph}^{\Sigma}(\mathcal{C}) \cong \Lambda$. Let contractions $c : \Lambda \to \Gamma, c' : \Lambda \to \Gamma'$ and

$$\Sigma_1 = (c \circ \gamma)^{-1}(E(\Gamma)), \Sigma_2 = (c' \circ \gamma)^{-1}(E(\Gamma'))$$

such that $\Sigma = \Sigma_1 \cup \Sigma_2$ with isomorphisms $\gamma_1 : \mathrm{Graph}^{\Sigma_1}(\mathcal{C}) \cong \Gamma$ and $\gamma_2 : \mathrm{Graph}^{\Sigma_2}(\mathcal{C}) \cong \Gamma'$. Hence ξ is both in $\overline{\mathscr{M}}_{\Gamma}(T)$ and $\overline{\mathscr{M}}_{\Gamma'}(T)$. Hence in $\overline{\mathscr{M}}_{\Gamma\Gamma'}(T)$

Conversely, as we have

$$\overline{\mathscr{M}}_{\Gamma\Gamma'}(T) = \begin{Bmatrix} (\xi, \xi', \phi) : \xi, \xi' \text{ are families of } \Gamma, \Gamma'\text{-marking stable} \\ P\text{-pointed genus } g \text{ curves over } T \text{ with } \phi : \xi \to \xi' \\ \text{a } T\text{-isomorphism} \end{Bmatrix}.$$

Then let $(\xi, \xi', \phi) \in \overline{\mathscr{M}}_{\Gamma\Gamma'}(T)$, hence we have $\gamma : \mathrm{Graph}^{\Sigma_1}(\mathcal{C}) \cong \Gamma$ and $\gamma' : \mathrm{Graph}^{\Sigma_2}(\mathcal{C}) \cong \Gamma'$. Hence we get $c, c' : \mathrm{Graph}^{\Sigma_1 \cup \Sigma_2}(\mathcal{C}) \to \Gamma, \Gamma'$ and $(\mathrm{Graph}^{\Sigma_1 \cup \Sigma_2}(\mathcal{C}), c, c') \in G_{\Gamma\Gamma'}$, hence we win. $\square$

5.1.3 Local structure of $\overline{\mathscr{M}}_{g,n}$ and $\overline{M}_{g,n}$

We also consider the case over $\mathbb{C}$. We will using the Kuranishi family and ν-log canonical Hilbert scheme to describe the local structure of the moduli stack and (coarse) space of the stable curves.

Recall that we have the local structure of the Deligne-Mumford stack and its coarse moduli space, that is, the Theorem 3.53 and Theorem 3.48 as follows.

Theorem A. Let $\mathscr{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Let $\pi : \mathscr{X} \to X$ be its coarse moduli space. For any closed point $x \in |\mathscr{X}|$ with geometric stabilizer G_x, we have an étale neighborhood $\mathrm{Spec}A^{G_x} \to X$ of $\pi(x) \in |X|$.
Theorem B. Let $\mathscr{X}$ be a separated Deligne-Mumford stack and $x \in \mathscr{X}(k)$ be a geometric point with stabilizer G_x. Then exists an affine and étale map

$$f : ([\mathrm{Spec}A/G_x], w) \to (\mathscr{X}, x)$$

where $w \in (\mathrm{Spec}A)(k)$ such that f induces an isomorphism of the stabilizer groups at w. Moreover, it can be arranged that $f^{-1}(BG_x) \cong BG_w$.

But now we will get a more coarse (but useful) local structure by using the Kuranishi family as follows. Actually as a set, $\overline{M}_{g,n}$ is a set of isomorphism class of the n-pointed stable curves. Hence by Definition 4.65, for a n-pointed stable curve we have a standard Kuranishi family $\xi : \mathcal{C} \to (X_0, x_0) \subset H_{\nu,g,n}$. Hence we have a natural map

$$\psi : X_0/G_{x_0} \to \overline{M}_{g,n}.$$

Recall some properties of X_0 in Definition 4.65:
• For any $y \in X_0$ we have $G_y := \mathrm{Aut}(\mathcal{C}_y; \sigma_i(y)) \cong \mathrm{stab}_{G_{x_0}}(y)$;
•• For any $y \in X_0$, there is a G_y-invariant analytic neighborhood U of y in X such that any isomorphism (of n-pointed curves) between fibers over U is induced by an element of G_y.

Theorem 5.12. *The map $\psi : X_0/G_{x_0} \to \overline{M}_{g,n}$ is étale. Moreover there are finite many such X_i and G_i covers $H_{\nu,g,n}$ such that the map*

$$\phi : Y := \coprod_i X_i/G_i \to \overline{M}_{g,n}$$

is étale and surjective.

Proof. We refer [8] Proposition XII.3.5. □

Theorem 5.13. *The canonical map*

$$\alpha : X := \coprod_i X_i \to \overline{\mathscr{M}}_{g,n}$$

is étale and surjective where X_i are Kuranishi families as before covers $H_{\nu,g,n}$.

Proof. We refer [8] Theorem XII.8.3. □

5.2 Line bundles and Picard groups of the moduli of curves

We will refer [8] chapter XIII and [5].

5.2.1 Line bundles on the moduli stack of stable curves

Example 5.14 (Hodge bundle)**.** *For any $S \to \overline{\mathscr{M}}_{g,n}$ which correspond to $\xi = (\pi : \mathcal{C} \to S)$, we let $\mathbb{E}_\xi := \pi_*\omega_\pi$. Hence induce a sheaf $\mathbb{E}$ over $\overline{\mathscr{M}}_{g,n}$ called the Hodge bundle. As the relative dualizing sheaf is functorial with respect to morphisms of families, this is a quasi-coherent sheaf. By the cohomology and base change, it is actually a vertor bundle of rank g as before. Let $\det \mathbb{E} = \bigwedge^g \mathbb{E}$ and we call it the Hodge line bundle. Usually we denote $\lambda := [\bigwedge^g \mathbb{E}] \in \mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$.*

Remark 5.15. *For the canonical map $\overline{\mathscr{M}}_{g,n} \to \overline{M}_{g,n}$ there are plenty of quasi-coherent sheaves on $\overline{\mathscr{M}}_{g,n}$ which do not come by pullback from the quasi-coherent sheaves on $\overline{M}_{g,n}$. For example, Hodge bundle as follows (see Remark 5.60 for more about Hodge bundle). More precisely, the Hodge bundle and its determinant do not descend to coherent sheaves on the moduli space $\overline{M}_{g,n}$ except in genus zero. For this we refer [8] Page 343-344.*

Example 5.16 (Generalization of the Hodge line bundle)**.** *For any $S \to \overline{\mathscr{M}}_{g,n}$ which correspond to $\xi = (\pi : \mathcal{C} \to S)$ and any $\nu \in \mathbb{Z}$, we let*

$$\Lambda(\nu)_\xi := \left(\bigwedge^{\max} R^1\pi_*\omega_\pi^{\otimes\nu}\right)^{-1} \otimes \bigwedge^{\max} \pi_*\omega_\pi^{\otimes\nu}.$$

Hence induce a line bundle $\Lambda(\nu)$ over $\overline{\mathscr{M}}_{g,n}$. Usually we denote $\lambda(\nu) := [\Lambda(\nu)] \in \mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$.

Actually when $\nu = 1$, by the same arguments in Lemma 2.14 we can show that $R^1\pi_*\omega_\pi \cong \mathscr{O}_S$. So we have $\Lambda(1)_\xi \cong \bigwedge^g \mathbb{E}_\xi$ canonically, hence $\Lambda(1) \cong \bigwedge^g \mathbb{E}$.

Example 5.17 (Point bundles)**.** *For $n > 0$ and for any $S \to \overline{\mathscr{M}}_{g,n}$ which correspond to $\xi = (\pi : \mathcal{C} \to S; \sigma_i)$, we let $(\mathscr{L}_i)_\xi := \sigma_i^*\omega_\pi$. Hence we get $\mathscr{L}_i$ be the line bundles over $\overline{\mathscr{M}}_{g,n}$. We usually Set*

$$\psi_i = [\mathscr{L}_i] \in \mathrm{Pic}(\overline{\mathscr{M}}_{g,n}), \psi = \sum_i \psi_i.$$

Remark 5.18. *As the Hodge bundle, in general, $\mathscr{L}_i$ can't descend to a line bundle on $\overline{M}_{g,n}$.*

Example 5.19 (Boundary divisors and bundles)**.** *As before, we have*

$$\partial\overline{\mathscr{M}}_{g,n} =: \mathscr{D} = \mathscr{D}_{irr} + \sum_{\mathcal{P}} \mathscr{D}_{\mathcal{P}},$$

where the sum runs through all stable bipartitions of $(g, \{1, ..., n\})$. We denote

$$\delta_{irr} = [\mathscr{O}(\mathscr{D}_{irr})] \in \mathrm{Pic}(\overline{\mathscr{M}}_{g,n}), \delta_{\mathcal{P}} = [\mathscr{D}_{\mathcal{P}}] \in \mathrm{Pic}(\overline{\mathscr{M}}_{g,n}).$$

5.2.2 Tangent bundle, cotangent bundle and normal bundle

Proposition 5.20. *Consider the moduli stack $\overline{\mathscr{M}}_{g,P}$, then tangent bundle $\mathscr{T} = T_{\overline{\mathscr{M}}_{g,P}}$ can be described as: for any $F = (f : X \to S, \{\sigma_p\}_{p\in P}) \in \overline{\mathscr{M}}_{g,P}(S)$, we have*

$$\mathscr{T}_F = f_*(\Omega_f^1 \otimes \omega_f(D))^\vee$$

where $D = \sum \sigma_p(S)$.

Proof. By Theorem 5.13, the Kuranishi families formed an étale covering. Hence consider a stable P-pointed curve $\{C; x_p\}$ and its Kuranishi family (see Theorem 4.50) $\mathcal{X} \to (U, u_0)$, then we have

$$T_{u_0}U \cong \mathrm{Ext}^1_{\mathscr{O}_C}(\Omega_C^1, \mathscr{O}_C(-\sum_p x_p)) \cong H^0(C, \Omega_C^1 \otimes \omega_C(\sum_p x_p))^\vee.$$

Hence we get the conclusion. □

Example 5.21 (Canonical bundle)**.** *Hence the cotangent bundle $\mathscr{T}^\vee$ given by $\mathscr{T}_F^\vee = f_*(\Omega_f^1 \otimes \omega_f(D))$. Hence we get the class of the canonical line bundle*

$$K_{\overline{\mathscr{M}}_{g,P}} := \left[\bigwedge^{\max} \mathscr{T}^\vee\right] \in \mathrm{Pic}(\overline{\mathscr{M}}_{g,P}).$$

Now we consider the normal bundle of $\xi_\Gamma : \overline{\mathscr{M}}_\Gamma \to \overline{\mathscr{M}}_{g,P}$ where Γ be a stable graph (For a map of smooth schemes $f : X \to Y$, we let $N_f = f^*T_Y/T_X$).

Example 5.22 (Single curves)**.** *Let N be a point of $\overline{\mathscr{M}}_\Gamma$ with image C in $\overline{\mathscr{M}}_{g,P}$. One can consider N as a partial normalization of C at some y_e. By Claim 2 in Remark 4.49 we get*

$$\begin{aligned} 0 \to &\mathrm{Ext}^1(\Omega_N^1, \mathscr{O}_N(-\widetilde{D} - R)) \to \mathrm{Ext}^1(\Omega_C^1, \mathscr{O}_C(-D)) \to \\ &\bigoplus_{e\in E(\Gamma)} \mathrm{Ext}^1(\Omega_{C,y_e}^1, \mathscr{O}_{C,y_e}) \to 0. \end{aligned}$$

where $D = \sum x_p$ with preimage $\widetilde{D}$ and R be the preimage of these y_e.

Easy to see that $\mathrm{Ext}^1(\Omega_N^1, \mathscr{O}_N(-\widetilde{D} - R))$ be the tangent space of $\overline{\mathscr{M}}_\Gamma$ at N and $\mathrm{Ext}^1(\Omega_C^1, \mathscr{O}_C(-D))$ be the tangent space of $\overline{\mathscr{M}}_{g,P}$ at C, hence the normal space to ξ_Γ at N is

$$\bigoplus_{e\in E(\Gamma)} \mathrm{Ext}^1(\Omega_{C,y_e}^1, \mathscr{O}_{C,y_e}) = \bigoplus_{e\in E(\Gamma)} T_{N,y_e'} \otimes T_{N,y_e''}$$

by Claim 2 in Remark 4.48 (or Claim 3 in Remark 4.49).

Proposition 5.23. *The normal bundle of $\xi_\Gamma : \overline{\mathscr{M}}_\Gamma \to \overline{\mathscr{M}}_{g,P}$ can be expreessed as*

$$N_{\xi_\Gamma} = \bigoplus_{\{l,l'\}\in E(\Gamma)} \eta_{v(l)}^*\mathscr{L}_l^\vee \otimes \eta_{v(l')}^*\mathscr{L}_{l'}^\vee$$

where $\eta_v : \overline{\mathscr{M}}_\Gamma \to \overline{\mathscr{M}}_{g_v,L_v}$ be the projections.

Proof. Let F in $\overline{\mathscr{M}}_\Gamma$ is the datum of a family $X_v \to S$ of stable L_v-pointed curves of genus g_v for each vertex v of Γ. We let $X = \coprod_v X_v$. For each $l \in L(E)$, we denote by σ_l the corresponding section of $X \to S$. The gluing construction yields a family $X' \to S$ of stable P-pointed genus g curves. Then the normal

$$N_{\xi_\Gamma, F} = \bigoplus_{\{l,l'\}\in E(\Gamma)} \sigma_l^* T_{X/S} \otimes \sigma_{l'}^* T_{X/S} = \bigoplus_{\{l,l'\}\in E(\Gamma)} \mathscr{L}_{l,F}^\vee \otimes \mathscr{L}_{l',F}^\vee$$

where $\mathscr{L}_i$ are point bundles. Hence we win. $\square$

Remark 5.24 (Excess intersection bundle)**.** *By Proposition 5.11, we consider*

$$\begin{array}{ccc} \overline{\mathscr{M}}_{\Gamma\Gamma'} = \coprod_{\Lambda\in G_{\Gamma\Gamma'}} \overline{\mathscr{M}}_\Lambda & \xrightarrow{\coprod \xi_{\Lambda\Gamma}} & \overline{\mathscr{M}}_\Gamma \\ \downarrow & & \downarrow \xi_\Gamma \\ \overline{\mathscr{M}}_{\Gamma'} & \xrightarrow{\xi_{\Gamma'}} & \overline{\mathscr{M}}_{g,P} \end{array}$$

Then the excess intersection bundle is

$$F_{\Gamma\Gamma'} = \bigoplus_{\Lambda\in G_{\Gamma\Gamma'}} F_{\Lambda\Gamma\Gamma'} := \bigoplus_{\Lambda\in G_{\Gamma\Gamma'}} \xi_\Gamma^*(N_{\xi_{\Gamma'}})/N_{\xi_{\Lambda\Gamma}}.$$

We can show that (as [8] XIII.(3.8))

$$F_{\Gamma\Gamma'} = \bigoplus_{\Lambda\in G_{\Gamma\Gamma'}} \bigoplus_{\{l,l'\}\in c^{-1}(E(\Gamma))\cap c'^{-1}(E(\Gamma'))} \eta_{v(l)}^* \mathscr{L}_l^\vee \otimes \eta_{v(l')}^* \mathscr{L}_{l'}^\vee.$$

Corollary 5.25. *We have*

$$\left[\bigwedge^{\max} N_{\xi_\Gamma}\right] = -\sum_{l\in H(\Gamma)} \eta_{v(l)}^* \psi_l$$

where $H(\Gamma)$ be the set of those half-edges of Γ which are not legs.

5.2.3 Determinant

• Basic linear algebra.

Definition 5.26. *A $\mathbb{Z}/2$-graded line bundle is a pair (L, r) where L be a line bundle over a scheme X and $r \in \{0, 1\}$. We define the determinant of a finite vector bundle F over X is a $\mathbb{Z}/2$-graded line bundle*

$$\det F := \left(\bigwedge^{\max} F, \mathrm{rank} F \pmod 2\right).$$

We say (L, r) is even/odd if r is even/odd. We define the tensor product of $\mathbb{Z}/2$-graded line bundles as $(L, r) \otimes (T, s) := (L \otimes T, r + s)$. Let $A := (L, r), B := (T, s)$ and define the canonical isomorphism

$$\tau_{A,B} : A \otimes B \to B \otimes A, l \otimes m \mapsto (-1)^{rs} m \otimes l.$$

Proposition 5.27. *(i) For $\mathbb{Z}/2$-graded line bundles A, B, C we have*

$$\tau_{A\otimes B,C} = (\tau_{A,C} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau_{B,C});$$

(ii) For an exact sequence $\mathcal{E} : 0 \to E \to F \to G \to 0$ of vector bundles, we have a canonical isomorphism $\phi_{\mathcal{E}} : \det E \otimes \det G \to \det F$;

(iii) Define $\mathbf{1}_X := (\mathscr{O}_X, 0)$ and $A^{-1} = (L^\vee, a)$ for a $\mathbb{Z}/2$-graded line bundle $A = (L, a)$. Then $A \otimes A^{-1} \cong \mathbf{1}_X, \alpha, \phi \mapsto \phi(\alpha)$ and

$$S_{A,B} : B^{-1} \otimes A^{-1} \cong (A \otimes B)^{-1}, \phi \otimes \psi \mapsto (\chi : \alpha \otimes \beta \mapsto \phi(\alpha)\psi(\beta));$$

(iv) We have $\tau^\vee_{B,A} \circ S_{A,B} = S_{B,A} \circ \tau_{A^{-1},B^{-1}}$.

Proof. Trivial by some easy linear algebra and calculation. □

Definition 5.28. *Let a finite complexes F^* of vector bundles on X, we define*

$$\det F^* := \bigotimes_{q\in\mathbb{Z}} (\det F^q)^{(-1)^q}.$$

Proposition 5.29. *(i) For a exact sequence of complexes $\mathcal{E} : 0 \to E^* \to F^* \to G^* \to 0$, we also have isomorphism*

$$\phi_{\mathcal{E}} : \det E^* \otimes \det G^* \cong \det F^*;$$

(ii) The determinant and $\phi_{\mathcal{E}}$ are functorial in the base space X;

(iii) Consider

$$\begin{array}{cccccccccc}
 & & & \mathcal{E}_1 & & \mathcal{E}_2 & & \mathcal{E}_3 & & \\
 & & & & & & & & & \\
 & & & 0 & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{R}_1 : & 0 & \longrightarrow & A^* & \longrightarrow & B^* & \longrightarrow & C^* & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{R}_2 : & 0 & \longrightarrow & A'^* & \longrightarrow & B'^* & \longrightarrow & C'^* & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{R}_3 : & 0 & \longrightarrow & A''^* & \longrightarrow & B''^* & \longrightarrow & C''^* & \longrightarrow & 0 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & 0 & & 0 & & 0 & &
\end{array}$$

then we have $\phi_{\mathcal{E}_2} \circ (\phi_{\mathcal{R}_1} \otimes \phi_{\mathcal{R}_3}) = \phi_{\mathcal{R}_2} \circ (\phi_{\mathcal{E}_1} \otimes \phi_{\mathcal{E}_3}) \circ (\mathrm{id} \otimes \tau_{C^*,A''^*} \otimes \mathrm{id})$;

(iv) If A^* be a finite acyclic complex of vector bundles on X, there is a canonical isomorphism $\det A^* \cong \mathbf{1}_X$. More generally, if $f : A^* \to B^*$ be a quasi-isomorphism of finite complexes of vector bundles, then there is an isomorphism $\det f : \det A^* \to \det B^*$ which depends only on the homotopy class of f;

(v) Consider

$$\begin{array}{cccccccccc}
\mathcal{E} : & 0 & \longrightarrow & A_1^* & \longrightarrow & A^* & \longrightarrow & A_2^* & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} & & \\
\mathcal{E}' : & 0 & \longrightarrow & B_1^* & \longrightarrow & B^* & \longrightarrow & B_2^* & \longrightarrow & 0
\end{array}$$

then $\det f \circ \phi_{\mathcal{E}} = \phi_{\mathcal{E}'} \circ (\det f_1 \otimes \det f_2)$*;*

(vi) Consider the exact sequences $\mathcal{E} : 0 \to A^* \xrightarrow{\alpha} B^* \to 0 \to 0$ *and* $\mathcal{E}' : 0 \to 0 \to B^* \xrightarrow{\beta} C^* \to 0$*, then*

$$\det \alpha = \phi_{\mathcal{E}} \circ (a \mapsto a \otimes 1), \det \beta = \phi_{\mathcal{E}'} \circ (b \mapsto 1 \otimes b).$$

Proof. These are more complicated linear algebra, we omit these here. We refer [8] XIII.4. □

• Constructions and properties.

Proposition 5.30 (Determinant of the cohomology of coherent sheaves)**.** *Aiming to construct the relative determinant of the cohomology here.*

•*Claim.* *Let* $f : X \to S$ ***be a flat morphism and let*** F ***be a coherent sheaf on*** X ***which is flat over*** S***. Let*** Z ***be the subset of*** X ***where*** F ***is not locally free, then*** Z ***does not contain any component of any fiber of*** f***.***

We omitted this fact and refer to [8] Lemma XIII.4.13.

•*Construction.* *Consider a family of nodal curves* $\pi : X \to S$*. Let* F *is flat over* S*. Let* S *covered by* U *such that there is an effective Cartier divisor* D *in* $\pi^{-1}(U)$ *which meets all the irreducible components of every fiber and does not contain any of them; in particular,* D *is relatively ample. We may replacing* D *with a multiple, then let* $R^1\pi_*F(D) = 0$*. By the Claim, we may also suppose that* F *is locally free at every point of* D*. We say such divisor admissible.*

Hence $F \subset F(D)$ *and let* $F(D)|_D := F(D)/F$*. By some cohomology and base change, we get* $\pi_*F(D), \pi_*F(D)|_D$ *are all locally free. We have*

$$0 \to \pi_*F \to \pi_*F(D) \to \pi_*F(D)|_D \to R^1\pi_*F \to 0,$$

hence the complex $E_D^* := (\pi_*F(D) \to \pi_*F(D)|_D)$ *computes the higher direct image of* F*. Hence we let locally* $d_\pi F = \det E_D^*$*.*

The independence on D *and the gluing map are not hard to construct and we omitted them, see [8] page 356. Hence we get the determinant* $d_\pi F$ *of the cohomology of* F *(relative to* π*).*

Remark 5.31. *The flatness of* F *over* S *is unnecessary but simplifies the construction.*

Proposition 5.32 (Determinant of the (hyper)cohomology of complexes)**.** *Consider a family of nodal curves* $\pi : X \to S$*. Similarly, Let* F^* *be a finite complex of coherent sheaves, flat over* S*. Let* U *be a sufficiently small open subset of* S*, and let* D *be a divisor in* $\pi^{-1}(U)$ *which is admissible for each one of the* F^i *and that* $F^i \to F^{i+1}$ *is a morphism of vector bundles at each point of* D *for each* i *(this is called admissible for* F^**). Hence we let* $E_D^{i,0} = \pi_*(F^i(D))$ *and* $E_D^{i,1} = \pi_*(F^i(D)|_D)$*, then we get a double complex* $E_D^{*,*}$*. Regard it as a single complex graded by total degree, and we locally define* $d_\pi F^*$ *to be its determinant.*

Proposition 5.33. *Consider a family of nodal curves* $\pi : X \to S$*.*

(i) For a coherent F *with* $\pi_*F, R^1\pi_*F$ *are locally free, then we have*

$$d_\pi F \cong \det(R^1\pi_*F)^{-1} \otimes \det(\pi_*F);$$

(ii) For a finite complex F^* with $\mathbb{R}^i\pi_*F^* := H^iR\pi_*F^*$ are locally free, then we have

$$d_\pi F^* = \bigotimes_{i\in\mathbb{Z}} \det(\mathbb{R}^i\pi_*F^*)^{(-1)^i}.$$

Proof. I just prove (i) since (ii) is similar.

We split

$$0 \to \pi_*F \to \pi_*F(D) \to \pi_*F(D)|_D \to R^1\pi_*F \to 0$$

into two sequences

$$0 \to \pi_*F \to E_D^0 \to Q \to 0, 0 \to Q \to E_D^1 \to R^1\pi_*F \to 0.$$

Then we have $\det(\pi_*F) \otimes \det Q \cong \det E_D^0$ and $\det(R^1\pi_*F) \otimes \det Q \cong \det E_1^0$. Hence we have locally

$$d_\pi(F) = \det(E_D^1)^{-1} \otimes \det(E_D^0) = \det(R^1\pi_*F)^{-1} \otimes \det(\pi_*F)$$

and well done. □

Remark 5.34. *These constructions are compatible with base change, hence for $s \in S$ we have*

$$d_\pi F^* \otimes \kappa(s) \cong \bigotimes_{q\in\mathbb{Z}} (\det \mathbb{H}^q(X_s, F_s^*))^{(-1)^q}.$$

Theorem 5.35. *Let $0 \to E^* \to F^* \to G^* \to 0$ be an exact sequence of finite complexes of coherent sheaves on X, all flat over S, then we have*

$$\phi : d_\pi(E^*) \otimes d_\pi(G^*) \cong d_\pi(F^*).$$

Proof. Not hard but it's hard to type and I omit it here. We refer [8] XIII.(4.17). □

• Determinant, relative duality and applications.

Theorem 5.36. *Consider a family of nodal curves $\pi : X \to S$ and a coherent sheaf F, we have*

$$d_\pi(\omega_\pi \otimes F^\vee) \cong d_\pi(F).$$

In particular, the Hodge bundle is $d_\pi(\omega_\pi) = d_\pi(\mathscr{O}_X)$.

Proof. This is also not hard to prove by checking the construction of the determinant of cohomology. Using some canonical exact sequences and diagrams this is almost trivial. I omit these here and we refer [8] page 360. □

Proposition 5.37 (Determinant and boundary of moduli)**.** *We will describe $\mathscr{O}(\mathscr{D})$ by using the determinant of cohomology.*

Proof. Let $\pi : X \to S$ be a family of connected nodal curves of genus g.
•**Claim 1.** Ω^1_π **is** S**-flat.**

WLOG we let S is smooth as these are pullbacked from a Kuranishi family. Shrinking S, we may assume that there exists an effective divisor D in X which cuts an ample divisor D_s on each fiber X_s and does not contain nodes of the fibers. We just need to show $\chi(X_s, \Omega^1_\pi(nD) \otimes \kappa(s))$ is independent of $s \in S$ for $n \gg 0$. By Corollary 4.23, we have

$$0 \to K \to \Omega^1_{X_s} \xrightarrow{\rho_s} \omega_{X_s} \to Q \to 0$$

where $\mathrm{supp}(K), \mathrm{supp}(Q) \subset \{\text{nodes}\}$. As they both have one-dimensional stalks by Claim 1,2 in Corollary 4.23, hence we have $\chi(X_s, \Omega^1_\pi(nD) \otimes \kappa(s)) = \chi(X_s, \omega_{X_s}(nD_s)) = 2g - 2 + n\deg(D)$.

•**Claim 2. Let** $L_\pi := d_\pi(\Omega^1_\pi \xrightarrow{\rho_\pi} \omega_\pi)$**, then** $L_\pi = d_\pi(\omega_\pi) d_\pi(\Omega^1_\pi)^{-1}$ **and induce** $\det(\rho_\pi) : d_\pi(\Omega^1_\pi) \to d_\pi(\omega_\pi)$ **which is a canonical section of** L_π**.**

As we have an exact sequence of complexes $0 \to \omega_\pi[0] \to (\Omega^1_\pi \xrightarrow{\rho_\pi} \omega_\pi) \to \Omega^1_\pi[1] \to 0$, we get $L_\pi = d_\pi(\omega_\pi) d_\pi(\Omega^1_\pi)^{-1}$. The map $\det(\rho_\pi) : d_\pi(\Omega^1_\pi) \to d_\pi(\omega_\pi)$ can be easily constructed step by step as the construction of d_π.

•**Claim 3. Various** L_π **and** $\det(\rho_\pi)$ **defines a line bundle** L **on** $\overline{\mathscr{M}}_{g,n}$ **with a canonical section** $\det(\rho)$**. As** ρ_π **is an isomorphism on smooth fibers, we get** $L \cong \mathscr{O}(\sum_i n_i \mathscr{D}_i)$ **where** $\mathscr{D}_i$ **are components of** $\mathscr{D}$ **with nonnegative integers** n_i**. We claim that all** $n_i = 1$ **and hence** $L \cong \mathscr{O}(\mathscr{D})$**.**

We consider the case when S is a disk (étale locally) centered at s and all the fibers of π are smooth except for X_s, which has a single node p. All we need is to calculate n_i, the order of vanishing of $\det \rho_\pi$ at s. I omit it here and refer [8] page 363. □

Proposition 5.38. *Let Γ be a connected P-pointed genus g graph. Let $H(\Gamma)$ for the set of the half-edges of Γ which are not legs. Suppose that for each v, we are given a family $\pi_v : X_v \to S$ of connected nodal L_v-pointed genus g_v curves. Let σ_l the corresponding section of π_v and $\mathscr{L}_l$ are point bundles on S where $l \in L_v$. Let $D_l = \sigma_l(S)$ and let $\pi : X \to S$ be the family gluing via Γ by X_v. Then*

$$\mathscr{O}(\mathscr{D})_\pi \cong \left(\bigotimes_{v \in V(\Gamma)} \mathscr{O}(\mathscr{D})_{\pi_v} \right) \otimes \left(\bigotimes_{h \in H(\Gamma)} \mathscr{L}_h^{-1} \right).$$

In particular, taking Chern classes we get

$$\xi_\Gamma^* \delta = \sum_{v \in V(\Gamma)} \eta_v^* \delta - \sum_{h \in H(\Gamma)} \eta_{v(h)}^* \psi_h$$

where $\eta_\Gamma : \overline{\mathscr{M}}_\Gamma \to \overline{\mathscr{M}}_{g,P}$ and $\eta_v : \overline{\mathscr{M}}_\Gamma \to \overline{\mathscr{M}}_{g_v, L_v}$.

Proof. Let $N = \coprod_v X_v \xrightarrow{\pi'} S$ with normalization $\nu : N \to X$. For $e = \{h, h'\} \in E(\Gamma)$

and let $\Sigma_e = \nu(D_h) = \nu(D_{h'})$, we have

$$0 \longrightarrow \omega_{\pi'} \longrightarrow \omega_{\pi'}\left(\sum_{h\in H(\Gamma)} D_h\right) \xrightarrow{Res} \bigoplus_{\{h,h'\}\in E(\Gamma)}(\mathscr{O}_{D_h} \oplus \mathscr{O}_{D_{h'}}) \longrightarrow 0$$

$$0 \longrightarrow \omega_{\pi} \longrightarrow \nu_*\left(\omega_{\pi'}\left(\sum_{h\in H(\Gamma)} D_h\right)\right) \longrightarrow \bigoplus_{e\in E(\Gamma)} \mathscr{O}_{\Sigma_e} \longrightarrow 0$$

Taking cohomology we get

$$d_{\pi'}(\omega_{\pi'}) \cong d_\pi(\nu_*\omega_{\pi'}) \cong d_\pi\left(\nu_*\omega_{\pi'}\left(\sum_{h\in H(\Gamma)} D_h\right)\right)$$
$$d_\pi(\omega_\pi) \cong d_\pi\left(\nu_*\omega_{\pi'}\left(\sum_{h\in H(\Gamma)} D_h\right)\right).$$

Hence

$$d_\pi(\omega_\pi) \cong d_{\pi'}(\omega_{\pi'}) \cong \bigotimes_{v\in V(\Gamma)} d_{\pi_v}(\omega_{\pi_v}).$$

On the other hand, we have $0 \to K \to \Omega^1_\pi \to \nu_*\Omega^1_{\pi'} \to 0$ which deduce

$$d_\pi(\Omega^1_\pi) \cong \left(\bigotimes_{e\in E(\Gamma)} \pi_* K_e\right) \otimes d_{\pi'}\Omega^1_{\pi'}.$$

•**Claim. We have** $\pi_* K_{\{h,h'\}} \cong \mathscr{L}_h \otimes \mathscr{L}_{h'}$**.**

Here we give a sketch of the claim. Consider $e = \{h, h'\}$ with local coordinates x, y, then K_e locally generated by $ydx(= -xdy)$, then we define it mapping to section $\sigma_h^*(dx) \otimes \sigma_{h'}^*(dy)$ of $\mathscr{L}_h \otimes \mathscr{L}_{h'}$. We omitted the verifing.

Finally, by the Claim 2,3 in Proposition 5.37 we get

$$\begin{aligned}
\mathscr{O}(\mathscr{D})_\pi = L_\pi &= d_\pi(\omega_\pi) \otimes d_\pi(\Omega^1_\pi)^{-1} \\
&= \left(\bigotimes_{v\in V(\Gamma)} d_{\pi_v}(\omega_{\pi_v})\right) \otimes \left(\left(\bigotimes_{e\in E(\Gamma)} \pi_* K_e\right) \otimes d_{\pi'}\Omega^1_{\pi'}\right)^{-1} \\
&= \left(\bigotimes_{v\in V(\Gamma)} \mathscr{O}(\mathscr{D})_{\pi_v}\right) \otimes \left(\bigotimes_{h\in H(\Gamma)} \mathscr{L}_h^{-1}\right) \otimes (d_{\pi'}\Omega^1_{\pi'})^{-1} \otimes \bigotimes_{v\in V(\Gamma)} d_{\pi_v}\Omega^1_{\pi_v} \\
&= \left(\bigotimes_{v\in V(\Gamma)} \mathscr{O}(\mathscr{D})_{\pi_v}\right) \otimes \left(\bigotimes_{h\in H(\Gamma)} \mathscr{L}_h^{-1}\right),
\end{aligned}$$

and we get the result. □

Remark 5.39. *We also get* $\xi_\Gamma^*\lambda = \sum_{v\in V(\Gamma)} \eta_v^*\lambda$.

5.2.4 Deligne pairing, a quick tour

Definition 5.40. *(a) Let C be a complete curve (need not be connected) and $D = \sum_p n_p p$ a divisor on C. If f is a rational function on C whose divisor (f) is disjoint from D, we set $f(D) := \prod_p f(p)^{n_p}$;*

(b) Let $\pi : X \to S$ be a family of nodal curves and D is an effective relative Cartier divisor not containing nodes of fibers, $\pi_*\mathscr{O}(D)$ is locally free, and there is a norm map $\mathrm{Norm}_{D/S} : \pi_*\mathscr{O}(D) \to \mathscr{O}_S$ *(as $D \to S$ is proper and quasi-finite, hence finite). We also induce* $\mathrm{Norm}_{D/S} : \pi_*\mathscr{O}(D)^\times \to \mathscr{O}_S^\times$*.*

Hence for an divisor $D = D_1 - D_2$ where D_i are effective, then we define

$$f(D) = \mathrm{Norm}_{D_1/S}(f)\mathrm{Norm}_{D_2/S}(f)^{-1}$$

which is well defined as if E_1, E_2 are all effective, then $f(E_1 + E_2) = f(E_1)f(E_2)$.

Proposition 5.41 (Weil reciprocity)**.** *(i)[Smooth case] Let C be a smooth proper curve (need not be connected) and f, g are rational functions which are nonzero on every component of C and with disjoint divisors. Then $f((g)) = g((f))$;*

(ii)[Nodal case] Let C be a possibly disconnected nodal curve, and let f and g be rational functions on C which do not vanish identically on any irreducible component of C and are regular and nonzero at all the nodes. Then, if the divisors of f and g are disjoint, we have $f((g)) = g((f))$;

(iii)[Relative case] Let $\pi : X \to S$ be a family of nodal curves and f and g are two meromorphic functions on X not vanish identically on any component of any fiber and be regular and nonzero at all the nodes, and their divisors be disjoint, then $f((g)) = g((f))$.

Proof. For (i) we refer [9] VI.B.2. For (ii), notice that what must be proved can reduces to the Weil reciprocity formula for the pullbacks of f and g to the normalization of C. For (iii), when S is reduced, we can do the same thing as single one. Otherwise, one can use the Kuranishi family and pullback. □

Definition 5.42 (Deligne pairing for single case)**.** *Let L, M are two line bundles over a nodal curve C. Let V be a vertor space generated by pairs (l, m) where l, m are rational sections of L, M, respectively, such that*

(a) l, m are nonzero on any component of C, and regular and nonzero at the nodes of C;

(b) the divisors (l) and (m) are disjoint.

Let $\langle L, M\rangle$ be the quotient of V modulo the equivalence relation generated by

$$(fl, m) \sim f((m))(l, m), \quad (l, gm) \sim g((l))(l, m)$$

where f and g are rational functions on C. This space is the Deligne pairing of L and M. The class of (l, m) denoted by $\langle l, m\rangle$.

Remark 5.43. *(i) Actually the meromorphic section l of L defined by a data (l_i, U_i) where $l_i \in \mathscr{K}_C(U_i)$ of covering $X = \bigcup_i U_i$ such that $l_i = \psi_{ij} \cdot l_j$ where $\psi_{ij} = \psi_i \circ \psi_j^{-1}$ are cocycles of trivializations $\phi_i : L|_{U_i} \cong \mathscr{O}_{U_i}$. In other words, l is a section of $L \otimes_{\mathscr{O}_X} \mathscr{K}_X$.*

Hence we have canonically divisor (l) *associated to* l *and we have trivially* $\mathscr{O}_X((l)) \cong L$ *(see [62] and [47]);*

(ii) Two equivalence relations are called L*-move and* M*-move, respectively.*

Proposition 5.44. *For any* L, M *on* C*, then* $\dim \langle L, M\rangle = 1$ *be a line.*

sketch. **•Claim 1. We have** $\dim \langle L, M\rangle \leq 1$**.**

For any $(l, m), (l', m')$, let μ be a meromorphic divisor of M disjoint of l, l'. Hence let $\mu = gm, m' = g'\mu, l' = fl$ where f, g, g' are rational functions, then

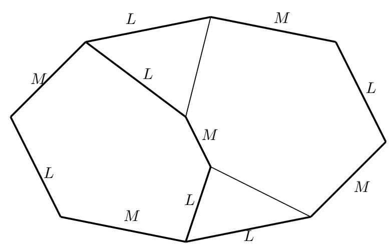
$$(l', m') \sim g'((l'))f((\mu))g((l))(l, m),$$

hence $\dim \langle L, M\rangle \leq 1$.

•Claim 2. A pair (l, m) **cannot be equivalent to a strict multiple of itself (a cycle).**

This is a very intersting proof by induction on the length of the cycle. After prove the case of 4 and 6 directly, we can let $n \geq 8$ and using Weil reciprocity.

This method break a n-move cycle into two cycles of length $n-2$, then one can use the induction. This proof is not so hard and much intersting, but I omit this and the detailed proof see [8] page 368. The main idea is the following diagram:



which tell us the cycle of 8 moves broken up in two cycles of 6. $\square$

Definition 5.45 (Deligne pairing for the families)**.** *A family* $\pi : X \to S$ *of nodal curves and* L *and* M *are line bundles on* X*. For any* $s \in S$ *we have a rank* 1 *free* $\mathscr{O}_{S,s}$*-module* $\langle L, M\rangle_s$ *by Proposition 5.44. For any open* $U \subset S$*, we define a sheaf*

$$U \mapsto \left\{ \{u_s \in \langle L, M\rangle_s \,|s \in U\} \left| \begin{array}{l} \text{for every } s \in U, \text{ there are a neighborhood} \\ U' \text{and meromorphic sections } l, m \text{ of } L, M \\ \text{over } \pi^{-1}(U') \text{ such that } u_t = \langle l, m\rangle \text{ for} \\ \text{every } t \in U'. \end{array} \right. \right\}.$$

denoted by $\langle L, M\rangle_\pi$*. This is a line bundle on* S*, called the Deligne pairing of* L *and* M*.*

Proposition 5.46. *Consider a family* $\pi : X \to S$ *of nodal curves and* L, L_1, L_2, L_3, M*,* M_1, M_2 *are line bundles on* X*.*

(i) We have canonical isomorphisms

$$\langle L_1, M\rangle_\pi \otimes \langle L_2, M\rangle_\pi \cong \langle L_1 \otimes L_2, M\rangle_\pi$$
$$\langle L, M_1\rangle_\pi \otimes \langle L, M_2\rangle_\pi \cong \langle L, M_1 \otimes M_2\rangle_\pi\,;$$

(ii) We have canonical isomorphisms $\langle L, \mathscr{O}_X\rangle_\pi \cong \mathscr{O}_S$ and $\langle \mathscr{O}_X, M\rangle_\pi \cong \mathscr{O}_S$;

(iii) Of course, we have the canonical isomorphism $\tau : \langle L, M\rangle_\pi \cong \langle M, L\rangle_\pi$ given by $\langle l, m\rangle \mapsto \langle m, l\rangle$. In particular when $L = M$, we have $\tau(-) = (-1)^{\deg L} \cdot (-)$.

Proof. See [8] XIII (5.4),(5.5) and Propisition 5.7. □

Theorem 5.47. *Consider a family $\pi : X \to S$ of nodal curves and L, M are line bundles on X. Then we have a canonical isomorphism*

$$\langle L, M\rangle_\pi \cong d_\pi(L \otimes M) \otimes d_\pi(L)^{-1} \otimes d_\pi(M)^{-1} \otimes d_\pi(\mathscr{O}_X)$$

compatible with base change.

Proof. See [8] XIII Theorem 5.8. □

Corollary 5.48. *(i) Let D be any relative divisor not passing through nodes of fibers of $\pi : X \to S$. The sheaves $\pi_*(\mathscr{O}_D)$ and $\pi_* M|_D$ are both locally free of rank equal to the degree of D over S. We may then define a line bundle on S as by setting*

$$\mathrm{Norm}_{D/S}(M|_D) := \mathscr{H}om(\det(\pi_*\mathscr{O}_D), \det(\pi_* M|_D)),$$

then we have

$$\langle \mathscr{O}_X(D), M\rangle_\pi \cong \mathrm{Norm}_{D/S}(M|_D).$$

(ii) In particular, if we have a section σ with $D = \sigma(S)$, then for any $M \in \mathrm{Pic}(X)$, we have

$$\langle \mathscr{O}_X(D), M\rangle_\pi \cong \sigma^* M.$$

Taking $M = \omega_\pi(D)$, we get

$$\langle \mathscr{O}_X(D), \omega_\pi\rangle_\pi \cong \langle \mathscr{O}_X(D), \mathscr{O}_X(D)\rangle_\pi^{-1}.$$

(iii) We have

$$c_1(\langle L, M\rangle_\pi) = \pi_*(c_1(L) \cdot c_1(M)).$$

Proof. (i) This is easy if we define a norm map $\mathrm{Norm}_{D/S} : \pi_*(M|_D) \to \mathrm{Norm}_{D/S}(M|_D)$ as $h \mapsto \det(\times h : \pi_*\mathscr{O}_D \to \pi_*(M|_D))$, then we get

$$\langle \mathscr{O}_X(D), M\rangle_\pi \cong \mathrm{Norm}_{D/S}(M|_D), \quad \langle 1, m\rangle_\pi \mapsto \mathrm{Norm}_{D/S}(m|_D).$$

(ii) Special case of (i).

(iii) This is a hard but difficult result, we refer [8] page 376, XIII.(5.20). □

Corollary 5.49 (Some kind of Riemann-Roch)**.** *Let $\pi : X \to S$ be a family of nodal curves, and let L be a line bundle on X. There is a canonical isomorphism of line bundles, compatible with base change:*

$$d_\pi(L)^2 \cong \left\langle L, L \otimes \omega_\pi^{-1}\right\rangle_\pi \otimes d_\pi(\mathscr{O}_X)^2.$$

Proof. As $\langle L, L \otimes \omega_\pi^{-1}\rangle_\pi \cong \langle L, L^{-1} \otimes \omega_\pi\rangle_\pi^{-1}$ by Proposition 5.46 (i)(ii), we then use Theorem 5.47 to $\langle L, L^{-1} \otimes \omega_\pi\rangle_\pi$ and we win. □

Example 5.50. *Consider a family of curves $\pi : X \to S$ plus sections σ_i, corresponding to divisors $D_i = \sigma_i(S)$. We denote $\widehat{\omega}_\pi := \omega_\pi(\sum_i D_i)$ and we get $\langle \widehat{\omega}_\pi, \widehat{\omega}_\pi \rangle_\pi \in \underline{\mathrm{Pic}}(S)$. As the Deligne pairing is well behaved under base change, this defines $\langle \widehat{\omega}, \widehat{\omega} \rangle$ on $\overline{\mathscr{M}}_{g,n}$ and we denote*

$$\kappa_1 = [\langle \widehat{\omega}, \widehat{\omega} \rangle] \in \mathrm{Pic}(\overline{\mathscr{M}}_{g,n}).$$

(For κ_a, the codimension a, can also be constructed)

Moreover, by Corollary 5.48, we get $[\langle \widehat{\omega}_\pi, \mathscr{O}_X(D_i) \rangle] = \psi_i$. More generally, we get

$$\left[\left\langle \widehat{\omega}_\pi^h \left(\sum_i a_i D_i \right), \widehat{\omega}_\pi^l \left(\sum_i b_i D_i \right) \right\rangle_\pi \right] = hl\kappa_1 - \sum_i a_i b_i \psi_i.$$

After this, if we let $\widetilde{\kappa}_1 = [\langle \omega, \omega \rangle] \in \mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$, we have

$$\widetilde{\kappa}_1 = \kappa_1 - \psi.$$

Finally, like Remark 5.39 we have $\xi_\Gamma^* \kappa_1 = \sum_{v \in V(\Gamma)} \eta_v^* \kappa_1$. The proof we refer [8] page 378.

5.2.5 The Picard group of moduli space of curves I

Theorem 5.51. *Consider $H_{\nu,g,n} \subset \underline{\mathrm{Hilb}}_{\mathbb{P}^{N-1}}^{P_\nu}$ be the Hilbert scheme of ν-log-canonically embedded n-pointed stable curves of genus g where $N = (2\nu - 1)(g-1) + \nu n$ and $P_\nu(t) = (2\nu t - 1)(g-1) + \nu n t$ for $\nu \geq 3$. Let $H'_{\nu,g,n} \subset H_{\nu,g,n}$ be the smooth locus. Hence we have $\overline{\mathscr{M}}_{g,n} \cong [H_{\nu,g,n}/\mathrm{PGL}(N)]$ and $\mathscr{M}_{g,n} \cong [H'_{\nu,g,n}/\mathrm{PGL}(N)]$. Then we have group isomorphisms:*

$$\mathrm{Pic}(\overline{\mathscr{M}}_{g,n}) \cong \mathrm{Pic}(H_{\nu,g,n}, \mathrm{PGL}(N)) \cong \mathrm{Pic}(H_{\nu,g,n})^{\mathrm{PGL}(N)},$$
$$\mathrm{Pic}(\mathscr{M}_{g,n}) \cong \mathrm{Pic}(H'_{\nu,g,n}, \mathrm{PGL}(N)) \cong \mathrm{Pic}(H'_{\nu,g,n})^{\mathrm{PGL}(N)}.$$

Proof. The first isomorphisms of these two statements are trivial. The second isomorphism need some GIT. We refer [73] for surjectivity and [8] Proposition XIII.6.1 for injectivity. □

Proposition 5.52. *For $\pi : \overline{\mathscr{M}}_{g,n} \to \overline{M}_{g,n}$ and $\vartheta : \mathscr{M}_{g,n} \to M_{g,n}$ we have exact sequences:*

$$0 \to \mathrm{Pic}(\overline{M}_{g,n}) \xrightarrow{\pi^*} \mathrm{Pic}(\overline{\mathscr{M}}_{g,n}) \to Q \to 0,$$
$$0 \to \mathrm{Pic}(M_{g,n}) \xrightarrow{\vartheta^*} \mathrm{Pic}(\mathscr{M}_{g,n}) \to R \to 0$$

where Q, R are torsion groups. More precisely, there is a positive integer k such that

$$k \cdot \mathrm{Pic}(\overline{\mathscr{M}}_{g,n}) \subset \mathrm{Pic}(\overline{M}_{g,n}) \text{ and } k \cdot \mathrm{Pic}(\mathscr{M}_{g,n}) \subset \mathrm{Pic}(M_{g,n}).$$

In particular, one has

$$\mathrm{Pic}(\overline{\mathscr{M}}_{g,n}) \otimes \mathbb{Q} \cong \mathrm{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}, \quad \mathrm{Pic}(\mathscr{M}_{g,n}) \otimes \mathbb{Q} \cong \mathrm{Pic}(M_{g,n}) \otimes \mathbb{Q}.$$

Proof. As the proof is the same at both cases, we just consider the case of $\mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$ and $\mathrm{Pic}(\overline{M}_{g,n})$. As $\overline{M}_{g,n}$ covered by $U_i = B_i/G_i$ where $X_i \to B_i$ are (standard algebraic) Kuranishi families with the automorphism groups of central fiber G_i. Let $L \in \mathrm{Pic}(\overline{M}_{g,n})$ pullback to $\overline{\mathscr{M}}_{g,n}$ is trivial hence has a nowhere vanishing global section. Hence gives a nowhere vanishing G_i-invariant section of the pullback of L to B_i by étale descent. Hence a nowhere vanishing section of L pullback to $\overline{\mathscr{M}}_{g,n}$, hence $\mathrm{Pic}(\overline{M}_{g,n}) \to \mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$ is injective.

Next we need to find a integer k such that for any $\mathscr{L} \in \mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$ we have $\mathscr{L}^k$ descents to a line bundle M on $\overline{M}_{g,n}$. Let $X = \coprod X_i, B = \coprod B_i$ where $X_i \to B_i$ are (standard algebraic) Kuranishi families with the automorphism groups of central fiber G_i, then $B \to \overline{\mathscr{M}}_{g,n}$ and $\coprod B_i/G_i \to \overline{M}_{g,n}$ are étale covers. Hence by étale descent we may let $\mathscr{L}$ as line bundle L over B with descent data to $B \to \overline{\mathscr{M}}_{g,n}$. Now take $b \in B$ and consider L_b, then $\mathrm{Aut}(X_b)$ act on L_b linearly. As L_b is just a one-dimensional vector space, hence this action is just multiplication by k_b-th roots of unity where $k_b := |\mathrm{Aut}(X_b)|$. Hence now we let $k = \prod_i |G_i|$ and then for any b, we have $k_b | k$ by the property of the standard Kuranishi family. Hence these groups act trivially over L^k and hence $\mathscr{L}^k$ descend to $\overline{M}_{g,n}$ by basic étale descent. □

5.2.6 The Picard group of moduli space of curves II

In this section we will mainly refer Enrico Arbarello and Maurizio Cornalba's classical paper [5] in the base field $\mathbb{C}$. But in the positive characteristic algebraically closed field k, we have the similar result, see [72]. Actually he prove more, that is, $\mathrm{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}_\ell \cong H^2_{\text{ét}}(\overline{M}_{g,n}, \mathbb{Q}_\ell)$ when ℓ is prime and invertible in k. But we do not care about these here.

• **Some preliminaries.**

Here we follows [81].

Definition 5.53 (Pencil)**.** *A pencil of hypersurfaces on a variety X is a projective line $\mathbb{P}^1 \subset |L|$, where L is a line bundle on X.*

Hence a pencil of hypersurfaces on a variety X gives us $\sigma_t \in H^0(X, L)$ for all $t \in \mathbb{P}^1$, up to a coefficient in $\mathbb{C}^\times$. These (well-)defines the hypersurfaces $X_t \subset X$ correspond to σ_t. So we denote $(X_t)_{t \in \mathbb{P}^1}$ as this pencil. Actually we can denote $\sigma_t = \sigma_0 + t\sigma_\infty$ for $t \in \mathbb{A}^1 \subset \mathbb{P}^1$. Hence the base locus of the pencil is $B = \bigcap_{t \in \mathbb{P}^1} X_t \subset X$ defined by σ_0, σ_∞. Let $X' = \mathrm{Bl}_B(X) \cong \{(x,t) \in X \times \mathbb{P}^1 : x \in X_t\}$, hence if we let $f : X' \to \mathbb{P}^1$, then $f^{-1}(t) \cong X_t$.

Definition 5.54 (Lefschetz pencil)**.** *A Lefschetz pencil $(X_t)_{t \in \mathbb{P}^1}$ is a pencil of hypersurfaces satisfies:*

(i) B is smooth with $\mathrm{codim}_X(B) = 2$;

(ii) X_t has at most one ordinary double point as singularity.

Remark 5.55 (Ordinary double point)**.** *Let X be an algebraic scheme over k with a closed $x \in X$.*

(i) If $k = \bar{k}$, then x is called an ordinary double point if

$$\widehat{\mathscr{O}}_{X,x} \cong k[[x_1, ..., x_n]]/(f)$$

where $f \in \mathfrak{m}^2$ such that $f = Q + R$ where Q be a nondegenerate quadratic form and $R \in \mathfrak{m}^3$ where $\mathfrak{m}$ be the maximal ideal of $k[[x_1, ..., x_n]]$;

(ii) For general k, $x \in X$ is called an ordinary double point if all points in $X \otimes_k \bar{k}$ lying over x are ordinary double points.

Next we will introduce something about K3 surfaces. We refer [12] chapter VIII or more general book [63] for more detailed arguments.

Definition 5.56. *A K3 surface over k is a proper nonsingular variety X of dimension two such that*

$$\bigwedge^2 \Omega_{X/k} \cong \mathscr{O}_X, H^1(X, \mathscr{O}_X) = 0.$$

Proposition 5.57 (see [12] Proposition VIII.13 or [63] Lemma II.2.1)**.** *Let X be a K3 surface and $C \subset S$ be a smooth curve of genus g, then $C^2 = 2g - 2$ and $h^0(X, \mathscr{O}_X(C)) = g + 1$.*

Proof. The statement $C^2 = 2g-2$ follows from adjunction formula. Again by adjunction formula we get

$$\omega_C = \omega_X \otimes \mathscr{O}_X(C) \otimes \mathscr{O}_C = \mathscr{O}_X(C) \otimes \mathscr{O}_C = \mathscr{O}_X(C)|_C.$$

Hence $H^0(C, \mathscr{O}_X(C)|_C) = H^0(C, \omega_C)$. As $H^1(X, \mathscr{O}_X) = 0$ and the exact sequence $0 \to \mathscr{O}_X \to \mathscr{O}_X(C) \to \mathscr{O}_X(C)|_C \to 0$ we get $h^0(X, \mathscr{O}_X(C)) = 1 + h^0(C, \omega_C) = g + 1$. By Riemann-Roch formula, we get $\chi(X, \mathscr{O}_X(C)) = g + 1$. As $h^2(X, \mathscr{O}_X(C)) = h^0(X, \mathscr{O}_X(-C)) = 0$, we get $h^0(X, \mathscr{O}_X(C)) \geq g + 1$. □

Theorem 5.58 (Existence of K3 surfaces)**.** *For any $g \geq 3$, there exists K3 surfaces S of degree $2g - 2$ embedded in $\mathbb{P}^g$.*

Proof. See [12] Theorem VIII.15. They construct K3 surfaces containing a very ample divisor D with $D^2 = 2g - 2$. □

• J. Harer's theorem and its corollaries.

Here we follows the paper [76] and the Appendix of the Enrico Arbarello and Maurizio Cornalba's paper [5]. We just summary the several results here and we refer the original papers [53] and [54] due to J. Harer by using the Teichmüller space (the construction one can see [48] and [8] chapter XV).

Theorem 5.59 (Harer's theorem)**.** *(i) The group $\mathrm{Pic}(M_g) \otimes \mathbb{Q}$ is freely generated by the λ;*

(ii) The group $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ is freely generated by the $\lambda, \Delta_{irr}, \Delta_i$;

(iii) The group $\mathrm{Pic}(M_{g,n}) \otimes \mathbb{Q}$ is freely generated by the λ, ψ_i;

(iv) The group $\mathrm{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$ is freely generated by the $\lambda, \psi_i, \Delta_{irr}, \Delta_i$.

Remark 5.60. *Note that we have found in Remark 5.15 that the Hodge class λ defined over $\overline{\mathscr{M}}_{g,n}$ can not descend to $\overline{M}_{g,n}$, so the Hodge class here we defined at the meaning of Proposition 5.52.*

Proposition 5.61 (See appendix in [5]). *The group* $\mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$ *and* $\mathrm{Pic}(\mathscr{M}_{g,n})$ *has no torsion.*

Corollary 5.62. *We have* $\mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$ *generated by rational coefficients classes* $\lambda, \psi_i, \delta_{irr}, \delta_i$ *and* $\mathrm{Pic}(\mathscr{M}_{g,n})$ *generated by rational coefficients classes* λ, ψ_i.

Proof. Follows from the Harer's theorem 5.59 and Proposition 5.61. □

• **The groups** $\mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$**and** $\mathrm{Pic}(\mathscr{M}_{g,n})$ **for** $g \geq 3$**.**

First we deal with the case of $n = 0$, as follows.

Theorem 5.63. *For* $g \geq 3$ *we have* $\mathrm{Pic}(\overline{\mathscr{M}}_g)$ *is freely generated by* $\lambda, \delta_{irr}, \delta_i$*; the group* $\mathrm{Pic}(\mathscr{M}_g)$ *is freely generated by* λ.

The most important thing is that we need to construct some special families of curves.

♣ **Construct four kinds of familes.**

▶ **Families of type I.** Λ_n **for** $2 \leq n \leq g$**.**

Pick a smooth K3 surface Y' of degree $2n-2$ in $\mathbb{P}^n$ by Theorem 5.58 and consider a Lefschetz pencil of hyperplane sections. As Y' is smooth, one might choose generic pencil of hyperplane sections by Bertini's theorem (see [81] corollary 2.10).

Let Bs be the base locus of the pencil and let $Y = \mathrm{Bl}_{\mathrm{Bs}}(Y')$. Let $\phi : Y \to B := \mathbb{P}^1$. The curves of the pencil appear in Y as fibers of ϕ and the exceptional curves appear as sections E_i of ϕ.

Fix a smooth curve Γ of genus $g - n$ and a point γ on it. Construct a new surface $X = (Y \sqcup \Gamma \times \mathbb{P}^1)/(E_1 \sim \{\gamma\} \times \mathbb{P}^1)$. Hence we get a family $\Lambda_n = (f : X \to B = \mathbb{P}^1)$. As we consider the Lefschetz pencil, we find that the fibers of $\phi : Y \to B$, hence the fibers of $f : X \to B$, are all nodal curves.

• **Describe** λ_{Λ_n}**.**

First we claim that

$$f_*\omega_f \cong \phi_*\omega_\phi \oplus (\mathscr{O}_B)^{g-n}.$$

Second we claim that $\mathrm{rank}(\phi_*\omega_\phi) = n$. As Y' be a K3 surface and the fiber of ϕ are the smooth curves $C \subset Y'$ correspond to the sections of Lefschetz pencil, hence $g(C) = p_a(C) = \frac{C^2}{2} + 1 = n$ by adjunction formula as the existence of K3 surface by Proposition 5.57 and Theorem 5.58 (hence flat by checking Hilbert polynomial. actually by our choice of Lefschetz pencil, all fibers of ϕ are smooth, hence so is ϕ). Hence $\mathrm{rank}(\phi_*\omega_\phi) = n$. Hence we get

$$\lambda_{\Lambda_n} = \bigwedge^g f_*\omega_f = \bigwedge^n \phi_*\omega_\phi.$$

• **Compute** $\deg \lambda_{\Lambda_n}$**.**

First, by the Riemann-Roch of vector bundles over curves (see St 0BS6) we get

$$\chi(B, \phi_*\omega_\phi) = \deg \lambda_{\Lambda_n} + n(1 - g(B)) = \deg \lambda_{\Lambda_n} + n.$$

Second, since $R^1\phi_*\omega_\phi = \mathscr{O}_B$ we get

$$\chi(\phi_!\omega_\phi) = \chi(\phi_*\omega_\phi) - \chi(\mathscr{O}_B).$$

Finally, by Leray spectral sequence $E_2^{p,q} = H^p(B, R^q\phi_*\omega_\phi) \Rightarrow H^{p+q}(Y, \omega_\phi)$ we get the $E_2 = E_\infty$ page:

$$\begin{tikzcd}
H^0(B, R^1\phi_*\omega_\phi) \arrow[drr] & 0 & 0 \\
H^0(B, \phi_*\omega_\phi) & H^1(B, \phi_*\omega_\phi) & 0
\end{tikzcd}$$

hence by the definition of $\phi_!$ we get $\chi(\phi_!\omega_\phi) = \chi(\omega_\phi)$. By Riemann-Roch of surfaces, we get

$$\chi(\omega_\phi) = \chi(\mathscr{O}_Y) + \frac{K_\phi^2 - K_\phi \cdot K_Y}{2}.$$

As ϕ is smooth, we get $\omega_Y \cong \phi^*\omega_B \otimes \omega_\phi$, hence $K_\phi \equiv_{\text{lin}} K_Y - \phi^*K_B$. Hence

$$\chi(\omega_\phi) = \chi(\mathscr{O}_Y) - \frac{\phi^*K_B^2 - K_Y \cdot \phi^*K_B}{2}.$$

By the construction of $\phi : Y \to B$, we get $\phi^*\omega_B \cong \mathscr{O}((2g(B) - 2)F)$ for a fiber F by the construction. Use the adjunction formula to F, we get $2g(F) - 2 = F^2 - F \cdot K_Y = -F \cdot K_Y$. Hence we get

$$\begin{aligned}\chi(\omega_\phi) &= \chi(\mathscr{O}_Y) - \frac{\phi^*K_B^2 - K_Y \cdot \phi^*K_B}{2} = \chi(\mathscr{O}_Y) - (g(B) - 1)(2g(F) - 2) \\ &= \chi(\mathscr{O}_Y) + 2n - 2 = 2n\end{aligned}$$

since Y is the blowing up of a K3 surface (hence birational to that K3 surface) which deduce $\chi(\mathscr{O}_Y) = \chi(\mathscr{O}_{Y'}) = 2$ as in this case it is $\mathscr{O}$-connected with vanishing higher direct image (this is a conclusion due to Hironaka in characteristic zero, more general, see [21]). Combining these, we get

$$\begin{aligned}\deg \lambda_{\Lambda_n} &= \chi(B, \phi_*\omega_\phi) - n = \chi(\phi_!\omega_\phi) + \chi(\mathscr{O}_B) - n \\ &\chi(\omega_\phi) + \chi(\mathscr{O}_B) - n = n + 1.\end{aligned}$$

We win!

▶ **Families of type II. F_n for $g \geq 3, 2 \leq 2n \leq g - 1$.**

Let smooth curves C_1, C_2, Γ of genus $n, g-n, 1$ and fix points $x_1 \in C_1, x_2 \in C_2, \gamma \in \Gamma$. Let $Y_1 = C_1 \times \Gamma, Y_2 = \mathrm{Bl}_{\{(\gamma,\gamma)\}}(\Gamma \times \Gamma)$ with exceptional divisor E and $Y_3 = C_2 \times \Gamma$. Let $A = \{x_1\} \times \Gamma, B = \{x_2\} \times \Gamma$ and Δ be the strict transform of the diagonal in Y_2 and S be the strict transform of $\{\gamma\} \times \Gamma$ in Y_2. Let

$$X = \frac{Y_1 \sqcup Y_2 \sqcup Y_3}{S \sim A, \Delta \sim B}$$

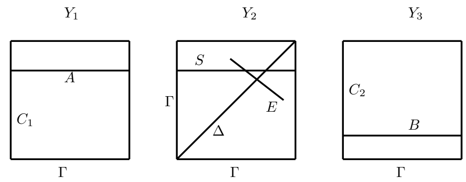
with $f : X \to \Gamma$ be the family, called F_n. The graphs of F_n and its fibers at $\gamma' \in \Gamma$ are as follows:

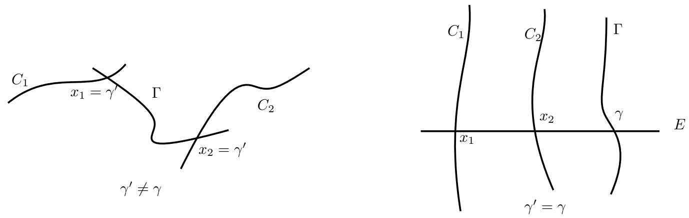
• **Compute** $\deg \lambda_{F_n}$**.**

First we have $f_*\omega_f \cong (H^0(\omega_{C_1}) \oplus H^0(\omega_{C_2}) \oplus H^0(\omega_\Gamma)) \otimes \mathscr{O}_\Gamma$. Hence $\deg \lambda_{F_n} = 0$.

• **Compute** $\deg(\delta_i)_{F_n}$**.**

By the arguments in [56] page 81, we have the following general principle:





Lemma 5.64. *Let $\pi : \mathcal{C} \to B$ be a family of stable curves over a smooth curve B which is obtained from a family $\phi : \mathscr{D} \to B$ of (not necessarily connected) nodal curves by identifying sections S_i, T_i pairwise. For each j, let Σ_j denote the image of S_j in $\mathcal{C}$. Suppose the locus of singular points of type i in the fibers of π is*

$$[p_1, ..., p_m] \cup \bigcup_j \Sigma_j$$

where the p_i are distinct points not belonging to $\bigcup_j \Sigma_j$. Then

$$(\delta_i)_\pi = \bigotimes_j (\phi_*(N_{S_j}) \otimes \phi_*(N_{T_j})) \left(\sum_l n_l \pi(p_l) \right)$$

where N_S be the normal bundle and $\mathcal{C}$ is of form $xy = t^{n_l}$ near p_l.

Now we will use this to compute $\deg(\delta_i)_{F_n}$. Actually by adjunction formula we get

$$A^2 = 2g(A) - 2 - A \cdot K_{Y_1} = -A \cdot (p^* K_{C_1} + q^* K_\Gamma) = 0,$$
$$B^2 = 2g(B) - 2 - B \cdot K_{Y_2} = -B \cdot (p^* K_{C_2} + q^* K_\Gamma) = 0.$$

By Proposition A.15, we have

$$\Delta^2 = 2g(\Delta) - 2 - \Delta \cdot K_{Y_2} = -\Delta \cdot (p^* K_\Gamma + q^* K_\Gamma + E) = -1,$$
$$S^2 = 2g(S) - 2 - S \cdot K_{Y_2} = -S \cdot (p^* K_\Gamma + q^* K_\Gamma + E) = -1.$$

Hence we have

$$\deg N_A = \deg N_B = 0, \deg N_S = \deg N_\Delta = -1.$$

Hence by the Lemma we get

$$\deg(\delta_{irr})_{F_n} = 0, \deg(\delta_1)_{F_n} = \begin{cases} 1, & n > 1; \\ 0, & g - n - 1 > n = 1; \\ -1, & g - n - 1 = n = 1 (g = 3). \end{cases}$$

$$\deg(\delta_n)_{F_n} = \begin{cases} -1, & g - n - 1 > n > 1; \\ 0, & g - n - 1 > n = 1; \\ -2, & g - n - 1 = n > 1; \\ -1, & g - n - 1 = n = 1 (g = 3), \end{cases} \quad \deg(\delta_{n+1})_{F_n} = -1 (\text{if } g - n - 1 > n).$$

And other cases are all 0.

▶ **Families of type III. The family F.**

Consider a general pencil of conics in $\mathbb{P}^2$ with four base points. Blowing up these points in the plane we get $\psi : X \to \mathbb{P}^2$ with exceptional lines $E_1, ..., E_4$. Moreover we consider the resulting conic bundle $\phi : X \to \mathbb{P}^1$. Hence we have

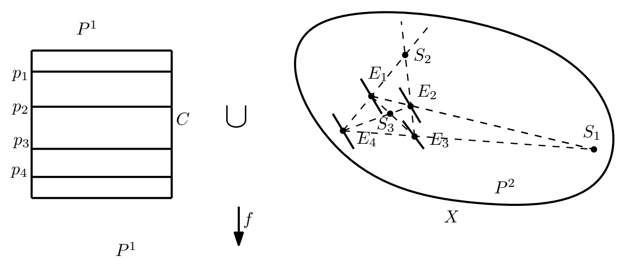
$$\omega_\phi = \omega_X \otimes \phi^* \mathscr{O}_{\mathbb{P}^1}(-2)^{-1} = \psi^*\omega_{\mathbb{P}^2} \otimes \mathscr{O}_X\left(\sum E_i\right) \otimes \phi^*\mathscr{O}_{\mathbb{P}^1}(2)$$
$$= \psi^*\omega_{\mathbb{P}^2} \otimes \mathscr{O}_X\left(\sum E_i\right) \otimes \psi^*\mathscr{O}_{\mathbb{P}^2}(4) \otimes \mathscr{O}_X\left(-2\sum E_i\right) = \psi^*\mathscr{O}_{\mathbb{P}^2}(1) \otimes \mathscr{O}_X\left(-\sum E_i\right).$$

Now we let C be a fixed smooth curve of genus $g-3$ with four fixed points $p_1, ..., p_4$ on it, let

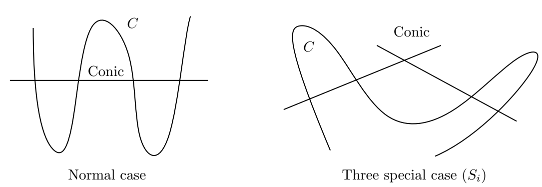
$$Y = \frac{X \sqcup (C \times \mathbb{P}^1)}{E_i \sim \{p_i\} \times \mathbb{P}^1 (i = 1, ..., 4)}$$

and consider $f : Y \to \mathbb{P}^1$ a family of curves of genus g. We call this family F.

We consider the fibers of F. First we draw the picture of the family F, then we find that there are exactly three special points such that the conics are not smooth, hence we have two different types of fibers. The following picture is the family $f : Y \to \mathbb{P}^1$:



Hence we have two kinds of fibers as follows:



• **Compute** $\deg \lambda_F$**.**

In fact $f_*\omega_f \to H^0(\omega_C(\sum p_i)) \otimes \mathscr{O}_{\mathbb{P}^1}$ is injective, hence an isomorphism. Hence $\deg \lambda_F = 0$.

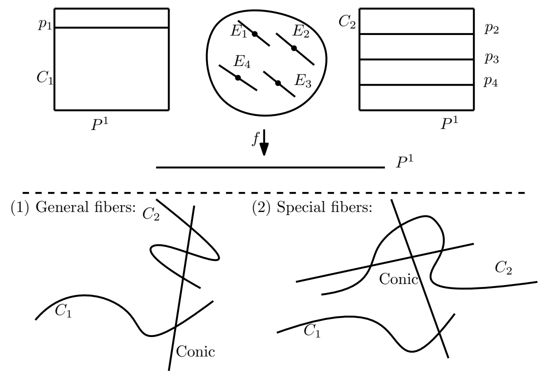
• **Compute** $\deg(\delta_i)_F$**.**

As that three special points, hence we get $\deg(\delta_{irr})_F = 3 + \sum \deg N_{E_i} = 3 + \sum E_i^2 = -1$. Moreover, it's easy to see that $\deg(\delta_i)_F = 0$ for $i > 0$.

▶ **Families of type IV. The family F'.**

Let C_1 be an smooth elliptic curve and C_2 be a smooth curves of genus $g-3$. Let $p_1 \in C_1$ and $p_2, p_3, p_4 \in C_2$. We consider the similar X in the construction of F, we let

$$Y = \frac{X \sqcup (C_1 \times \mathbb{P}^1) \sqcup (C_2 \times \mathbb{P}^1)}{E_i \sim \{p_i\} \times \mathbb{P}^1, i = 1, ..., 4},$$



to get a family of stable curves $f : Y \to \mathbb{P}^1$. We call this family F', as follows: There are two kinds of fibers as before.

• **Compute** $\deg \lambda_{F'}$**.**

Similar as F, we get $f_*\omega_f$ is trivial. Hence $\deg \lambda_{F'} = 0$.

• **Compute** $\deg(\delta_i)_{F'}$**.**

As that three special points, hence we get $\deg(\delta_{irr})_{F'} = 3+\sum_{i\geq 2} E_i^2 = 0$. Moreover, we have $\deg(\delta_1)_{F'} = \deg N_{E_1} = E_1^2 = -1$. Finally we get $\deg(\delta_i)_{F'} = 0$ for all $i > 1$.

♣ **Back to the proof of the theorem.**

Let $k = \lfloor g/2 \rfloor$ and let $G_i = (\mathcal{C}_i \to S_i)$ are $k+2$ families of stable curves. We denote the matrix

$$\eta(G_1, ..., G_{k+2}) = \begin{pmatrix} \deg \lambda_{G_1} & \deg(\delta_{irr})_{G_1} & \cdots & \deg(\delta_k)_{G_1} \\ \deg \lambda_{G_2} & \deg(\delta_{irr})_{G_2} & \cdots & \deg(\delta_k)_{G_2} \\ \vdots & \vdots & & \vdots \\ \deg \lambda_{G_{k+2}} & \deg(\delta_{irr})_{G_{k+2}} & \cdots & \deg(\delta_k)_{G_{k+2}} \end{pmatrix}.$$

Proof of the theorem. For our familes of curves we find that λ, δ_i are linearly independent. By Harer's result (Corollary 5.62) we have that $\mathrm{Pic}(\overline{\mathscr{M}}_g)$ is gemnerated by the rational coefficients of the linear combinations of λ, δ_i. So we let $\xi \in \mathrm{Pic}(\overline{\mathscr{M}}_g)$ with $\xi = a\lambda + b_0\delta_{irr} + \sum b_i\delta_i$ where $a, b_i \in \mathbb{Q}$. Now we first let that we have constructed two different sets of $k+2$ families of stable curves G_i such that two $\det \eta$s are relative prime. Let $d_i = \deg \xi_{G_i}$, then

$$\begin{pmatrix} d_1 \\ \vdots \\ d_{k+2} \end{pmatrix} = \eta \begin{pmatrix} a \\ b_0 \\ \vdots \\ d_k \end{pmatrix}.$$

As $d_i \in \mathbb{Z}$, then so are $a \det \eta, b_0 \det \eta, ..., b_k \det \eta$. As two $\det \eta$s are relative prime, then $a, b_i \in \mathbb{Z}$ and we win! Now we just need to construct these two different sets of $k + 2$ families.

• **When** g **is odd and** $g = 2m + 1$**.**

We consider $\eta_n := \eta(\Lambda_n, F, F_1 ..., F_m)$ where n is an integer between 2 and $k = \lfloor g/2 \rfloor$.

When have

$$\det \eta_n = \det \begin{pmatrix} n+1 & \cdots & & & & & & \\ 0 & -1 & \cdots & & & & & \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdots & \\ 0 & 0 & 1 & -1 & -1 & 0 & \cdots & \\ & & 1 & 0 & -1 & -1 & \cdots & \\ \vdots & & & & & & & \vdots \\ & & 1 & 0 & \cdots & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & \cdots & & 0 & -2 \end{pmatrix} = (-1)^{m+1}(n+1).$$

Taking $n = 2, 3$ and well done.

• **When g is even and $g = 2m + 2$.**

We consider $\eta_n := \eta(\Lambda_n, F, F', F_1 ..., F_m)$ where n is an integer between 2 and $k = \lfloor g/2 \rfloor$. When have

$$\det \eta_n = \det \begin{pmatrix} n+1 & \cdots & & & & & & \\ 0 & -1 & \cdots & & & & & \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & \\ 0 & 0 & 0 & -1 & 0 & 0 & \cdots & \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdots & \\ & & 1 & 0 & -1 & -1 & 0 & \cdots \\ \vdots & & & & & & & \vdots \\ & & 1 & 0 & \cdots & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & \cdots & & 0 & -2 \end{pmatrix} = (-1)^m(n+1).$$

Taking $n = 2, 3$ and well done. $\square$

Now we come to the general case. Here we just give a sketch and the detailed proof we refer the section 3 in the original paper [5].

Theorem 5.65. *For every $g \geq 3$, the group $\mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$ is freely generated by $\lambda, \psi_i, \delta_j$ and $\mathrm{Pic}(\mathscr{M}_{g,n})$ is freely generated by λ, ψ_i.*

Remark 5.66. *As the marked points here are in order (instead of the case $\overline{\mathscr{M}}_{g,P}$), we now need to make the boundary divisor more explicitly. The class δ_{irr} is the locus that the partial normalization is connected. The class $\delta_{\alpha;i_1,...,i_a}$ are the locus that the partial normalization have two connected components, one of them is of genus α with marked points $p_{i_1}, ..., p_{i_a}$ and another one is of genus $g - \alpha$ with remaining marked points. Of course we will let $0 \leq \alpha \leq \lfloor g/2 \rfloor$, $0 \leq a \leq n$, $i_1 < \cdots < i_a$ and $a \geq 2$ when $\alpha = 0$.*

♣ **Step 1. Define the forgatting map $\vartheta : \mathrm{Pic}(\overline{\mathscr{M}}_{g,n}) \to \mathrm{Pic}(\overline{\mathscr{M}}_{g,n+1})$.**

Actually this of course induced by the forgetful map (is some kind of blow down). Moreover along this map, we have the following fundamental relations.

$$\begin{cases} \vartheta(\lambda) = \lambda, & \\ \vartheta(\psi_i) = \psi_i - \delta_{0;i,n+1}, & i = 1, ..., n, \\ \vartheta(\delta_{irr}) = \delta_{irr}, & \\ \vartheta(\delta_\alpha) = \delta_\alpha, & \text{if } \alpha = g/2, n = 0, \\ \vartheta(\delta_{\alpha;i_1,\dots,i_a}) = \delta_{\alpha;i_1,\dots,i_a} + \delta_{\alpha;i_1,\dots,i_a,n+1}, & \text{otherwise.} \end{cases}$$

♣ **Step 2. Preparation I.**

Pick a smooth family $F = (f : \mathcal{C} \to S, \sigma_i) \in \mathscr{M}_{g,n}(S)$ and consider the pullback of σ_i in $\mathcal{C} \times_S \mathcal{C} \to \mathcal{C}$ as σ_i'. Let

$$X = \mathrm{Bl}_{\bigcup_i (\Delta \cap \sigma_i'(\mathcal{C}))}(\mathcal{C} \times_S \mathcal{C})$$

and consider the diagram

$$\begin{tikzcd} X = \mathrm{Bl}_{\bigcup_i (\Delta \cap \sigma_i'(\mathcal{C}))}(\mathcal{C} \times_S \mathcal{C}) \arrow[r] \arrow[dr, "\phi"] & \mathcal{C} \times_S \mathcal{C} \arrow[r] \arrow[d, shift right] & \mathcal{C} \arrow[d, "f"', shift right] \\ & \mathcal{C} \arrow[r, "f"'] \arrow[u, shift right, bend right, "\sigma_i'"'] \arrow[ul, bend left, "\tau_i, \widehat{\Delta}"] & S \arrow[u, shift right, bend right, "\sigma_i"'] \end{tikzcd}$$

where $\widehat{\Delta}, \tau_i$ are strict transform of Δ, σ_i'. We let $F' = (\phi : X \to \mathcal{C}, \tau_i, \widehat{\Delta})$.

Definition 5.67. *Let $L \in \mathrm{Pic}(\mathscr{M}_{g,n+1})$. We shall say that L is trivial on smooth curves if $L|_{F'}$ is trivial whenever S consists of a single point.*

Lemma 5.68. *Let L be a line bundle on $\overline{\mathscr{M}}_{g,n+1}$ If L is trivial on smooth curves there exists a line bundle $\mathscr{L}$ on $\overline{\mathscr{M}}_{g,n}$ such that $L \equiv \vartheta(\mathscr{L})\mathrm{mod}$ boundary classes. Conversely, if there is $\mathscr{L}$ on $\overline{\mathscr{M}}_{g,n}$ such that $L - \vartheta(\mathscr{L})$ is an integral linear combination of boundary classes other than the $\delta_{0;i,n+1}$, then L is trivial on smooth curves.*

Proof. See [5] Lemma 2. □

♣ **Step 3. Preparation II.**

Let X be a smooth K3 surface of degree $d = 2g-2$ in $\mathbb{P}^g$ such that $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot L$ where L be a hyperplane section, by [63] Example II.3.9. Pick a Lefschetz pencil of hyperplane sections on X. Blowing up the base locus to get Y' with exceptional curves $E_1, ..., E_d$ as sections of $Y' \to \mathbb{P}^1$. Hence $\mathrm{Pic}(Y')$ freely generated by a fiber and the E_i by Proposition A.16.

Notice that as one varies the Lefschetz pencil the monodromy action on the base points of the pencil, and hence on the E, is given by the full symmetric group.

Let $Y = Y' - \bigcup\{\text{Singular fibers}\}$ and $\mathbb{P}$ be the projection of Y over $\mathbb{P}^1$. Let $\psi : Y \to \mathbb{P}$. We write E_i instead of $E_i \cap Y$. Hence we get the E_i freely generate $\mathrm{Pic}(Y)$ as we have the exact sequence

$$\mathbb{Z}^k \to \mathrm{Pic}(Y') \to \mathrm{Pic}(Y) \to 0$$

where k be the numbers components of singular fibers.

♣ **Step 4. Proof for $n \leq 2g-2$ by induction on n.**

As $n = 0$ is proved we let so is n when $n \leq 2g-3$. We just need to show that $\mathrm{Pic}(\overline{\mathscr{M}}_{g,n+1})$ is generated, over $\mathbb{Z}$, by $\vartheta(\mathrm{Pic}(\overline{\mathscr{M}}_{g,n}))$, ψ_{n+1} and the boundary classes. Let $\mu \in \mathrm{Pic}(\overline{\mathscr{M}}_{g,n+1})$. As $n \leq 2g-2 = d$, we let

$$f : \mathscr{Y} := \mathrm{Bl}_{\bigcup_{i=1}^{n}(\Delta \cap E_i)}(Y \times_{\mathbb{P}} Y) \to Y, \widehat{\Delta}, \widehat{E}_1, ..., \widehat{E}_n$$

as the construction in **Preparation I**.

As μ_f is an integral linear combination of $E_1, ..., E_d$, ny monodromy, the coefficients of $E_{n+1}, ..., E_d$ are all equal, that is,

$$\mu_f = \sum_{i \leq n} a_i E_i + a_{n+1} \sum_{i > n} E_i.$$

On the other hand we can express $(\psi_i)_f, (\psi_{n+1})_f, (\delta_{0;j,n+1})_f$ as the combinations of E_i. So if we let

$$\mu = \sum \alpha_j \psi_j + \beta\lambda + \sum \gamma_j \delta_{0;j,n+1} + \cdots$$

where $\alpha_j, \beta, \gamma_j \in \mathbb{Q}$ by Harer's theorem, then we can get some relations (we will omit it here). In particular, we get $\alpha_{n+1}, \alpha_j + \gamma_j \in \mathbb{Z}$ for $j \leq n$.

Set

$$\mu' = \mu - \alpha_{n+1}\psi_{n+1} - \sum (\alpha_j + \gamma_j)\delta_{0;j,n+1},$$

then by these relations, we get $(\mu')_f = 0$, and similarly, on any fibers of f. On the other hand, since we have

$$\mu = \alpha_{n+1}\psi_{n+1} + \sum \alpha_j \vartheta(\psi_j) + \beta\vartheta(\lambda) + \sum (\alpha_j + \gamma_j)\delta_{0;j,n+1} + \cdots$$

by several relations in **Step 1**. Hence μ' a $\mathbb{Q}$-coefficients linear combination of classes in $\vartheta(\mathrm{Pic}(\overline{\mathscr{M}}_{g,n}))$ and boundary classes not of the form $\delta_{0;j,n+1}$. By Lemma 5.68 there exists $\xi \in \mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$ such that $\mu' \equiv \vartheta(\xi)(\text{mod boundary classes})$, hence

$$\mu \equiv \alpha_{n+1}\psi_{n+1} + \vartheta(\xi)(\text{mod boundary classes})$$

and we win!

♣ **Step 5. Proof for $n > 2g-2$ by induction on n.**

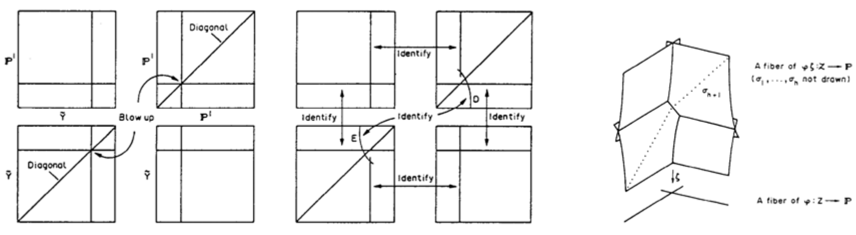
We assume that this is proved for some $n \geq 2g-2$ and we consider $n+1$. The main idea is similar as the previous case and we just give a construction of the family of curves we considered here and omit all details.

Consider the same $\psi : Y \to \mathbb{P}, E_1, ..., E_d$ and let $Q = \mathbb{P}^1 \times \mathbb{P} \to \mathbb{P}$ with sections $D_{2g-4}, ..., D_n$. Let

$$\phi : Z := \frac{Y \sqcup Q}{E_{2g-3} \sim D_{2g-4}}, E_1, ..., E_{2g-4} \to \mathbb{P}, D_{2g-3}, ..., D_n.$$

Consider $\zeta : \mathscr{X} \to Z, \sigma_1, ..., \sigma_{n+1}$ constructed via $Z \times_{\mathbb{P}} Z$ as follows

Like the previous case, we can have some relations and then, we will use the Lemma 5.68.



Step 1 ($\tilde{Y}$ is a fiber of $\psi: Y \to \mathbb{P}$) Step 2 (E and D are the exceptional divisors)

5.2.7 The tautological & canonical class

Here we will follow the section XIII.7 in [8].
• **Situation A.** Let $j : Y \to Z$ be a codimension r closed immersion of smooth schemes with $\mathscr{G}$ be a coherent sheaf on Y. By Grothendieck-Riemann-Roch theorem we get

$$\mathrm{ch}(j_*\mathscr{G}) = \mathrm{ch}(j_!\mathscr{G}) = j_*(\mathrm{ch}(\mathscr{G})\mathrm{td}(Y))\mathrm{td}(Z)^{-1}.$$

Hence we get $c_i(j_*\mathscr{G}) = 0$ when $i < r$ and $c_r(j_*\mathscr{G}) = (-1)^{r-1}(r-1)!\mathrm{rank}(\mathscr{G})[Y]$ by the codimension reasons.
• **Situation B.** For any $(f : X \to H, \tau_1, ..., \tau_n) \in \overline{\mathscr{M}}_{g,n}(H)$ where X, H are smooth and any coherent sheaf $\mathscr{F}$ on X, by Grothendieck-Riemann-Roch theorem we get

$$\mathrm{ch}(f_!\mathscr{F}) = f_*(\mathrm{ch}(\mathscr{F}) \cdot \mathrm{td}(\Omega^\vee_{X/H})).$$

Consider the degree 1 terms, we get

$$c_1(f_!\mathscr{F}) = f_*\left(\frac{c_1(\mathscr{F})^2}{2} - c_2(\mathscr{F}) - \frac{c_1(\mathscr{F})c_1(\Omega^1_f)}{2} + \frac{c_1(\Omega^1_f)^2 + c_2(\Omega^1_f)}{12}\right).$$

Next let Σ be the locus of nodes with ideal sheaf $\mathscr{I}$, then by Corollary 4.23 we have

$$0 \to \Omega^1_f \to \omega_f \to \omega_f \otimes \mathscr{O}_\Sigma \to 0$$

as X is smooth. Now consider $j : \Sigma \to X$ and $\mathscr{G} = \omega_f \otimes \mathscr{O}_\Sigma$ in **Situation A** and Whitney formula we get $c_1(\Omega^1_f) = c_1(\omega_f)$ and $c_2(\Omega^1_f) = [\Sigma]$.

Theorem 5.69 (Mumford)**.** *For $2g-2+n > 0$ we have $\kappa_1 = 12\lambda+\psi-\delta$ in* $\mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$.

Proof. By the previous situations, let $\mathscr{F} = \omega_f$ we get

$$c_1(f_!\omega_f) = f_*\left(\frac{c_1(\omega_f)^2}{2} - \frac{c_1(\omega_f)c_1(\omega^1_f)}{2} + \frac{c_1(\omega_f)^2 + [\Sigma]}{12}\right) = f_*\left(\frac{c_1(\omega_f)^2 + [\Sigma]}{12}\right).$$

By Corollay 5.48(iii) and the definition of κ_1 (Example 5.50) we get

$$\lambda = \frac{\kappa_1 - \psi + \delta}{12} \Rightarrow \kappa_1 = 12\lambda + \psi - \delta$$

in $\mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$. □

Theorem 5.70 (Mumford)**.** *For $2g-2+n>0$ we have $K_{\overline{\mathscr{M}}_{g,n}} = 13\lambda+\psi-2\delta$ in $\mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$.*

Proof. In the case of **Situation B** we let $f: X \to H$ with divisor of sections D and $\mathscr{F} = \Omega_f^1 \otimes \omega_f(D)$. By Proposition 4.53 and Serre duality we get $f_!\mathscr{F} = f_*\mathscr{F}$. Hence again by Example 5.21 we get

$$K_{\overline{\mathscr{M}}_{g,n}} = c_1(f_*(\Omega_f^1 \otimes \omega_f(D))) = c_1(f_*\mathscr{F}) = c_1(f_!\mathscr{F}).$$

By the same work in **Situation B** we have $c_1(\Omega_f^1 \otimes \omega_f(D)) = c_1(\omega_f^2(D))$ and $c_2(\Omega_f^1 \otimes \omega_f(D)) = [\Sigma]$. Hence again we have

$$K_{\overline{\mathscr{M}}_{g,n}} = 13\lambda+\psi-2\delta$$

in $\mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$. □

Corollary 5.71. *For $g \geq 1$ and $g+n \geq 4$, we have $K_{\overline{M}_{g,n}} = 13\lambda+\psi-2\delta-\delta_{1,\emptyset}$ in $\mathrm{Pic}(\overline{M}_{g,n})$.*

Proof. We need to following results due to [8] Proposition XII.2.5:
• **Fact.** If $g \geq 1$ and $g+n \geq 4$, consider the locus parameterizing curves with nontrivial automorphism group Ξ, then the only divisor components of Ξ is $\Delta_{1,\emptyset}$.

Back to the problem, then this follows $K_{\overline{\mathscr{M}}_{g,n}} = 13\lambda+\psi-2\delta$ in $\mathrm{Pic}(\overline{\mathscr{M}}_{g,n})$, this fact and Riemann-Hurwitz Theorem! □

Remark 5.72. *Note that when we talking about δ instead of Δ over coarse moduli space $\overline{M}_{g,n}$, we means that $\delta_1 = \frac{1}{2}\Delta_1$ and $\delta = \Delta_{irr} + \frac{1}{2}\Delta_1 + \Delta_2 + \cdots$. Hence here $K_{\overline{M}_g} = 13\lambda - 2\Delta + \frac{1}{2}\Delta_1$.*

5.2.8 A glimpse of ample & nef divisors and F-conjecture

Here we will summary (without proofs) some results about ample divisors over the coarse moduli space $\overline{M}_{g,n}$. We will follows the idea in [45] here.

Theorem 5.73 (Cornalba-Harris [24], 1988)**.** *The class $a\lambda - b\delta \in \mathrm{Pic}(\overline{M}_g)\otimes\mathbb{Q}$ has non-negative degree on every curve in $\overline{M}_g$ not contained in the boundary $\Delta = \overline{M}_g \backslash M_g$ if and only if $a \geq (8+\frac{4}{g})b$ and is ample if and only if $a > 11b > 0$.*

Remark 5.74. *By Lemma 6.1 in [25], the Hodge class λ is big and nef. Note that by this result, λ itself is not ample, but since it is big it is a sum of an ample and an effective divisor.*

Definition 5.75. *(i) The strata consisting of curves with $3g-4+n$ nodes form curves in $\overline{M}_{g,n}$ called F-curves (in honor of Faber and Fulton);*

(ii) The locus of flag curves is the image $\overline{F}_{g,n}$ of the morphism

$$\overline{M}_{0,g+n}/\mathfrak{S}_g \to \overline{M}_{g,n}$$

obtained by attaching g copies of the pointed rational elliptic curve at the g-unordered points.

Remark 5.76. *For (i), since the locus of curves with k nodes has codimension k in $\overline{M}_{g,n}$ and $\dim(\overline{M}_{g,n}) = 3g - 3 + n$, so we just consider the 1-dimensional locus.*

Actually by classical Nakai-Moishezon criterion we know that a divisor D on $\overline{M}_{g,n}$ is ample if and only if $D^{\dim K} \cdot K > 0$ for all integral subscheme $K \subset \overline{M}_{g,n}$. But Fulton's Conjecture asserts more remarkable thing:

Conjecture 1 (F-Conjecture)**.** *A divisor D on $\overline{M}_{g,n}$ is ample if and only if $D \cdot C > 0$ for every F-curve on $\overline{M}_{g,n}$.*

In the paper [45] they showed that we just need to consider the case $n = 0$:

Theorem 5.77 (Gibney-Keel-Morrison, 2001)**.** *A divisor D on $\overline{M}_{g,n}$ is nef if and only if D has non-negative intersection with all the F-curves and the restriction $D|_{\overline{F}_{g,n}}$ is nef. In particular, the F-conjecture for $g = 0$ implies the F-conjecture for all g.*

(But although $n = 0$, this problem is also open and difficult) In particuler, this result can deduce many ad hoc examples. We may call the divisor which has non-negative intersection number with F-curves are called F-nef. Hence the F-conjecture asserts that the F-nef cone of divisors is the same as the nef cone of divisors of $\overline{M}_{g,n}$.

Corollary 5.78. *Let D be an F-nef divisor $a\lambda - \sum b_i\delta_i$ on $\overline{M}_g$. Assume further for each coefficient b_i, $1 \le i \le \lfloor g/2 \rfloor$, that either $b_i = 0$ or $b_i \ge b_{irr}$. Then D is nef.*

Proof. See [45] Proposition 6.1. □

Corollary 5.79. *(i) The ray $10\lambda - 2\delta + \delta_{irr}$ is nef on $\overline{M}_g$ for all $g \ge 2$;*
(ii) (Cornalba-Harris). The class $11\lambda - \delta$ is nef on $\overline{M}_g$ for all $g \ge 2$.

Proof. See [45] Corollary 6.2, 6.3. □

We should also remark that the F-conjecture is known for small genus and small numbers of points thanks to the work of Keel, McKernan, Farkas and Gibney:

Theorem 5.80 (Keel-McKernan with Gibney-Keel-Morrison and Farkas)**.** *The F-conjecture holds for $\overline{M}_{g,n}$ when the pair (g, n) is of form:*
(i) (g, n) for $g + n \le 7$;
(ii) $(g, 0)$ for $g \le 24$;
(iii) $(g, 1)$ for $g \le 8$ or $(6, 2)$.

Proof. See Corollary (0.4) in [45], Theorem 1 in [37] and the results in [44]. In fact, Gibney has reduced the conjecture on a given $\overline{M}_g$ to an entirely combinatorial question which can be checked by computer. □

5.3 The Kodaira dimension of moduli space of curves

5.3.1 Summary of the results of kodaira dimension

Here as an introduction we will summary some results of the types of $\overline{M}_{g,n}$ where we will prove and we may not prove.

First, $\overline{M}_{g,n}$ is uniruled or even unirational for some small values of g and n:

Theorem 5.81 (Summaried in [14]). *Here we have a table about these. Let $\overline{M}_{g,n}$ is rational if $0 \leq n \leq a(g)$; is unirational if $0 \leq n \leq b(g)$ and uniruled if $0 \leq n \leq c(g)$:*

g	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$a(g)$	12	14	15	12	8										
$b(g)$	12	14	15	12	15	11	8	9	3	10	1	0	2		
$c(g)$	12	14	15	14	15	13	12	10	9	10	5	3	2	2	0

Remark 5.82. *Hence the Kodaira dimension of $\overline{M}_g$ is negative for $g \leq 15$.*

Let's back to consider the Kodaira dimension and general typeness of $\overline{M}_{g,n}$.

Theorem 5.83 (Belorousski [13], 1998; Bini-Fontanari [16], 2004). *We have*

$$\kappa(\overline{M}_{1,n}) = \begin{cases} -\infty, & 1 \leq n \leq 10; \\ 0, & \quad n = 11; \\ 1, & \quad n \geq 12. \end{cases}$$

Corollary 5.84 (Bini-Fontanari [16], 2004). *For $n \geq 1$, $\overline{M}_{1,n}$ is never of general type.*

Theorem 5.85 (Summaried in [76]). *Let $\overline{M}_{g,n}$ is of general type for all $n \geq m(g)$ given in the following table:*

g	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$m(g)$	16	15	16	15	14	13	11	12	11	11	10	10	9	9	9	7	6	4	4	1

Now we consider the case when $n = 0$, that is, the space $\overline{M}_g$.

Theorem 5.86 (Mumford-Eisenbud-Harris [56][33], 1987; Farkas-Jensen-Payne [38], 2020). *The space $\overline{M}_g$ is of general type when $g \geq 22$.*

Remark 5.87. *(i) The Mumford-Eisenbud-Harris theorem proved that $\overline{M}_g$ is of general type when $g \geq 24$ and has positive Kodaira dimension when $g = 23$. Further more, Farkas-Jensen-Payne proved that $\overline{M}_g$ is of general type when $g = 22, 23$;*

(ii) The remaining cases are $16 \leq g \leq 21$. Actually the Kodaira dimension of $\overline{M}_g$ are still open for $16 \leq g \leq 21$ (Chang and Ran also argued that $\overline{M}_{16}$ is uniruled, but Tseng recently found a fatal computational error in this argument arXiv:1905.00449, and this case is again open).

The first main aim of the chapter is to show the Mumford-Eisenbud-Harris theorem, that is, $\overline{M}_g$ is of general type when $g \geq 24$. We will refer [32] and [33] by using limit linear series instead of admissible covers in [56].

5.3.2 The theorem of Harris-Mumford-Eisenbud

We omitted the theory of limite linear series and focus on the whole structure of the proof.

Theorem 5.88 (Mumford-Eisenbud-Harris [56][33], 1987)**.** *The space $\overline{M}_g$ is of general type when $g \geq 24$.*

Remark 5.89. *In papers [56][33] they show that M_g is of general type when $g \geq 24$. But their criterion also can derived that $\overline{M}_g$ is of general type.*

To prove this, just need to show that $K_{\overline{M}_g}$ is big (see Definition A.17). Recall that by Corollary 5.71, we get

$$K_{\overline{M}_g} = 13\lambda - 2\delta - \delta_1 = 13\lambda - 2\delta_{irr} - 3\delta_1 - 2\sum_{i=2}^{\lfloor g/2 \rfloor} \delta_i.$$

We also know that big if and only if it is numerically equivalent to the sum of an ample and an effective divisor.

Theorem 5.90 (Criterion in [33])**.** *The space $\overline{M}_g$ is of general type if there exists an effective divisor D over it with class*

$$D = a\lambda - b_0\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i\delta_i$$

such that

$$\frac{a}{b_1} < \frac{13}{3}, \quad \frac{a}{b_i} < \frac{13}{2} \text{ for all } i.$$

Proof. By Theorem 5.73 we know that $(11+\varepsilon)\lambda - \delta$ is ample, then hence λ is big. Now we let we have such an effective divisor D on $\overline{M}_g$. Let $c = \frac{p}{q}$ be a rational number such that

$$\max\left\{\frac{2}{b_0}, \frac{3}{b_1}, \frac{2}{b_i}\right\} < c < \frac{13}{a},$$

and we find that

$$qK_{\overline{M}_g} - pD = (13q - ap)\lambda + (pb_0 - 2q)\delta_{irr} + (pb_1 - 3q)\delta_1 + \sum_{i=2}^{\lfloor g/2 \rfloor}(pb_i - 2q)\delta_i.$$

As λ is big and D is effective, then $K_{\overline{M}_g}$ is big. Hence $\overline{M}_g$ is of general type. $\square$

So we need to find such effective divisor D for any $g \geq 24$.

♣ **Construction A. Brill-Noether divisors D_s^r.**

If $g+1$ is composite, we can write $g = (r+1)(s-1) - 1$ for $s \geq 3, r > 0$ and let $d = rs - 1$. Let $S = \{[C] \in M_g : [C] \text{ admits } \mathfrak{g}_d^r\}$. Consider D_s^r be the union of the codimension 1 components of $\overline{S} \subset \overline{M}_g$. This divisor called Brill-Noether divisor since in this case the Brill-Noether number $\rho = g - (r+1)(g-d+r) = -1$.

Theorem A.(Brill-Noetherian Ray Theorem) In this situation, there exists some rational number $c > 0$ such that

$$D^r_s = c\left((g+3)\lambda - \frac{g+1}{6}\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i\right).$$

♣ **Construction B. Petri divisors E^r_s.**

If $g+1>2$ is not composite (in particular g is even), then we let $g=(r+1)(s-1)$ and $d=rs$. Let

$$T = \left\{[C]\in M_g \,\middle|\, \begin{matrix}[C] \text{ admits } L=(\mathscr{L},V) \text{ be a } \mathfrak{g}^r_d \text{ such that the map} \\ V\otimes H^0(C,K_C\otimes L^{-1})\to H^0(C,K_C) \text{ is not injective}\end{matrix}\right\}.$$

Then consider E^r_s be the union of the codimension 1 components of $\overline{T}\subset\overline{M}_g$. In this case the Brill-Noether number $\rho=0$.

Theorem B. If $g=2(d-1)$ then we have

$$E^1_d = c\left(e\lambda - f_0\delta_{irr} - \sum_{i=1}^{g/2} f_i\delta_i\right)$$

where $c=2\frac{(2d-4)!}{d!(d-2)!}, e=6d^2+d-6$ and $f_0=d(d-1), f_1=(2d-3)(3d-2), f_2=3(d-2)(4d-3)$ and for $e\le i\le g/2$ we have $f_i>f_{i-1}$.

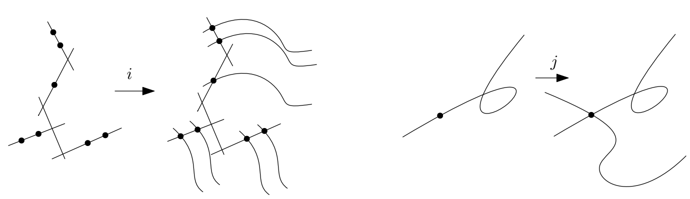
♣ **The proof of the main theorem.**

Assume that we have proved the **Theorem A** and **B**, then we give an easy proof of the main theorem 5.88 by using the Theorem 5.90.

Proof of Theorem 5.88. First, for $g\ge 28$ and even, consider $E^1_{(g/2)+1}$ and one can easy to calculate that it satisfied the Theorem 5.90; Second, for $g\ge 24$ and odd, then $D^1_{(g+1)/2}$ satisfied the Theorem 5.90; For $g=24$, then D^4_6 satisfied the Theorem 5.90 and for $g=26$, then D^2_{10} satisfied the Theorem 5.90. Hence for any $g\ge 24$, there exists such divisor. Hence for any $g\ge 24$ the space $\overline{M}_g$ is of general type. □

♣ **Preparation for the proof of Theorem A and B.**

We just prove **Theorem A** and give a sketch of **Theorem B** since it is so complicated. Consider $i:\overline{M}_{0,g}\to\overline{M}_g$ be the map by attaching g copies of a fixed pointed elliptic curve at each of the marked points; And $i:\overline{M}_{2,1}\to\overline{M}_g$ be the map by attaching a fixed general smooth pointed curve of genus $g-2$ at the marked point:



Let $W\subset\overline{M}_{2,1}$ be the closure of the locus for which the marked point is Weierstrass point of the underlying curves in $M_{2,1}$.

Theorem 5.91. *Let $D \subset \overline{M}_g$ be an effective divisor as*

$$D = a\lambda - b_0\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i\delta_i.$$

(i) If $\mathrm{supp}(j^*D) \subset W$*, then $a = 5b_1 - 2b_2$ and $b_0 = \frac{b_1}{2} - \frac{b_2}{6}$. Further, if we write $j^*D = qW$ for some rational number q, then $b_2 = 3q$.*

(ii) If $i^*D = 0$, then

$$b_i = \frac{i(g-i)}{g-1} b_1 \text{ for } i = 2, ..., \left\lfloor \frac{g}{2} \right\rfloor.$$

Proof. See [33] Theorem 2.1 and Theorem 3.1 or the final several parts of the section 6.F in [55]. $\square$

♣ **Proof of Theorem A.**

Corollary 5.92. *Let $D \subset \overline{M}_g$ be an effective divisor as $D = a\lambda - b_0\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} b_i\delta_i$ satisfies both (i)(ii) in the Theorem 5.91, then there exists some rational c such that*

$$D = c\left((g+3)\lambda - \frac{g+1}{6}\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i\right).$$

Proof. Almost trivial. Use the second relation to write b_2 in terms of b_1. Then use the first to show that $a/b_0 = 6 + 12/(g+1)$. Then show that if $a = g+3$, then $b_0 = (g+1)/6$ and $b_1 = 1$. The remaining coefficients are then immediate from the second set of relations. $\square$

Proposition 5.93. *Let $g = (r+1)(s-1) - 1$ for $s \geq 3, r > 0$, then D_s^r doesn't meet either $i(\overline{M}_{0,g})$ or $j(\overline{M}_{2,1} \backslash W)$. Moreover, $j^*(D_s^r)$ is a positive multiple of the class of W.*

Proof. See [33] Proposition 4.1. and [55] Proposition 6.68. $\square$

Proof of Theorem A. By the first statement in Proposition 5.93 and Corollary 5.92, we get

$$D_s^r = c\left((g+3)\lambda - \frac{g+1}{6}\delta_{irr} - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i\right)$$

for some rational c. To show $c > 0$, we use the second statement in Proposition 5.93. This statement shows that the coefficient cb_2 of δ_2 in the Theorem is positive. As D_s^r is effective, hence $c > 0$. Well done. $\square$

♣ **Sketch of Theorem B.**

Let $g = 2d - 2 = 2k$ and let $E_d^1 = a\lambda - b_0\delta_{irr} - \sum_{i=1}^{d-1} b_i\delta_i$. Since when $g \leq 2$ is trivial, we let $g \geq 4$.

• **Step A.** First show that $j^* E_d^1 \subset W$. By Theorem 5.91 we get

$$R_A: \quad a = 5b_1 - 2b_2 \text{ and } b_0 = \frac{b_1}{2} - \frac{b_2}{6}.$$

• **Step B.** Choose some kind of family of curves $C_j \subset \overline{M}_{0,g}$ and restricting to them. Then we get some relations:

$$R_{B0}: \quad b_j - 2b_{j+1} + b_{j+2} = -2b_1 + b_2 + 2\frac{(2k-2)!}{k!(k-1)!} - \begin{cases} 2\frac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k+l-1)!}, & j = 2l \text{ even;} \\ 0, & j \text{ odd.} \end{cases}$$

(Where b_{k+1} is interpreted as $= b_{k-1}$) These give the following formulas for $b_2, ..., b_k$ in terms of b_1:

$$R_{B1}: \quad b_2 = \frac{4k-4}{2k-1}b_1 - 2\frac{(2k-1)!}{(k-2)!(k+1)!};$$

for $3 \le i \le k$ we have

$$R_{B2}: \quad \begin{aligned} b_i = &-i(i-2)b_1 + \frac{i(i-1)}{2}b_2 + (i-2)(i-1)\frac{(2k-2)!}{k!(k-1)!} \\ &- \sum_{l-1}^{\lfloor (i-2)/2 \rfloor} 2(i-1-2l)\frac{(2l)!(2k-2-2l)!}{(l+1)!l!(k-l)!(k+l-1)!}. \end{aligned}$$

Hence the relations R_A, R_{B1}, R_{B2} express all the coefficients in terms of b_1. But since R_{B1}, R_{B2} are inhomogeneous, this is not enough to check the criterion given in the introduction.

• **Step C.** We consider another 1-dimensional family of curves and restrict E_d^1 into that family, then we can get

$$R_C: \quad (4k-2)b_0 - b_1 = (4k+2)\frac{(2k-1)!}{(k-2)!(k+1)!}.$$

• **The final proof of the Theorem B.**

Proof of Theorem B. Combining R_A, R_{B1}, R_C we can get

$$a = c(6k^2 + 13k + 1), b_0 = ck(k+1), b_1 = c(2k-1)(3k+1), b_2 = 3c(k-1)(4k+1)$$

where $c = 2\frac{(2k-2)!}{(k+1)!(k-1)!}$. Moreover, by R_{B0} we get

$$\begin{aligned} b_j - 2b_{j+1} + b_{j+2} &\le -2b_1 + b_2 + 2\frac{(2k-2)!}{k!(k-1)!} \\ &\le -12k\frac{(2k-2)!}{(k+1)!(k-1)!} < 0, \end{aligned}$$

so the sequence of b_i is convex. But since $b_{k+1} = b_{k-1}$ and taking $j = k-1$ gives $b_k > b_{k-1}$, hence we get Theorem B. $\square$

5.4 About Hassett-Keel program

We will work over algebraically closed field of characteristic zero.

5.4.1 A glimpse of Hassett-Keel program of $\overline{M}_g$

Here we follows the survey [36] and the introduction of the paper [60][61] and [4][2][3]. These are called Hassett-Keel program aiming to give modular interpretations of the log canonical models of $\overline{M}_g$, with the ultimate goal of giving a modular interpretation of the canonical model for the case $g \gg 0$.

By the Theorem of Mumford-Eisenbud-Harris, we get $\overline{M}_g$ is of general type when $g \gg 0$. Fixed this kind of g. By the paper BCHM[18] we get the canonical ring

$$R(\overline{M}_g) = \bigoplus_{n \geq 0} H^0(\overline{M}_g, nK_{\overline{M}_g})$$

is finitely generated. Hence we can consider its canonical model, more generally, its log canonical models

$$\overline{M}_g(\alpha) = \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0\left(\overline{\mathscr{M}}_g, \left\lfloor n(K_{\overline{\mathscr{M}}_g} + \alpha\delta)\right\rfloor\right)\right)$$

for $\alpha \in \mathbb{Q} \cap [0, 1]$.

Before consider these, we first introduce some more stability of curves other than Deligne-Mumford's.

Definition 5.94 (Some types of elliptic chains)**.** *(i) Elliptic tail be a connected subcurve of genus* 1 *which meets the rest curves at only one node;*

(ii) Elliptic bridge be a connected subcurve of genus 1 *which meets the rest curves at only two nodes;*

(iii) Open elliptic chain of length l is a 2*-pointed subcurve (C', p, q) such that $C = E_1 \cup_{a_1} \cup \cdots \cup_{a_{l-1}} E_l$ where E_i are connected genus one curves such that $E_i \cap E_{i+1}$ is a node and $E_i \cap E_j - \emptyset$ for $|i - j| > 1$ and $p \in E_1, q \in E_l$ are smooth points;*

(iv) Open tacnoded elliptic chain of length l is a 2*-pointed subcurve (C', p, q) such that $C = E_1 \cup_{a_1} \cup \cdots \cup_{a_{l-1}} E_l$ where E_i are connected genus one curves with nodes, cusps, or tacnodes as singularities such that $E_i \cap E_{i+1}$ is a tacnode and $E_i \cap E_j - \emptyset$ for $|i - j| > 1$ and $p \in E_1, q \in E_l$ are smooth points with $\omega_{C'}(p + q)$ is ample.*

Definition 5.95. *Let C be a projective connected curve of arithmetic genus $g \geq 3$, with nodes, cusps, and tacnodes as singularities. We say C admits an open (tacnodal) elliptic chain if there is an open (tacnodal) elliptic chain (C', p, q) and a morphism $i : C' \to C$ such that*

(i) i is an isomorphism over $C' \backslash \{p, q\}$ onto its image;

(ii) $i(p), i(q)$ are nodes of C; we allow the case $i(p) = i(q)$, in which case C is said to be a closed (tacnodal) elliptic chain.
C admits a weak tacnodal elliptic chain if there exists $i : C' \to C$ as above with the second condition replaced by

(ii') $i(p)$ is a tacnode and $i(q)$ is a node of C;

(ii") $i(p) = i(q)$ is a tacnode of C, in which case C is said to be a closed weak tacnodal elliptic chain.

Definition 5.96 (pseudo-stable)**.** *A complete curve is pseudo-stable if*

(1) it is connected, reduced, and has only nodes and ordinary cusps as singularities;

(2) admits no elliptic tails and its the canonical sheaf of the curve is ample.

Definition 5.97 (c-stable)**.** *A complete curve is c-semistable if*

(1) C has nodes, cusps, and tacnodes as singularities and ω_C is ample;

(2) a connected genus one subcurve meets the rest of the curve in at least two points (not counting multiplicity).

It is said to be c-stable if it is c-semistable and has no tacnodes or elliptic bridges.

Remark 5.98. *A curve is c-stable if and only if it is pseudo-stable and has no elliptic bridges.*

Definition 5.99 (h-stable)**.** *A complete curve is h-semistable if it is c-semistable and admits no tacnodal elliptic chains. It is said to be h-stable if it is h-semistable and admits no weak tacnodal elliptic chains.*

Actually in [60][61], Hassett and Hyeon showing that:

Theorem 5.100 (Hassett-Hyeon)**.**

$$\overline{M}_g(\alpha) \cong \begin{cases} \overline{M}_g, & \text{if } \alpha \in (9/11, 1] \\ \overline{M}_g^{ps}, & \text{if } \alpha \in (7/10, 9/11] \\ \overline{M}_g^{c}, & \text{if } \alpha = 7/10 \\ \overline{M}_g^{h}, & \text{if } \alpha \in (2/3 - \varepsilon, 7/10) \end{cases}$$

where $\overline{M}_g^{ps}, \overline{M}_g^{c}, \overline{M}_g^{h}$ are the moduli spaces of pseudostable, c-semistable, and h-semistable curves, respectively.

In these works, new projective moduli spaces of curves are constructed using GIT. Actually one of the most appealing features of the Hassett-Keel program is the way that it ties together different compactifications of M_g obtained by varying the parameters in the GIT constructions of Mumford's.

In the series of papers [4][2][3], recall that in Part II, the theory of Deligne-Mumford stabilization has three steps:

(a) Prove that the functor of stable curves is a proper Deligne-Mumford stack $\overline{\mathscr{M}}_{g,n}$;

(b) Use the Keel-Mori theorem to show that $\overline{\mathscr{M}}_{g,n}$ has a coarse moduli space $\overline{M}_{g,n}$;

(c) Find some line bundle on $\overline{\mathscr{M}}_{g,n}$ descends to an ample line bundle on $\overline{M}_{g,n}$.

In these papers, they proved a general existence theorem for good moduli spaces of non-separated algebraic stacks that can be viewed as a generalization of the Keel-Mori theorem. Hence we can run the modified version of the previous standard three-step procedure to construct moduli interpretations for the log canonical models:

$$\overline{M}_{g,n}(\alpha) = \mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0\left(\overline{\mathscr{M}}_g, \left\lfloor n(K_{\overline{\mathscr{M}}_g} + \alpha\delta + (1-\alpha)\psi)\right\rfloor\right)\right)$$

Actually, for all $\alpha > 2/3 + \varepsilon$, where $0 < \varepsilon \ll 1$, we have

(a) Construct an algebraic stack $\overline{\mathscr{M}}_{g,n}(\alpha)$ of α-stable curves (see [4]);

(b) Construct a good moduli space $\overline{\mathscr{M}}_{g,n}(\alpha) \to \overline{\mathbb{M}}_{g,n}(\alpha)$;

(c) Show that $K_{\overline{\mathscr{M}}_g} + \alpha\delta + (1-\alpha)\psi$ on $\overline{\mathscr{M}}_{g,n}(\alpha)$ descends to an ample line bundle on $\overline{\mathbb{M}}_{g,n}(\alpha)$, and conclude that $\overline{\mathbb{M}}_{g,n}(\alpha) \cong \overline{M}_{g,n}(\alpha)$.

Here in the intervals $(9/11, 1), (7/10, 9/11), (2/3, 7/10)$ and $(2/3-\varepsilon, 2/3)$, the definition of α-stability does not change, hence so are $\overline{\mathscr{M}}_{g,n}(\alpha)$ and $\overline{M}_{g,n}(\alpha)$.

5.4.2 Log canonical models of Deligne-Mumford stacks

This section taken from the appendix of paper [60]. In this section, a scheme means a separated scheme of finite type over k and a stack means a separated Deligne-Mumford stack of finite type over k.

Definition 5.101. *A birational morphism of stacks is a morphism $f : \mathscr{X}_1 \to \mathscr{X}_2$ such that there exist dense open substacks $\mathscr{U}_i \subset \mathscr{X}_i$ with $\mathscr{U}_2 = f^{-1}(\mathscr{U}_1)$ and $f : \mathscr{U}_1 \to \mathscr{U}_2$ an isomorphism.*

There is a largest open substack $\mathscr{U} \subset \mathscr{X}_1$ such that f an isomorphism, and we let the complement is exceptional locus $\mathrm{Exc}(f)$. For any closed substack $\mathscr{D} \subset \mathscr{X}_2$ such that $f(\mathscr{U}) \cap \mathscr{D}$ dense in $\mathscr{D}$, the birational transform $f_*^{-1}\mathscr{D}$ is the closure of $f^{-1}(f(\mathscr{U}) \cap \mathscr{D})$ in $\mathscr{X}_1$.

We say that $\mathscr{X}_1$ and $\mathscr{X}_2$ are properly birational if there exists a stack $\mathscr{Y}$ and birational proper morphisms $g_i : \mathscr{Y} \to \mathscr{X}_i$.

Theorem 5.102 (Resolution of singularities)**.** *For a reduced stack $\mathscr{X}$, there exists a smooth stack $\mathscr{X}'$ and a birational proper map $f : \mathscr{X}' \to \mathscr{X}$ such that*

(i) locus $\mathrm{Exc}(f)$ can be taken to be a normal crossings divisor;

(ii)If $\mathscr{Z} \hookrightarrow \mathscr{X}$ is a closed substack with ideal sheaf $\mathscr{I}_{\mathscr{Z}} \subset \mathscr{O}_{\mathscr{X}}$, then there exists an divisor $\mathscr{D}' \subset \mathscr{X}'$ such that $f^*\mathscr{I}_{\mathscr{Z}} \cong \mathscr{O}_{\mathscr{X}'}(-\mathscr{D}')$ and $\mathrm{Exc}(f) \cup \mathscr{D}'$ is simple normal crossings.

Proof. This need the functorial resolving singularities and such procedures commute with étale maps (see Page 329 in [59]), here we give an idea. One have an étale presentation $R \rightrightarrows U$. After resolving R, U to be R', U', we can get $R' \rightrightarrows U'$ for a smooth stack $\mathscr{X}'$. □

Proposition 5.103. *Let X be a connected normal separated scheme of finite type over k. Let $D = \sum_j a_j D_j$ be a $\mathbb{Q}$-divisor on X, with the D_j distinct and reduced and $a_j \in [0, 1]$. Then the following properties of pair (X, D) are local on the étale topology:*

(i) X is normal;

(ii) D_j are codimension one reduced closed subschemes;
(ii) for some $m > 0$ the divisor $m(K_X + D)$ is Cartier.
We will call (X, D) admissible if it satisfying (i)(ii)(iii).
Pick a coherent sheaf $\mathscr{F}$ over X, then following properties are local on the étale topology:
(a) $\mathscr{F}$ is locally free;
(b) $\mathscr{F}$ with S_k condition for $k > 0$.
If X integral with a integral Weil divisor, then $\mathscr{O}_X(D)$ be a reflexive S_2 of rank 1 and the formation of $\mathscr{O}_X(D)$ commutes with étale maps.
Let (X, D) admissible, then being terminal, canonical, klt, plt, lc are local on the étale topology.

Proof. These are the results of descent theory, we refer St 0238 and [66] 5.20. □

Definition 5.104. *Let $(\mathscr{X}, \mathscr{D})$ is proper and admissible, then define its log canonical ring is*

$$R(\mathscr{X}, \mathscr{D}) = \bigoplus_{m \geq 0} \Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}}(mK_{\mathscr{X}} + \lfloor m\mathscr{D} \rfloor)).$$

We define $(\mathscr{X}, \mathscr{D})$ is terminal, canonical, klt, plt, lc if it admits an étale presentation with this property. We say it is strictly lc if it is lc and $\mathscr{X} \setminus \bigcup_j \mathscr{D}_j$ is canonical.

Similar as the normal birational geometry as in [66], we have:

Proposition 5.105. *The admissible pair $(\mathscr{X}, \mathscr{D})$ is terminal, canonical, klt, and lc if and only if there exists a log resolution $f : \mathscr{X}' \to \mathscr{X}$ such that*
(i) $\mathrm{Exc}(f) = \bigcup \mathscr{E}_j$ is a divisor, $\mathrm{Exc}(f) \cup f^{-1}\mathscr{D}$ is simple normal crossings and $\sum_j f_*^{-1}\mathscr{D}_j$ is smooth;
(ii) we have

$$K_{\mathscr{X}'} + \sum_j a_j f_*^{-1}\mathscr{D}_j \sim_{\mathbb{Q}} f^*(K_{\mathscr{X}} + \mathscr{D}) + \sum_i d_i \mathscr{E}_i$$

such that ($a_j < 1$ and $d_i > 0$), ($a_j \leq 1$ and $d_i \geq 0$), ($a_j < 1$ and $d_i > -1$) and ($a_j \leq 1$ and $d_i \geq -1$).

Proof. See [66] Corollary 2.32. □

Remark 5.106. *These $d_i =: d(\mathscr{E}_i; \mathscr{X}, \mathscr{D})$ is called discrepancy if replaced by any proper birational $\mathscr{Y} \to \mathscr{X}$. We will define $d(\mathscr{D}_j; \mathscr{X}, \mathscr{D}) = -a_j$ and $d(\mathscr{D}_0; \mathscr{X}, \mathscr{D}) = 0$ for divisors in $\mathscr{X}$ and not in $\mathscr{D}$.*

Definition 5.107. *Two admissible pairs $(\mathscr{X}, \mathscr{D})$ and $(\mathscr{X}', \mathscr{D}')$ are properly birational if there exists an admissible pair $(\mathscr{Y}, \mathscr{B})$ and proper birational morphisms $f : \mathscr{Y} \to \mathscr{X}$ and $f' : \mathscr{Y} \to \mathscr{X}'$ such that $K_{\mathscr{Y}} + \mathscr{B} - f^*(K_{\mathscr{X}} + \mathscr{D})$ is f-exceptional and effective, similar as $K_{\mathscr{X}'} + \mathscr{D}'$.*

Proposition 5.108. *If two admissible pairs $(\mathscr{X}, \mathscr{D})$ and $(\mathscr{X}', \mathscr{D}')$ are properly birational, then the natural morphism is an isomorphism of graded rings:*

$$R(\mathscr{X}, \mathscr{D}) \cong R(\mathscr{X}', \mathscr{D}').$$

Proof. This is directly, see [60] Proposition A.12. $\square$

Proposition 5.109. *Let $(\mathscr{X}, \mathscr{D})$ be a proper lc (or klt) pair with coarse moduli space $\pi : \mathscr{X} \to X$. Then there exists an effective $\mathbb{Q}$-divisor $D' = \sum_\ell c'_\ell D'_\ell, 0 \leq c_\ell \leq 1$ of X such that*

(i) pair (X, D') is lc (or klt);

(ii) for each $m > 0$ such that $m(K_X + D')$ is integral and Cartier, we have

$$m(K_{\mathscr{X}} + \mathscr{D}) = \pi^* m(K_X + D');$$

(iii) for any $m \geq 0$ we have

$$\Gamma(X, mK_X + \lfloor mD' \rfloor) \cong \Gamma(\mathscr{X}, mK_{\mathscr{X}} + \lfloor m\mathscr{D} \rfloor).$$

Together these we have

$$\pi^* : R(X, D') \cong R(\mathscr{X}, \mathscr{D}).$$

Proof. Omitted, see [60] Proposition A.13. $\square$

Corollary 5.110 (Basepoint-freeness for stacks)**.** *Let $(\mathscr{X}, \mathscr{D})$ be a proper klt pair and $K_{\mathscr{X}} + \mathscr{D}$ is nef and big (i.e., $K_X + D'$ is nef and big). Consider the stack and coarse moduli space*

$$\mathscr{Y} := [(\mathrm{Spec} R(\mathscr{X}, \mathscr{D}) \backslash 0)/\mathbb{G}_m], Y := \mathrm{Proj} R(\mathscr{X}, \mathscr{D}) = \mathrm{Proj} R(X, D'),$$

where the action of $\mathbb{G}_m$ arises from the grading, then there is a morphism of stacks $\psi : \mathscr{X} \to \mathscr{Y}$ inducing on coarse moduli spaces the contraction from X to the log canonical model of (X, D').

Proof. Use [66] Theorem 3.3. $\square$

5.4.3 The first result for $9/11 < \alpha \leq 1$

Here we just consider $g \geq 4$. First we need to point out the following well-known result:

Lemma 5.111. *For $g \geq 4$ we consider the canonical map $f : \overline{\mathscr{M}}_g \to \overline{M}_g$, then we have*

$$f^* \left(K_{\overline{M}_g} + \alpha(\Delta_{irr} + \Delta_2 + \cdots + \Delta_{\lfloor g/2 \rfloor}) + \frac{1+\alpha}{2} \Delta_1 \right) = K_{\overline{\mathscr{M}}_g} + \alpha\delta.$$

Proof. Similar as Corollary 5.71, the only divisor components of the locus parameterizing curves with nontrivial automorphism group is Δ_1. Hence by Riemann-Hurwitz theorem we have

$$\begin{aligned} & f^* \left(K_{\overline{M}_g} + \alpha(\Delta_{irr} + \Delta_2 + \cdots + \Delta_{\lfloor g/2 \rfloor}) + \frac{1+\alpha}{2} \Delta_1 \right) \\ &= K_{\overline{\mathscr{M}}_g} - \delta_1 + \alpha(\delta_{irr} + \delta_2 + \cdots + \delta_{\lfloor g/2 \rfloor}) + \frac{1+\alpha}{2} f^*\Delta_1 \\ &= K_{\overline{\mathscr{M}}_g} - \delta_1 + \alpha(\delta_{irr} + \delta_2 + \cdots + \delta_{\lfloor g/2 \rfloor}) + (1+\alpha)\delta_1 = K_{\overline{\mathscr{M}}_g} + \alpha\delta, \end{aligned}$$

so we get the conclusion. $\square$

Fact. By Lemma 5.111, the universal property of the coarse moduli space implies that sections of invertible sheaves on $\overline{\mathscr{M}}_g$ are all pullbacks of sections of the corresponding reflexive sheaves on $\overline{M}_g$. Hence by Proposition 5.109 we get the log canonical model of $\overline{\mathscr{M}}_g$ with respect to $K_{\overline{\mathscr{M}}_g} + \alpha\delta$ can thus be identified with the log canonical model of $\overline{M}_g$ with respect to (we will use both of them)

$$K(\overline{M}_g; \Delta, \alpha) := K_{\overline{M}_g} + \alpha(\Delta_{irr} + \Delta_2 + \cdots + \Delta_{\lfloor g/2 \rfloor}) + \frac{1+\alpha}{2}\Delta_1.$$

Theorem 5.112 (Mumford-Cornalba-Harris)**.** *If $9/11 < \alpha \leq 1$, then $\overline{M}_g(\alpha) \cong \overline{M}_g$.*

Proof. By Remark 5.72 and Theorem 5.73, we know that the divisor

$$\begin{aligned} a\lambda - \Delta + \frac{1}{2}\Delta_1 &= \frac{a}{13}\left(K_{\overline{M}_g} + \left(2 - \frac{13}{a}\right)(\Delta_{irr} + \Delta_2 + \cdots) + \left(\frac{3}{2} - \frac{13}{2a}\right)\Delta_1\right) \\ &= \frac{a}{13}K\left(\overline{M}_g; \Delta, \left(2 - \frac{13}{a}\right)\right) \end{aligned}$$

is ample if and only if $a > 11$. Hence $K(\overline{M}_g; \Delta, \alpha)$ is ample if and only if $9/11 < \alpha \leq 1$. As $\overline{M}_g$ is proper, we get $\overline{M}_g(\alpha) \cong \overline{M}_g$ by [46] Proposition 13.48. □

Remark 5.113. *As $(\overline{\mathscr{M}}_g, \alpha\delta)$ is lc and proper, the log canonical model (in sense of [66] Definition 3.50) is unique and is $\overline{M}_g$ here by [66] Theorem 3.52.*

5.4.4 The main results for $7/10 < \alpha \leq 9/11$

We will focus on the paper [60] and work out this paper. Here we just consider $g \geq 4$.

When $\alpha = 9/11$, the divisor $K(\overline{M}_g; \Delta, \alpha)$ is not ample and pair $(\overline{\mathscr{M}}_g, \alpha\delta)$ is a klt pair. Hence using the log MMP argument, we need to find some extremal ray to obain a contraction!

By Theorem 5.73, when $\alpha = 9/11$, then $K(\overline{M}_g; \Delta, \alpha) = 11\lambda - \delta$ is nef and big (when $g \geq 22$ as then it's log general type). By the base-point free theorem we get it is semi-ample.

Fix a (C_2, p) in $\overline{M}_{g-1,1}$ and consider

$$\overline{M}_{1,1} \times \overline{M}_{g-1,1} \to \overline{M}_g.$$

Consider $C = C_1 \cap_p C_2$ be curves with elliptic tails and (C_1, p) in $\overline{M}_{1,1} \cong \mathbb{P}^1$. Hence these curves parameterizing by a rational curve $R(C_2, p) \subset \overline{M}_g$.

Lemma 5.114. *In $\overline{M}_g$, the class $R = [R(C_2, p)]$ is independent of (C_2, p) and we have $\lambda \cdot R = 1/12$, $\delta_{irr} \cdot R = 1$, $\delta_1 \cdot R = -1/12$ and $\delta_i \cdot R = 0$ for all $i \geq 2$. In particular we have $(K_{\overline{\mathscr{M}}_g} + 9/11\delta) \cdot R = 0$.*

Proof. This follows from the analysis in [24] and introductions in [60]. □

In fact, in [45] (Proposition 6.4) they find that:

Proposition 5.115. *When $g \geq 5$, the only divisorial contraction $\gamma : \overline{M}_g \to X$ with $\rho(\overline{M}_g/X) = 1$ with X projective is the blowdown of the elliptic tails, contraction of R.*

Now what is X? Actually the construction of γ as follows. As $D := K(\overline{M}_g; \Delta, \alpha)$ is semi-ample, replacing a multiple we get a morphism $f : \overline{M}_g \to P := \mathbb{P}(\Gamma(\overline{M}_g, D))$. By taking Stein factorization, we get

$$\overline{M}_g \to \underline{\mathrm{Spec}}_P f_* \mathscr{O}_{\overline{M}_g} \to P = \mathbb{P}(\Gamma(\overline{M}_g, D)).$$

The morphism $\overline{M}_g \to \underline{\mathrm{Spec}}_P f_* \mathscr{O}_{\overline{M}_g}$ is the contraction map. We claim that

$$\underline{\mathrm{Spec}}_P f_* \mathscr{O}_{\overline{M}_g} \cong \mathrm{Proj} \bigoplus_n \Gamma(\overline{M}_g, nD).$$

First we need to find a canonical morphism $r : \overline{M}_g \to \mathrm{Proj} \bigoplus_n \Gamma(\overline{M}_g, nD)$ and analyze its properties. This is a standard scheme theory. For simplicity, we denote $A_d = \Gamma(\overline{M}_g, dD)$ and $A = \bigoplus_n \Gamma(\overline{M}_g, nD)$. Then the map $\mathscr{O}_{\overline{M}_g} A \to \bigoplus_d \mathscr{O}(dD) = \mathrm{Sym}(\mathscr{O}(D))$ induce the morphism

$$r : \overline{M}_g \cong \mathrm{Proj}\mathrm{Sym}(\mathscr{O}(D)) \to \underline{\mathrm{Proj}} \mathscr{O}_{\overline{M}_g} A = \mathrm{Proj} A \times \overline{M}_g \to \mathrm{Proj} A$$

which induce $r^{-1}(D_+(f)) = (\overline{M}_g)_f$ and the restriction map $r_f : (\overline{M}_g)_f \to D_+(f)$ yield $A_{(f)} \cong \Gamma((\overline{M}_g)_f, \mathscr{O}_{\overline{M}_g})$ where $f \in A_+$ and $(\overline{M}_g)_f$ means the non-vanishing locus of f (for more details of the general case of this, we refer the previous part of the Proposition 13.48 in[46]).

Next, we have the canonical map $\pi : \mathrm{Proj} \bigoplus_n \Gamma(\overline{M}_g, nD) \to \mathbb{P}(\Gamma(\overline{M}_g, D))$ induced by surjection $\mathrm{Sym}\Gamma(\overline{M}_g, D) \to \bigoplus_n \Gamma(\overline{M}_g, nD)$. Easy to see that we have the following commutative diagram:

$$\begin{tikzcd}
\underline{\mathrm{Spec}}_P f_* \mathscr{O}_{\overline{M}_g} \arrow[dr, "h"'] & \overline{M}_g \arrow[l, "g"'] \arrow[r, "r"] \arrow[d, "f"] & \mathrm{Proj} \bigoplus_n \Gamma(\overline{M}_g, nD) \arrow[dl, "\pi"] \\
& P = \mathbb{P}(\Gamma(\overline{M}_g, D)) &
\end{tikzcd}$$

Let f induced by the sections f_i, then pick the standard coordinates x_i of P we have $h^{-1}(D'_+(x_i)) = D_+(f_i) = \mathrm{Spec} A_{(f_i)} \cong \mathrm{Spec}\Gamma((\overline{M}_g)_{f_i}, \mathscr{O}_{\overline{M}_g})$. One the other hand, since $f^{-1}(D'_+(x_i)) = (\overline{M}_g)_{f_i}$, we also have $h^{-1}(D'_+(x_i)) = \mathrm{Spec}\Gamma((\overline{M}_g)_{f_i}, \mathscr{O}_{\overline{M}_g})$. As the intersection parts are automatically coincident, we get the claim. Hence we can restate the result:

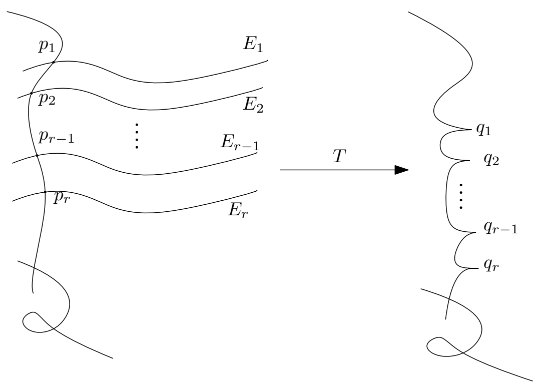
Proposition 5.116. *When $g \geq 5$, the only divisorial contraction of $\overline{M}_g$ is $\gamma : \overline{M}_g \to \overline{M}_g(9/11)$ with $\overline{M}_g(9/11)$ projective is the blowdown of the elliptic tails Δ_1.*

As γ is a divisorial contraction and $\overline{M}_g$ is a quotient of smooth variety by a finite group (by E. Looijenda, see [70]), the space $\overline{M}_g(9/11)$ is $\mathbb{Q}$-factorial. Then we may ask that is $\overline{M}_g(9/11)$ the coarse moduli space of moduli stack of some kind of curves which is the compactification of moduli space of smooth curves? Actually this is one of our main theorems. Here we state the main theorems we will prove.

Theorem 5.117. *Let $\overline{\mathscr{M}}_g^{ps}$ be the stack of pseudostable curves. Then there is a morphism of stacks*

$$\mathcal{T} : \overline{\mathscr{M}}_g \to \overline{\mathscr{M}}_g^{ps}$$

which is an isomorphism in the complement of δ_1. And for a stable curve $C \in \delta_1(\mathrm{Spec}k)$, the curve $\mathcal{T}(\mathrm{Spec}k)(C)$ is obtained by replacing each elliptic tail of C with a cusp:



The coarse moduli space $\overline{M}_g^{ps} \cong \overline{M}_g(9/11)$ and the induced map

$$T : \overline{M}_g \to \overline{M}_g^{ps}$$

coincides with the extremal contraction $\gamma : \overline{M}_g \to \overline{M}_g(9/11)$.

Theorem 5.118. *For $7/10 < \alpha \leq 9/11$, then $\overline{M}_g(\alpha) \cong \overline{M}_g^{ps}$ as projective varieties.*

- **Results about the moduli stack of pseudostable curves.**
  Define the moduli stack of pseudostable curves $\overline{\mathscr{M}}_g^{ps}$ as

$$\overline{\mathscr{M}}_g^{ps}(S) := \left\{ f : \mathcal{C} \to S \,\middle|\, \begin{array}{l} f \text{ is proper and flat and the geometric fibers} \\ \text{of } f \text{ are pseudostable of genus } g. \end{array} \right\}.$$

The main reference we use about the moduli stack of pseudostable curves is paper [75].

Theorem 5.119 (D. Schubert, 1991)**.** *The stack $\overline{\mathscr{M}}_g^{ps}$ is a separated Deligne-Mumford stack of finite typre over k. By Keel-Mori's theorem, they have a coarse moduli space $\overline{M}_g^{ps}$. Actually we have $\overline{\mathscr{M}}_g^{ps} \cong [\mathbf{chow}_3^s/\mathrm{PGL}_{5g-5}]$ and $\overline{M}_g^{ps} \cong \mathbf{chow}_3 /\!\!/ \mathrm{SL}_{5g-5}$ where $\mathbf{chow}_3$ be the Chow variety and $\mathbf{chow}_3^s$ be the GIT-stable locus.*

Remark 5.120. *We may use the Hilbert scheme instead of Chow variety, we refer [61].*

- **Constructing the morphism of moduli stacks.**

Lemma 5.121 (Single case). *Let C be a genus $g > 2$ stable curve of with elliptic tails $E_1, ..., E_r$ and let D be the union of the components of C p_i the node where E_i meets D, away from the elliptic tails an. Then there exists a unique curve $\mathcal{T}(C)$ characterized by the following properties:*

(a) there is a birational morphism $\nu : D \to \mathcal{T}(C)$, which is an isomorphism away from $p_1, ..., p_r$;

(b) ν is bijective and maps each $p_i \in D$ to a cusp $q_i \in \mathcal{T}(C)$.

There is a unique replacement morphism $\xi_C : C \to \mathcal{T}(C)$ with $\xi_C|_D = \nu$ and $\xi_C|_{E_i}$ are constant. Note that $\mathcal{T}(C)$ has arithmetic genus g.

Proof. It suffices to determine the subrings $\mathscr{O}_{\mathcal{T}(C),q_i} \subset \mathscr{O}_{D,p_i}$ (some kind of anti-normalization). Let $\mathfrak{m}_{D,p_i}$ be the maximal ideal and let $\mathscr{O}_{\mathcal{T}(C),q_i}$ generated by the constants and $\mathfrak{m}_{D,p_i}^2$ as an algebra. Then the maximal ideal of it generated by two elements x, y such that $x^2 = y^3$ (let t be the uniformizer, then $\mathfrak{m}_{D,p_i}^2$ generated by t^2, t^3). Conversely, any germ of a cuspidal curve normalized by (D, p_i) is obtained in this way. Hence ν is the normalization of the cusps $q_1, ..., q_r$. □

••Step I. Reduce to the concrete case.

Now we need to construct the (1-)morphism of stacks

$$\mathcal{T} : \overline{\mathscr{M}}_g \to \overline{\mathscr{M}}_g^{ps}$$

assigns to each stable curve $f : \mathcal{C} \to S$ a pseudostable curve $\mathcal{T}(f) : \mathcal{T}(\mathcal{C}) \to S$.

The main tool is the (family-case) replacement S-morphism $\xi_{\mathcal{C}} : \mathcal{C} \to \mathcal{T}(\mathcal{C})$ satiefies the following properties:

(a) Over the open subset $S_0 \subset S$ mapping to the complement of δ_1, the $\xi_{\mathcal{C}}|_{S_0}$ is an isomorphism;

(b) for the $s \in S$ mapping to δ_1, the morphism $\xi_{\mathcal{C}_s}$ is the morphism as Lemma 5.121;

(c) $\xi_{\mathcal{C}}$ is compatible with base change and isomorphisms.

Pick a étale cover U of $\overline{\mathscr{M}}_g$ (as it is a Deligne-Mumford stack) with representation $R \rightrightarrows U$ and the universal family $\pi : Z \to U$ where R encodes the isomorphisms among the fibers of π.

Proposition 5.122. *To construct $\mathcal{T}$ and ξ over $\overline{\mathscr{M}}_g$, it suffices to construct $\mathcal{T}(\pi) : \mathcal{T}(Z) \to U$ and ξ_Z compatibly with the isomorphism relation R.*

Proof. For any family of stable curves $f : \mathcal{C} \to S$ correspond to $\rho_f : S \to \overline{\mathscr{M}}_g$, consider the cartesian of stacks:

$$\begin{tikzcd} S' := S \times_{\overline{\mathscr{M}}_g} U \arrow[r, "\mathrm{pr}_U"] \arrow[d, "\mathrm{pr}_S"'] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d, "\rho_\pi"] \\ S \arrow[r, "\rho_f"] & \overline{\mathscr{M}}_g \end{tikzcd}$$

and we get the stable curves

$$\begin{tikzcd} \mathrm{pr}_S^*\mathcal{C} \arrow[rr, "\cong"] \arrow[dr, "\mathrm{pr}_S^* f"'] & & \mathrm{pr}_U^* Z \arrow[dl, "\mathrm{pr}_U^*\pi"] \\ & S' & \end{tikzcd}$$

After base change, we get $\mathrm{pr}_U^*\mathcal{T}(\pi) : \mathrm{pr}_U^*\mathcal{T}(Z) \to S'$ and $\mathrm{pr}_U^*\xi_Z : \mathrm{pr}_U^* Z \to \mathrm{pr}_U^*\mathcal{T}(Z)$. Hence they correspond to $\mathrm{pr}_S^* f : \mathrm{pr}_S^*\mathcal{C} \to S'$. As pr_S is the base change of ρ_π, it is étale, hence faithfully flat. As $\mathcal{T}$ is compatible with isomorphisms, by the descent theory we get

$$\mathcal{T}(f) : \mathcal{T}(\mathcal{C}) \to S, \quad \xi_{\mathcal{C}} : \mathcal{C} \to \mathcal{T}(\mathcal{C})$$

as desired. $\square$

••**Step II.** Setup step. Find a appropriate line bundle.

Consider

$$\begin{tikzcd} Z \arrow[r, "\mu_\pi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{U}_g = \overline{\mathscr{M}}_{g,1} \supset \delta_{1,\{1\}} \arrow[d, "\mathrm{univ}"] \\ U \arrow[r, "\rho_\pi"] & \overline{\mathscr{M}}_g \end{tikzcd}$$

with elliptic tail $\delta_{1,\{1\}}$. Let $E := \mu_\pi^*\delta_{1,\{1\}}$ (is Cartier, omitted) and $W = \rho_\pi^*\delta_1$ and consider $L := \omega_\pi \otimes \mathscr{O}(E) = \omega_\pi(E)$. Like Lemma 2.14, we find that over $U \backslash W$ we have

$$k_n := \mathrm{rank}\pi_* L^{\otimes n} = \begin{cases} g & n = 1, \\ (2n-1)(g-1) & n \geq 2. \end{cases}, \quad R^1\pi_* L^{\otimes n} \cong \begin{cases} \mathscr{O}_U & n = 1, \\ 0 & n \geq 2. \end{cases}$$

Hence we need to discover that happens near W as we will use $\pi_* L^{\otimes n}$ to define our maps.

••**Step III.** Locally freeness I. Some cohomology.

Note that the locally freeness and cohomology can descend along the failthfully flat extension. Fix $u_0 \in W$, then $Z_{u_0} := C$ has r elliptic tails E_i and the remaining component we denote D. After some faithfully flat extension, we may assume that π has sections $s_1, ..., s_r : U \to Z$ such that $s_i(u_0) \in E_i$ are smooth points.

Now we have

$$0 \to L^{\otimes n} \to L^{\otimes n}(s_1 + ... + s_r) \to L^{\otimes n}(s_1 + ... + s_r)|_{\{s_1,...,s_r\}} \to 0$$

Over this fiber C, we have

$$0 \to L^{\otimes n}|_C \to L^{\otimes n}(s_1 + ... + s_r)|_C \to L^{\otimes n}(s_1 + ... + s_r)|_{\{s_1(u_0),...,s_r(u_0)\}} \to 0$$

and the last term support over a finite sets $\{s_1(u_0), ..., s_r(u_0)\}$. On the other hand, we can show that $H^1(C, L^{\otimes n}(s_1 + ... + s_r)|_C) = 0$ (as in [60] section 3.3 using long exact sequence, we omitted the details here). Then we get an exact sequence

$$0 \to \pi_* L^{\otimes n} \to F^0 \to F^1 \to R^1\pi_* L^{\otimes n} \to 0$$

with $F^0 := \pi_*(L^{\otimes n}(s_1 + ... + s_r))$ and $F^1 := \pi_*(L^{\otimes n}(s_1 + ... + s_r)|_{s_1+...+s_r})$ locally free of rank r_0, r_1. Let $Q = \mathrm{coker}(\pi_* L^{\otimes n} \to F^0)$ is a subsheaf of a locally-free sheaf, hence is locally-free of rank $r_0 - k_n$ away from a subset $Y \subset U$ of codimension≥ 2. Note that Y is contained in the locus where $R^1\pi_* L^{\otimes n}$ fails to be locally free, and thus is a subset of W.

••**Step IV.** Locally freeness II. Some limit linear series.

Let B be a spectrum of a DVR with closed point 0 and generic point b. Consider the map $\beta : (B, 0) \to (U, u_0)$ such that $0 \mapsto u_0$ and $b \notin W$. Consider the fiber product

$$\begin{tikzcd} Z_B \arrow[r, "\beta'"] \arrow[d, "\pi_B"'] \arrow[dr, phantom, "\ulcorner"] & Z \arrow[d, "\pi"] \\ B \arrow[r, "\beta"] & U \end{tikzcd}$$

and let $L_B = (\beta')^* L$. As $\pi_{B,*} L_B^{\otimes n}$ is locally free of rank k_n as it is a subsheaf of $\beta^* F_0$ and thus torsion free and then flat (over Dedekind domain).

We need the follwoing result:

Proposition 5.123 ([26] 4.3iii)**.** *For $n \geq 1$, let $V_n := \pi_{B,*} L_B^{\otimes n}|_0$ can be naturally identified with*

$$\Gamma\left(D, \omega_D^{\otimes n}\left((2n-2)\sum_{j=1}^{r} p_j\right)\right) + \mathrm{span}(\sigma_1^n, ..., \sigma_r^n)$$

as a subspace of $H^0(C, L^{\otimes n}|_C)$ where σ_j be a section of V_1 such that $\sigma_i(p_i) \neq 0$ and σ_j vanishes to order two at p_i for $i \neq j$.

Corollary 5.124. *For $n \geq 2$, the linear series V_n defines $C \to \mathbb{P}^{k_n - 1}$ with image $\mathcal{T}(C)$. The induced $C \to \mathcal{T}(C)$ is the morphism in Lemma 5.121.*

Proof. For stable curve $(D, p_1, ..., p_r)$, unless $g(D) = 2, r = 0$ (which can not occur here) the line bundle $(\omega_D(p_1 + ... + p_r))^2$ is very ample. Hence V_n can induce an embedding of $D \backslash \{p_1, ..., p_r\}$ into the projective space. By Proposition 5.123 we get that the vanishing sequence of V_n near p_j are all $(0, 2, 3, ...)$ and then induce p_j into cusps. As p_j separated by σ_j^n, these cusps are different ones. As these sections are constant ($\neq 0$) along the E_i, so each E_i is collapsed to the corresponding cusp. $\square$

••Step V. Locally freeness III. Finish the locally freeness.

Proposition 5.125. *For integer $n \geq 1$, the sheaf $\pi_* L^{\otimes n}$ is locally free of rank k_n.*

Proof. We denote $\underline{\mathrm{Grass}}(m, F)$ be the geometric Grassmannian represented the functor of rank m subbundles of F. Recall the exact sequence

$$0 \to \pi_* L^{\otimes n} \to F^0 \xrightarrow{\varphi} F^1 \to R^1 \pi_* L^{\otimes n} \to 0$$

with F^0 and F^1 locally free of rank r_0, r_1. Let $Q = \mathrm{coker}(\pi_* L^{\otimes n} \to F^0)$ is locally-free of rank $r_0 - k_n$ away from a subset $Y \subset U$ of codimension≥ 2.

Then we induce the morphism

$$\begin{aligned} \tau : U \backslash Y &\to \underline{\mathrm{Grass}}(r_0 - k_n, (F^0)^\vee) \times \underline{\mathrm{Grass}}(r_0 - k_n, F^1) \\ &\hookrightarrow \mathbb{P}\left(\bigwedge^{r_0 - k_n} (F^0)^\vee\right) \times \mathbb{P}\left(\bigwedge^{r_0 - k_n} F^1\right) \\ &\hookrightarrow \mathbb{P}\left(\mathscr{H}om\left(\bigwedge^{r_0 - k_n} F^0, \bigwedge^{r_0 - k_n} F^1\right)\right). \end{aligned}$$

Consider as a rational map

$$U \dashrightarrow \mathbb{P}\left(\mathscr{H}om\left(\bigwedge^{r_0-k_n} F^0, \bigwedge^{r_0-k_n} F^1\right)\right),$$

it is given by the section $\bigwedge^{r_0-k_n}\varphi$ and the indeterminacy locus of it is precisely the zero locus of $\bigwedge^{r_0-k_n}\varphi$ defined by $(r_0-k_n)\times(r_0-k_n)$-minors of φ (which is one of the Fitting ideals $\mathscr{I}$ of $R^1\pi_*L^{\otimes n}$). Blowing up the base locus as follows:

$$\begin{tikzcd}
B \arrow[r, "\beta_i'"] \arrow[dr, "\beta_i"'] & U' := \mathrm{Bl}_{\mathscr{I}}(U) \arrow[d, "\sigma"] \arrow[drr, "\tau'"] & & \\
& U \arrow[rr, dashed, "\tau"] & & \underline{\mathrm{Grass}}(r_0-k_n, (F^0)^\vee) \times \underline{\mathrm{Grass}}(r_0-k_n, F^1)
\end{tikzcd}$$

Now we claim that σ is an isomorphism.

As U normal (it is étale over smooth stack $\overline{\mathscr{M}}_g$) and σ is proper birational, using the Zariski's main theorem, we just need to show that σ is quasi-finite. Let it is not quasi-finite and for two different closed points $x_1, x_2 \in U'$ such that τ is not defined at $u := \sigma(x_1) = \sigma(x_2)$. Pick a DVR $B = \mathrm{Spec}\Delta$ with closed point 0 and generic point η. Then let $\beta_i' : B \to U'$ such that $0 \mapsto x_i$ and η is not in the locus of δ_1. Let $\beta_i = \sigma \circ \beta_i'$ and using the functorial, we find that $\tau' \circ \beta_i'$ correspond to

$$0 \to K_i \to \beta_i^* F^0 \to Q_i \to 0,$$

where K_i is of rank k_n. Consider the fiber product

$$\begin{tikzcd}
Z_B \arrow[r, "\mu_i"] \arrow[d, "\pi_i"'] \arrow[dr, phantom, "\ulcorner"] & Z \arrow[d, "\pi"] \\
B \arrow[r, "\beta_i"] & U
\end{tikzcd}$$

then we get the exact sequence

$$0 \to \pi_{i,*}\mu_i^* L^{\otimes n} \to \beta_i^* F^0 \xrightarrow{\beta_i^*\varphi} \beta_i^* F^1 \to R^1(\pi_i)_*\mu_i^* L^{\otimes n} \to 0.$$

Note that $\beta_i^*F^0/\pi_{i,*}\mu_i^*L^{\otimes n}$ is locally free and $\pi_{i,*}\mu_i^*L^{\otimes n}$ and K_i are agree over $B\backslash\{0\}$ as subbundles of $\beta_i^*F^0$. Hence they are isomorphic over entire B. By Proposition 5.123, $(\pi_{i,*}\mu_i^*L^{\otimes n})_0$ are identified in $H^0(Z|_{\beta_i(0)}, L^{\otimes n}|_{Z|_{\beta_i(0)}})$, then $\tau' \circ \beta_1'(0) = \tau' \circ \beta_2'(0)$. Then any points in the fiber over u maps to the same point and then τ should regular at u which is impossible! Hence σ is an isomorphism and we get the claim.

From the claim we know that there exists a subbundle $K \subset F^0$ over U such that $K|_{U\backslash Y} \cong \pi_*L^{\otimes n}|_{U\backslash Y}$. Let $j : U\backslash Y \hookrightarrow U$, then $j_*(\pi_*L^{\otimes n}|_{U\backslash Y}) \cong \pi_*L^{\otimes n}$ as it is reflexive and $\mathrm{codim}_U(Y) \geq 2$. Hence we have

$$K \cong j_*(K|_{U\backslash Y}) = j_*(\pi_*L^{\otimes n}|_{U\backslash Y}) \cong \pi_*L^{\otimes n}$$

be a vector bundle. $\square$

••**Step VI.** Finish the construction.

Proposition 5.126. *For $n \geq 2$, the sections of $L^{\otimes n}$ relative to π induced $Z \to \mathbb{P}(\pi_* L^{\otimes n})$ over U which factors as*

$$Z \xrightarrow{\xi_Z} \mathcal{T}(Z) \hookrightarrow \mathbb{P}(\pi_* L^{\otimes n}).$$

Proof. Now by Proposition 5.125, $\pi_* L^{\otimes n}$ is locally free. By the argument in Proposition 5.123 and Corollary 5.124, using the Stein factorization we get

$$Z \xrightarrow{\xi_Z} \mathcal{T}(Z) \hookrightarrow \mathbb{P}(\pi_* L^{\otimes n}).$$

By the functoriality of dualizing sheaf and $\delta_{1,\{1\}}$ on the moduli stack $\overline{\mathscr{M}}_{g,1}$, we find that this construction is compatible with isomorphism relation and commutes with base extension. □

•**Moduli of pseudostable curves as log canonical models.**

Proof of Theorem 5.117. By Proposition 5.126, we just need to identify $\overline{M}_g^{ps}$ and $\overline{M}_g(9/11)$.

By Lemma 5.121, we know that $\mathcal{T} : \overline{\mathscr{M}}_g \to \overline{\mathscr{M}}_g^{ps}$ implies

(a) T is isomorphism over $\overline{M}_g \backslash \Delta_1$;

(b) T takes locus of elliptic tails into cusps as in Lemma 5.121;

(c) $\mathcal{T}(C) = \mathcal{T}(C')$ if and only if the number of elliptic tails are the same and the remaining pointed stable curves $(D; p_1, ..., p_r) \cong (D'; p'_1, ..., p'_r)$.

Hence T is also an extremal contraction of the locus of elliptic tails. By Remark 1.26 in [66], we know that $\gamma : \overline{M}_g \to \overline{M}_g(9/11)$ are the same as T. Well done. □

Now we turn to prove Theorem 5.118. Let δ^{ps} be the boundary divisor of $\overline{\mathscr{M}}_g^{ps}$, it is the image of δ under $\mathcal{T}$. We will write $K_{\overline{\mathscr{M}}_g} + \alpha\delta$ instead of $K(\overline{M}_g; \Delta, \alpha)$ over $\overline{M}_g$ since they are the same things, after a pull-back.

Lemma 5.127 (Log Discrepancy Formula)**.**

$$K_{\overline{\mathscr{M}}_g} + \alpha\delta = T^*(K_{\overline{\mathscr{M}}_g^{ps}} + \alpha\delta^{ps}) + (9 - 11\alpha)\delta_1.$$

Proof. First, it's easy to see that

$$K_{\overline{\mathscr{M}}_g} + \alpha\delta = T^*(K_{\overline{\mathscr{M}}_g^{ps}} + \alpha\delta^{ps}) + c\delta_1.$$

Then by Lemma 5.114 we find that

$$\begin{aligned}(K_{\overline{\mathscr{M}}_g} + \alpha\delta) \cdot R &= 13\lambda \cdot R + (\alpha - 2)R \cdot \delta \\ &= 13/12 + (\alpha - 2)(1 - 1/12) = \frac{11\alpha - 9}{12}\end{aligned}$$

and $\delta_1 \cdot R = -1/12$. Hence $c = 9 - 11\alpha$. □

Hence if $(\overline{\mathscr{M}}_g, \alpha\delta - (9-11\alpha)\delta_1)$ is lc pair and $\alpha \leq 9/11$, then so is $(\overline{\mathscr{M}}_g^{ps}, \alpha\delta^{ps})$. This is right as reducing the coefficient of δ_1 can only increase the discrepancies of divisors lying over δ_1; this does not affect whether the singularities are log canonical. Now we just need to analyze whether $K_{\overline{\mathscr{M}}_g^{ps}} + \alpha\delta^{ps}$ is ample for $7/10 < \alpha < 9/11$.

We have

$$T^*\mathrm{NS}(\overline{M}_g^{ps}) = R^\perp \subset \mathrm{NS}(\overline{M}_g),$$

then $\mathrm{Amp}(\overline{M}_g^{ps}) = \mathrm{Int}(R^\perp \cap \mathrm{Nef}(\overline{M}_g))$. Actually, if the Fulton's Conjecture 1 holds, then we finish the proof. Of course we can not use this for now and fortunately, we do not need the full strength of the conjecture and we just need to use the following result (Corollary 5.78):

Proposition 5.128 ([45] Proposition 6.1)**.** *Let $D = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i\delta_i$ over $\overline{M}_g$ such that if $1 \leq i \leq g/2$ then either $b_i = 0$ or $b_i > b_0$. If D has non-negative intersection with all F-curves, then D is nef.*

Actually the intersection numbers of $D = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i\delta_i$ with the F-curves has the following results (let $b_i = b_{g-i}$ for $i > g/2$):

(a)	$a/12 - b_0 + b_1/12$	family of elliptic tails
(b)	b_0	
(c)	b_i	for $g-2 \geq i$
(d)	$2b_0 - b_{i+1}$	for $g-2 \geq 2i$
(e)	$b_i + b_j - b_{i+j}$	for $i, j \geq 1, i+j \leq g-1$
(f)	$b_i + b_j + b_k + b_l - b_{i+j} - b_{i+k} - b_{i+l}$	for $i, j, k \geq 1, i+j+k+l = g$

Proof of Theorem 5.118. We just need to show that for $7/10 < \alpha \leq 9/11$, the divisor

$$D := K_{\overline{\mathscr{M}}_g} + \alpha\delta - (9-11\alpha)\delta_1 = 13\lambda + (\alpha - 2)\delta - (9-11\alpha)\delta_1$$

lies in $\mathrm{Int}(R^\perp \cap \mathrm{Nef}(\overline{M}_g))$. Then we just need to check Proposition 5.128. In our case, we have $a = 13$ and $b_i = \begin{cases} 2-\alpha, & i \neq 1; \\ 11-12\alpha, & i = 1 \end{cases}$. Then our divisor satisfied the conditions in Proposition 5.128, and we just need to check the table above!

(i) We do not consider (a) as it is elliptic tails;
(ii) for (b)(c)(f), if $\alpha \leq 11/12$ then they are positive;
(iii) for (d), if $\alpha > 7/10$ it is positive;
(iv) for (e), if $\alpha > 7/10$ it is positive.
Hence well done! □

5.5 More geometry of moduli space of curves

5.5.1 A glimpse of some results using Teichmüller theory

Theorem 5.129 (Boggi-Pikaart, 2000, [19])**.** *The stack $\overline{\mathscr{M}}_{g,n}$ has no non-trivial étale cover, hence it is simply connected.*

Corollary 5.130. *The coarse moduli space $\overline{M}_{g,n}$ is simply connected.*

Theorem 5.131 (Harer-Zagier 1986, [52]; Bini-Harer 2011, [17])**.** *For non-negative integers $g, n, n > 2-2g$, we have: (i) The orbifold Euler characteristic of $\mathscr{M}_{g,n}$ is*

$$\chi(\mathscr{M}_{g,n}) = (-1)^n \frac{(2g-1)B_{2g}}{(2g)!}(2g+n-3)!;$$

(ii) The orbifold Euler characteristic of $\overline{\mathscr{M}}_{g,n}$ is

$$\chi(\overline{\mathscr{M}}_{g,n}) = \sum_{G \text{ of type } (g,n)} \frac{\prod_v \chi(\mathscr{M}_{g_v,L_v})}{\sharp \mathrm{Aut}(G)}.$$

5.5.2 Intersection theory of moduli space of curves

We will refer [43] and [8] chapter XVII. We work over $\mathbb{C}$.
•Chow rings of moduli space of curves.

In the classical book [42] section 8.3 (Example 8.3.12) we know that the intersection product in the Chow ring only defined on the non-singular case and the rational coefficient Chow ring can defined over the quotient variety X/G of a non-singular variety X by a finite group G of automorphisms with $p : X \to X/G$, that is,

$$V \cdot W := \frac{p_*(p^*[V] \cdot p^*[W])}{\sharp G}.$$

(In fact we have $\mathrm{CH}^*(X/G) \otimes \mathbb{Q} = (\mathrm{CH}^*(X) \otimes \mathbb{Q})^G$.)

Now we consider smooth quotient X/G and its stack $[X/G]$ with canonical map $p : [X/G] \to X/G$. We define $\mathrm{CH}^*([X/G])_{\mathbb{Q}} := \mathrm{CH}^*(X/G) \otimes \mathbb{Q}$. Concerning Chern classes of vector bundles on a stack $[X/G]$, recall that such a vector bundle can be viewed as a G-equivariant bundle E on X, meaning that E is a vector bundle on X plus a lifting of the G-action on X. The Chern classes of E are naturally G-invariant elements of $\mathrm{CH}^*(X)$ and therefore give well-defined Chern classes $c_i(E) \in \mathrm{CH}^*([X/G])_{\mathbb{Q}}$.

Now back to the moduli of curves. Now $M_{g,n}$ and $\overline{M}_{g,n}$ are all singular! Foundations of these due to D. Mumford. But $M_{g,n}$ can be written as a non-singular variety(the ℓ-level structure, omit it here) quotient by a finite group, hence we have define as above. But $\overline{M}_{g,n}$ is not that easy. Mumford use that, $\overline{M}_{g,n}$ is étale locally as a non-singular variety (Kuranishi family) quotient by a finite group by Theorem 5.12 and globally it is a quotient of Cohen-Macaulay variety by a finite group, to define the rational Chow ring. However, E. Looijenda (cf. [70]) showed that $\overline{M}_{g,n}$ is also be a globally quotient of a smooth variety $\overline{\mathfrak{M}}_g/G$ by a finite group (non abelian ℓ-level structure)! Hence we can define $\mathrm{CH}^*(M_g) \otimes \mathbb{Q}$ and $\mathrm{CH}^*(\overline{M}_g) \otimes \mathbb{Q}$ now.

For moduli stacks, unfortunately, although this works for coarse moduli space, the moduli stack in general not quotients of smooth varieties modulo finite groups! Let then $M \cong \mathfrak{M}/G$ be a moduli space of (stable or smooth) curves, and $\mathscr{M}$ the corresponding moduli stack. We then have natural morphisms

$$[\mathfrak{M}/G] \xrightarrow{\alpha} \mathscr{M} \xrightarrow{\beta} M = \mathfrak{M}/G$$

by the definition of coarse moduli space. Hence we define

$$\mathrm{CH}^*(\mathscr{M})_{\mathbb{Q}} := \mathrm{CH}^*([\mathfrak{M}/G])_{\mathbb{Q}} = \mathrm{CH}^*(M)_{\mathbb{Q}}.$$

Noet that here we have

$$\mathrm{CH}^1(\mathscr{M})_{\mathbb{Q}} = \mathrm{CH}^1(M)_{\mathbb{Q}} = \mathrm{Pic}(\mathscr{M}) \otimes \mathbb{Q} = \mathrm{Pic}(M) \otimes \mathbb{Q}.$$

•Basic classes.

Fixed the moduli stack $\overline{\mathscr{M}}_{g,P}$ with sections $\sigma_p : \overline{\mathscr{M}}_{g,P} \to \mathscr{U}_{g,P} \cong \overline{\mathscr{M}}_{g,P\cup\{x\}}$. Let $\Sigma_p := \mathrm{Im}(\sigma_p)$ and $\Sigma = \sum_{p\in P} \Sigma_p$. Fix the dualizing sheaf ω_π with universal family $\pi : \mathscr{U}_{g,P} \to \overline{\mathscr{M}}_{g,P}$.

Example 5.132 (Mumford-Morita-Miller)**.** *(1) Let* $\psi_p := \sigma_p^* c_1(\omega_\pi) \in \mathrm{CH}^1(\overline{\mathscr{M}}_{g,P})_{\mathbb{Q}}$*;*

(2) Let $\kappa_i := \pi_*(c_1(\omega_\pi(\Sigma))^{i+1}) \in \mathrm{CH}^i(\overline{\mathscr{M}}_{g,P})_{\mathbb{Q}}$.

Remark 5.133. *(i) We have* $\kappa_0 = (2g-2+\sharp P)[\overline{\mathscr{M}}_{g,P}]$

(ii) Also we let $\tilde{\kappa}_i = \kappa_i - \sum \psi_p^i$.

Example 5.134 (λ-classes)**.** *We let* $\lambda_i := c_i(\pi_*\omega_\pi) = c_i(\pi_!\omega_\pi) \in \mathrm{CH}^i(\overline{\mathscr{M}}_{g,P})_{\mathbb{Q}}$*. More generally, we let*

$$\lambda_i(\nu) := c_i(\pi_!\omega_\pi^\nu) \in \mathrm{CH}^i(\overline{\mathscr{M}}_{g,P})_{\mathbb{Q}}.$$

Hence we may define these over the coarse moduli space.

•Tautological relations.

••Relations after forgetful maps.

Proposition 5.135. *Let* $\pi : \overline{\mathscr{M}}_{g,P\cup\{x\}} \to \overline{\mathscr{M}}_{g,P}$ *and let* $a_t \in \mathbb{Z}_{\geq 0}$ *labelled by* $t \in P\cup\{x\}$*.*

(i) (String Equation)

$$\pi_*\left(\prod_{p\in P} \psi_p^{a_p}\right) = \sum_{a_p>0} \psi_p^{a_p-1} \prod_{q\neq p} \psi_q^{a_q};$$

(ii) (Generalized Dilaton Equation)

$$\pi_*\left(\psi_x^{a_x+1} \prod_{p\in P} \psi_p^{a_p}\right) = \kappa_{a_x} \prod_{p\in P} \psi_p^{a_p}.$$

Proof. Omitted, see Proposition XVII.4.9 in [8]. □

Remark 5.136. *The standard dilaton equation is the special case of the generalized one in which* $a_x = 0$*, that is,*

$$\pi_*\left(\psi_x \prod_{p\in P} \psi_p^{a_p}\right) = (2g-2+\sharp P) \prod_{p\in P} \psi_p^{a_p}.$$

Corollary 5.137. *Let* $\pi : \overline{\mathscr{M}}_{g,P\cup\{x\}} \to \overline{\mathscr{M}}_{g,P}$*, then*

(i) $\pi^*(\kappa_a) = \kappa_a - \psi_x^a$*;*

(ii) $\pi_{x,*}(\kappa_a) = \kappa_{a-1}$*;*

(iii) $\pi^*(\psi_p a) = \psi_p - \delta_{0,\{p,x\}}$*;*

(iv) If we let $\pi_n : \overline{\mathscr{M}}_{g,n} \to \overline{\mathscr{M}}_{g,n-1}$*, then*

$$(\pi_{k+1}\cdots\pi_n)_*(\psi_1^{a_1}\cdots\psi_k^{a_k}\psi_{k+1}^{a_{k+1}+1}\cdots\psi_n^{a_n+1}) = \psi_1^{a_1}\cdots\psi_k^{a_k}\sum_{\sigma\in\mathfrak{S}_k}\kappa_\sigma$$

where $\kappa_\sigma = \kappa_{\sharp\gamma_1}\cdots\kappa_{\sharp\gamma_{\nu(\sigma)}}$ *with* σ *as a product of* $\nu(\sigma)$ *disjoint cycles* γ_i*.*

Proof. Omitted, (i)(ii) follows from string equation and see XVII.4.17,18 in [8]. (iii) follows from XVII.4.6 in [8]. (iv) follows directly from dilaton equation. □

Remark 5.138. *Now combine (iv) and string equation, we find that the intersection numbers of the classes* ψ_i *and* κ_i *on a fixed* $\overline{\mathscr{M}}_{g,n}$ *is completely determined by the intersection theory of the* ψ_i *alone on all the* $\overline{\mathscr{M}}_{g,\nu}$ *with* $\nu \geq n$*, and conversely. This is a generalization of E. Witten's remark.*

Proposition 5.139 (Some relations of boundaries)**.** *Let* $\pi : \overline{\mathscr{M}}_{g,P\cup\{x\}} \to \overline{\mathscr{M}}_{g,P}$ *and* Γ *be a dual graph with* Γ_ν *is the* $P\cup\{x\}$*-marked graph obtained from* Γ *by letting* $P_\nu \cup \{x\}$ *be the index set for the vertex* ν*. Then*

$$\pi^*\delta_\Gamma = \sum_{\nu\in V}\delta_{\Gamma_\nu}.$$

In particular, we have $\pi^*\delta_{irr} = \delta_{irr}$ *and* $\pi^*\delta_{a,A} = \delta_{a,A} + \delta_{a,A\cup\{x\}}$*.*

Proof. Trivial. □

••Relations after general gluing maps.

Proposition 5.140. *Consider the morphisms*

$$\begin{array}{ccc} \overline{\mathscr{M}}_\Gamma = \prod_{v\in V(\Gamma)}\overline{\mathscr{M}}_{g_v,L_v} & \xrightarrow{\eta_v} & \overline{\mathscr{M}}_{g_v,L_v} \\ \Big\downarrow{\scriptstyle \xi_\Gamma} & & \\ \overline{\mathscr{M}}_{g,P} & & \end{array}$$

Then:

(i) $\xi_\Gamma^*\psi_p = \psi_p$*;*

(ii) $\xi_\Gamma^*\kappa_a = \sum_{v\in V(\Gamma)}\eta_v^*\kappa_a$*;*

(iii) by the notations as before, we have

$$\begin{array}{ccc} \overline{\mathscr{M}}_{\Gamma\Gamma'} := \overline{\mathscr{M}}_\Gamma \times_{\overline{\mathscr{M}}_{g,P}} \overline{\mathscr{M}}_{\Gamma'} & & \\ \Big\| & & \\ \coprod_{\Lambda\in G_{\Gamma\Gamma'}}\overline{\mathscr{M}}_\Lambda & \xrightarrow{\coprod\xi_{\Lambda\Gamma}} & \overline{\mathscr{M}}_\Gamma \end{array}$$

then

$$\xi_\Gamma^*\delta_{\Gamma'} = \Sigma_{\Lambda\in G_{\Gamma\Gamma'}}\xi_{\Lambda\Gamma,*}\left(\prod_{\{l,l'\}\in E(|\Lambda|)}\left(-\eta_{v(l)}^*\psi_l - \eta_{v(l')}^*\psi_{l'}\right)\right).$$

Remark 5.141. *The special case of the relations of gluing maps we refer Lemma XVII.4.35,36 in [8].*

•The tautological ring and relative theorems and conjectures.
Here we refer [78] and chapter XVII and XX in [8].

Definition 5.142. *The system of tautological rings $(R^*(\overline{\mathscr{M}}_{g,n})) \subset (\mathrm{CH}^*(\overline{\mathscr{M}}_{g,\nu})_{\mathbb{Q}})_{g,n}$ is the smallest system of $\mathbb{Q}$-algebras satisfying*
(i) $\psi_1, ..., \psi_n \in R^*(\overline{\mathscr{M}}_{g,n})$;
(ii) the system is closed under pushforwards by morphisms $\pi, \xi_{irr}, \xi_{a,A}$.

Define

$$F_g := \sum_{n\geq 0}\frac{1}{n!}\sum_{\sharp\mathbf{k}=3g-3+n}\left(\int_{\overline{\mathscr{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\right)t_{k_1}\cdots t_{k_n}$$

and $F := \sum_g F_g\lambda^{2g-2}$. This is Witten's free energy. For convenience, we will set $\lambda = 1$.

Theorem 5.143 (Witten's Conjecture (Kontsevich's Theorem))**.** *We have the following system of PDEs:*

$$(2n+1)\frac{\partial^3 F}{\partial t_n\partial t_0^2} = \left(\frac{\partial^2 F}{\partial t_{n-1}\partial t_0}\right)\left(\frac{\partial^3 F}{\partial t_0^3}\right) + 2\left(\frac{\partial^3 F}{\partial t_{n-1}\partial t_0^2}\right)\left(\frac{\partial^2 F}{\partial t_0^2}\right) + \frac{1}{4}\left(\frac{\partial^5 F}{\partial t_{n-1}\partial t_0^4}\right).$$

Sketch of Kontsevich's Proof. Kontsevich's Proof consisted of two parts. The first part was to prove a combinatorial formula for the gravitational descendents. Let $G_{g,n}$ be the set of isomorphism classes of trivalent ribbon graphs of genus g with n faces and together with a numbering $\mathrm{Faces}(G) \cong [n]$. Denote by $V(G)$ the set of vertices of a graph $G \in G_{g,n}$. Let us introduce formal variables $\lambda_i, i \in [n]$. For an edge $e \in \mathrm{Edges}(G)$, let $\lambda(e) = \frac{1}{\lambda_i+\lambda_j}$ where i and j are the numbers of faces adjacent to e. Then we have (Kontsevich's combinatorial formula):

$$\sum_{\sharp\mathbf{k}=3g-3+n}\int_{\overline{\mathscr{M}}_{g,n}}\psi_1^{k_1}\cdots\psi_n^{k_n}\prod_{\ell=1}^n\frac{(2k_\ell-1)!!}{\lambda_\ell^{2k_\ell+1}} = \sum_{G\in G_{g,n}}\frac{2^{\sharp\mathrm{Edges}(G)-\sharp V(G)}}{\sharp\mathrm{Aut}(G)}\prod_{e\in\mathrm{Edges}(G)}\lambda(e).$$

The second step of Kontsevich's proof was to translate the combinatorial formula into a matrix integral. Then, by using non-trivial analytical tools and the theory of τ-functions of the KdV hierarchy, he was able to prove that $\exp(F)$ is a τ-function of the KdV hierarchy and, hence, the free energy F satisfies our equation. □

Remark 5.144. *(i) Using these PDEs along with a straightforward geometric fact known as the string equation and the initial condition $\int_{\overline{\mathscr{M}}_{0,3}} 1 = 1$, as $\overline{\mathscr{M}}_{0,3}$ is a point, all top intersections are quickly recursively determined;*

(ii) There was a later reformulation of Witten's conjecture using Virasoro equations, see [8];

(iii) Okounkov and Pandharipande in 2001 showed the Witten's conjecture using the ELSV formula which relates intersection numbers with Hurwitz numbers (as a corollary, they get $R^{3g-3+n}(\overline{\mathscr{M}}_{g,n}) \cong \mathbb{Q}$);

(iv) Mirzakhani in 2004 showed the Witten's conjecture with a formula which relates intersection numbers with volumes of moduli spaces.

There are several conjectures about Poincaré duality of moduli of curves:

Conjecture 2 (Poincaré duality conjecture for stable curves (Hain-Looijenga))**.** *Let $d = \dim \overline{\mathscr{M}}_{g,n} = 3g-3+n$, then*

(I) $R^i(\overline{\mathscr{M}}_{g,n}) = 0$ for $i > d$ (which is obvious);

(II) $R^d(\overline{\mathscr{M}}_{g,n}) \cong \mathbb{Q}$ (ture by ELSV formula);

(III) the natural pairing $R^i(\overline{\mathscr{M}}_{g,n}) \times R^{d-i}(\overline{\mathscr{M}}_{g,n}) \to R^d(\overline{\mathscr{M}}_{g,n}) \cong \mathbb{Q}$ is perfect.

This conjecture was motivated by an earlier conjecture by Faber:

Conjecture 3 (Faber's conjecture for smooth curves)**.** *(I) $R^*(\mathscr{M}_g)$ is a dimension $g-2$ Poincaré ring;*

(II) The $\lfloor g/3 \rfloor$ classes $\kappa_1, ..., \kappa_{\lfloor g/3 \rfloor}$ generated the ring , with no relations in degree $\lfloor g/3 \rfloor$ (is true in cohomology by Morita, Ionel; Ionel's proof should extend to the Chow ring without difficulty);

(III) We have

$$\psi_1^{d_1+1} \cdots \psi_k^{d_k+1} = \frac{(2g-3+k)!(2g-1)!!}{(2g-1)! \prod_{j=1}^k (2d_j+1)!!} = \sum_{\sigma \in \mathfrak{S}_k} \kappa_\sigma$$

which defined as before (proved by Getzler, Pandharipande, Givental).

Remark 5.145. *E. Looijenga showed that $R^i(\mathscr{M}_g) = 0$ for $i > g-2$ and $R^{g-2}(\mathscr{M}_g) \cong \mathbb{Q}$.*

5.5.3 Cohomology of moduli space of curves

•Collections of results about cohomology groups.

We will consider about the rational (singular)-cohomology (or homology) groups of moduli space of smooth or stable curves over $\mathbb{C}$. We note that as in [30], we have $H^i(\mathscr{M}) \otimes \mathbb{Q} \cong H^i(M, \mathbb{Q}) := H^i(M(\mathbb{C}), \mathbb{Q})$, so we may just consider the coarse moduli space.

Theorem 5.146 (Harer 1986)**.** *For $i > 4g-5$, we have $H_i(M_{g,n}, \mathbb{Q}) = 0$ for $i > c(g,n)$ where*

$$c(g,n) = \begin{cases} n-3, & g = 0; \\ 4g-5, & g > 0, n = 0; \\ 4g-4-n, & g > 0, n > 0. \end{cases}$$

Proof. Using the construction of Teichmüller method, see chapter XIX Theorem 2.2 in [8]. □

Theorem 5.147. *The following hold true:*

(i) (Mumford, Harer) $H^1(M_{g,n};\mathbb{Q}) = 0$ for any $g \geq 1$ and any n such that $2g - 2 + n > 0$;

(ii) (Harer) $H^2(M_{g,n};\mathbb{Q})$ is freely generated by $\kappa_1, \psi_1, ..., \psi_n$ for any $g \geq 3$ and any n. $H^2(M_{2,n};\mathbb{Q})$ is freely generated by $\psi_1, ..., \psi_n$ for any n, while $H^2(M_{1,n};\mathbb{Q})$vanishes for all n.

Proof. For the modern proof using Deligne's spectral sequence, we refer Theorem 10 in [7]. □

Theorem 5.148 (Arbarello-Cornalba [6], 1998; Bergström-Faber-Payne [15], 2022; Canning-Larson-Payne [20], 2022)**.** *(i) The cohomology groups $H^k(\overline{M}_{g,n}) = 0$ for all odd $k \leq 9$;*

(ii) The cohomology groups $H^2(\overline{M}_{g,n})$ generated by κ_1, ψ_i and δ_{irr} and $\delta_{a,A}$ for $0 \leq a \leq g$ and $2a - 2 + \sharp A \geq 0$ and $2(g-a) - 2 + \sharp A^c \geq 0$, with relations

(ii-a) If $g > 2$ all relations are generated by $\delta_{a,A} = \delta_{g-a,A^c}$;

(ii-b) If $g = 2$ all relations are generated by $\delta_{a,A} = \delta_{g-a,A^c}$ and

$$5\kappa_1 = 5\psi + \delta_{irr} - 5\sum_A \delta_{0,A} + 7\sum_A \delta_{1,A};$$

(ii-c) If $g = 1$ all relations are generated by $\delta_{a,A} = \delta_{g-a,A^c}$ and

$$\kappa_1 = \psi - 5\sum_A \delta_{0,A}, 12\psi_p = \delta_{irr} + 12\sum_{p\in S,\sharp S\geq 2} \delta_{0,S}, p \in \{1, ..., n\};$$

(ii-d) If $g = 0$ all relations are generated by $\delta_{a,A} = \delta_{g-a,A^c}$ and

$$c(g,n) = \begin{cases} \kappa_1 = \sum_{x,y\notin A}(\sharp A - 1)\delta_{0,A}, & x, y \in \{1, ..., n\}, x \neq y; \\ \psi_z = \sum_{z\in A;x,y\notin A} \delta_{0,A}, & x, y, z \in \{1, ..., n\}, x, y, z \text{ distinct}; \\ \delta_{irr} = 0. & \end{cases}$$

(iii) The group $H^{11}(\overline{M}_{g,n})$ nonzero if and only if $g = 1$ and $n \geq 11$, and in this case we have $H^{11}(\overline{M}_{1,n}) \cong (H^{11}(\overline{M}_{1,11}))^{\oplus\binom{n-1}{10}}$;

(iv) Assume $k \leq 11$. Let $g_1, ..., g_k$ be distinct positive integers, and set $g = 1 + g_1 + \cdots + g_k$. Then $H^{11+2k}(\overline{M}_{g,n}) \neq 0$ for $n \geq 11 - k$.

Proof. (i)(ii) In paper [6], E. Arbarello and M. Cornalba showed when $k = 1, 2, 3, 5$ and in paper [15], Jonas Bergström, Carel Faber and Sam Payne showed when $k = 7, 9$.

(iii)(iv) In paper [20], Samir Canning, Hannah Larson and Sam Payne showed these. □

Remark 5.149. *The funny point is that, in papers [15] and [20], they using some number-theoric method, such as Hasse-Weil zeta functions.*

A Appendix. Useful results in basic algebraic geometry

A.1 Some corollaries of semi-continuity theorem

Review A.1 (Cohomology and Base Change, see [58] III.12.11). *Let $f : X \to Y$ be a proper and finitely presented morphism of schemes with a finitely presented sheaf on X which is flat over Y. Let a point $y \in Y$ and $i \in \mathbb{Z}$, the comparison map $\phi_y^i : R^i f_* F \otimes \kappa(y) \to H^i(X_y, F_y)$ is surjective. Then*

(i) There is an open neighborhood $V \subset Y$ of y such that for any morphism $g : Y' \to V$ of schemes, the comparison map $\phi_{Y'}^i : g^* R^i f_* F \to R^i f'_* (g')^* F$ is an isomorphism. In particular ϕ_y^i is an isomorphism;

(ii) ϕ_y^{i-1} is surjective if and only if $R^i f_* F$ is locally free in a neighborhood of y.

Review A.2 (Grauert's Corollary). *(See [1] A.7.16) Let $f : X \to Y$ be a flat proper morphism of noetherian schemes such that $h^0(X_y, \mathscr{O}_y) = 1$ for all $y \in Y$ ($\Leftrightarrow \mathscr{O}_Y = f_* \mathscr{O}_X$ and stable under base-change) (resp. the geometric fibers are integral).*

For a line bundle L on X, consider the functor $(Sch/Y) \to (Sets)$ by sending $T \to Y$ to $\{*\}$ if L_T is the pullback of a line bundle on T and to $\emptyset$ otherwise. Then this functor is representable by a locally closed (resp. closed) subscheme of Y.

A.2 Artin approximation and its corollaries

Definition A.3. *Let $A \to B$ be a map of noetherian rings. It is called geometrically regular if it is flat and for every prime ideal $\mathfrak{p} \subset A$ and any finite field extension $K/\kappa(\mathfrak{p})$, the fiber $B \otimes_A K$ is regular.*

A noetherian local ring R is called a G-ring if $A \to \widehat{A}$ is geometrically regular.

Theorem A.4 (Artin approximation, see [1] A.10.9). *Let S be a scheme and $s \in S$ be a point such that $\mathscr{O}_{S,s}$ is G-ring. Let $F : (Sch/S) \to (Sets)$ be a colimit preserving contravariant functor (commutes with systems of $\mathscr{O}_S$-algebras) and $\widehat{\xi} \in F(\mathrm{Spec}\widehat{\mathscr{O}}_{S,s})$. For any integer $N \geq 0$, there exists an étale morphism $(S', s') \to (S, s)$ and $\xi' \in F(S')$ with $\kappa(s) = \kappa(s)'$ such that the restrictions of $\widehat{\xi}$ and ξ' to $\mathrm{Spec}(\mathscr{O}_{S,s}/\mathfrak{m}_s^{N+1})$ are equal.*

Corollary A.5 (See [1] A.10.13). *Let X_1, X_2 be schemes of finite type over S and let $s \in S$ be a point such that $\mathscr{O}_{S,s}$ is a G-ring. If $x_1 \in X_1, x_2 \in X_2$ are points over s such that $\widehat{\mathscr{O}}_{X_1,x_1}$ and $\widehat{\mathscr{O}}_{X_2,x_2}$ are isomorphic as $\mathscr{O}_{S,s}$-algebras, then there exists a common residually-trivial étale neighborhood as*

$$\begin{tikzcd} & (X_3, x_3) \arrow[dl] \arrow[dr] & \\ (X_1, x_1) & & (X_2, x_2) \end{tikzcd}$$

A.3 Some birational geometry of surfaces

Here we give some well-known results of birational geometry of surfaces.

Theorem A.6 (Minimal Resolutions)**.** *Let X be a surface. There exists a unique projective birational morphism $\pi : \widetilde{X} \to X$ from a smooth surface such that every other resolution $Y \to X$ factors as $Y \to \widetilde{X} \to X$ (or equivalently such that $K_X \cdot E \geq 0$ for every π-exceptional curve E).*

Proof. See [67] Theorem 2.16. □

Theorem A.7 (Embedded Resolutions of Curves in Surfaces)**.** *Let X be a surface and $X_0 \subset X$ be a curve. There is a finite sequence of blow-ups at reduced points of X_0 yielding a projective birational morphism $\widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and such that the preimage $\widetilde{X}_0$ of X_0 has set-theoretic normal crossings, i.e. $(\widetilde{X}_0)_{red}$ is nodal.*

Proof. See [67] Theorem 1.47. □

Theorem A.8 (Castelnuovo's Contraction Theorem)**.** *Let X be a smooth projective surface and E a smooth rational (-1)-curve. Then there is a projective birational morphism $X \to Y$ to a smooth surface and a point $y \in Y$ such that $f^{-1}(y) = E$ and $X \backslash E \to Y \backslash \{y\}$ is an isomorphism.*

Proof. See [67] Theorem 2.14. □

Corollary A.9 (Existence of Relative Minimal Models)**.** *A smooth surface X admits a projective birational morphism $X \to X_{\min}$ to a smooth surface such that every projective birational morphism $X_{\min} \to Y$ to a smooth surface is an isomorphism. In particular $X_{\min}$ has no smooth rational (-1)-curves.*

A.4 Miscellany

Review A.10. *Let k be a field and X be a proper geometrically connected and geometrically reduced k-scheme, then $\Gamma(X, \mathscr{O}_X) = k$.*

Proof. This is almost trivial. See [46] Proposition 12.66 or St 0BUG in [11]. □

Review A.11 (Openness of ampleness)**.** *Let $X \to S$ be a proper morphism of schemes and L be a line bundle over X. Let S is noetherian. If for some $s \in S$, the fiber L_s over X_s is ample (resp. very ample), then exists an open neighborhood U of s such that L_U is ample (resp. very ample) over X_U.*

Proposition A.12 (Openness of nefness)**.** *Let X be a proper and flat scheme over a DVR R and $\mathscr{E}$ be a vector bundle on X. Let $0, \eta \in \mathrm{Spec}R$ be the closed and generic points. If $\mathscr{E}|_0$ is nef, then so is $\mathscr{E}|_\eta$.*

Proposition A.13 (St 0C45)**.** *Let X be a locally Noetherian scheme of dimension 1 with normalization $f : \widetilde{X} \to X$. Then*

(1) f is integral (finite if X is reduced locally finite type over a field), surjective, and induced a bijection on irreducible components;

(2) there exists a factorization $\widetilde{X} \to X_{red} \to X$ such that the morphism $\widetilde{X} \to X_{red}$ is the normalization of X_{red} and it is birational;

(3) for every closed point $x \in X$ we have $(f_*\mathscr{O}_{\widetilde{X}})_x$ is the integral closure of $\mathscr{O}_{X,x}$ in total ring of fractions of $(\mathscr{O}_{X,x})_{red} = \mathscr{O}_{X_{red},x}$;

(4) $\widetilde{X}$ is a disjoint union of integral normal Noetherian schemes.

Proposition A.14 (0B5V)**.** *Let $f : X \to Y$ be a morphism of schemes proper over Noetherian ring R. Let L be a line bundle over Y and assume f is finite and surjective. Then L is ample if and only if f^*L is ample.*

Proposition A.15 (Canonical bundle and blowing ups)**.** *Let X be a regular variety and let Y be a regular subvariety of codimension $r \geq 2$. Let $\pi : X' = \mathrm{Bl}_Y X \to X$ be the blowing up with exceptional divisor E, then*

$$\omega_{X'} \cong \pi^*\omega_X \otimes \mathscr{O}_{X'}((r-1)E).$$

Proof. This is Exercise II.8.5 in [58]. □

Proposition A.16 (Picard groups and blowing ups)**.** *Let X be a regular variety and let Y be a regular subvariety of $\mathrm{codim}_X(Y) = r \geq 2$. Let $\pi : X' = \mathrm{Bl}_Y X \to X$ be the blowing up of X along Y and let E be the exceptional divisor. Then the map $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X')$ given by functorialtiy of the Picard group and the map $\mathbb{Z} \to \mathrm{Pic}(X')$ defined by $n \mapsto nE$ define an isomorphism $\mathrm{Pic}(X) \oplus \mathbb{Z} \cong \mathrm{Pic}(X')$.*

Proof. Let $U = X - Y$ and we have $\mathrm{Pic}(X) \cong \mathrm{Pic}(U)$ is similar as $\mathrm{Pic}(X) \xrightarrow{\pi^*} \mathrm{Pic}(X') \to \mathrm{Pic}(U)$. Hence $\mathrm{Pic}(X) \xrightarrow{\pi^*} \mathrm{Pic}(X') \to \mathrm{Pic}(X)$ is identity. Consider $\mathbb{Z} \to \mathrm{Pic}(X') \to \mathrm{Pic}(X) \to 0$ is exact, we just need to find a splitting for $\mathbb{Z} \to \mathrm{Pic}(X')$.

The closed immersion induce $\mathrm{Pic}(X') \to \mathrm{Pic}(E)$. As E is a projective bundle over Y, then $\mathrm{Pic}(E) \cong \mathrm{Pic}(Y) \oplus \mathbb{Z}$ as regularness by [58] Exercise II.7.9(a). Hence we get

$$f : \mathbb{Z} \to \mathrm{Pic}(X') \to \mathrm{Pic}(E) \cong \mathrm{Pic}(Y) \oplus \mathbb{Z} \to \mathbb{Z}$$

which sends $1 \mapsto \mathscr{O}_{X'}(E) \cong \mathscr{O}_{X'}(-1) \mapsto \mathscr{O}_E(-1) \mapsto -1$. Hence consider $-f$ and we win! □

Definition A.17. *Fix a variety X over a field.*

(i) Define the canonical ring $R(X) = \bigoplus_{m \geq 0} H^0(X, mK_X)$, we define the Kodaira dimension as

$$\kappa(X) := \begin{cases} -\infty, & \textit{if } R(X) = \mathbb{C}, \\ \mathrm{trdeg}_{\mathbb{C}}\mathrm{Frac}(R(X)) - 1, & \textit{otherwise}; \end{cases}$$

(ii) Define X is of general type if $\kappa(X) = \dim X$ (This if and only if K_X is big).

Definition A.18. *Let X be a proper algebraic variety of dimension n over an algebraically field k.*

(a) We call X is rational if there is a birational map $X \dashrightarrow \mathbb{P}^n$;

(b) we call X is unirational if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$;

(c) we call X is uniruled if there is a variety Y of dimension $n-1$ with a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$.

Proposition A.19. *Let $f : C \to B$ be a smooth proper flat and finitely presented morphism of relative dimension $= 1$ with geometric fibres isomorphic to $\mathbb{P}^1$.*

(i) If f admits a section $s : B \to C$, then there exists a rank 2 vector bundle $\mathscr{E}$ on B such that C is isomorphic to the projective bundle $C \cong \mathbb{P}(\mathscr{E})$ over B;

(ii) If f admits two disjoint sections $s_1, s_2 : B \to C$, then the bundle $\mathscr{E}$ splits as a direct sum $\mathscr{E} \cong \mathscr{L}_1 \oplus \mathscr{L}_2$ of line bundles;

(iii) If f admits three disjoint sections $s_1, s_2, s_3 : B \to C$, then we can take $E = \mathscr{O}_B^{\oplus 2}$ above, so that $C \cong B \times \mathbb{P}^1$.

Proof. (i) Note that since $f : C \to B$ is proper and the composition $f \circ s$ is a closed embedding, then s is a closed embedding. Since the image of s is an effective Cartier divisor when restricted to each fibre, by Tag 062Y it defines a relative effective Cartier divisor. Let $\mathscr{L} = \mathscr{O}_C(s)$ and we claim that $\mathscr{E} = f_*\mathscr{L}$ is a locally free sheaf on B of rank 2. This is directly follows from Review A.1. The fact $C \cong \mathbb{P}(\mathscr{E})$ follows by the similar proof in [58] Proposition V.2.2;

(ii) By (i) we can obtain line bundles $\mathscr{L}_i = \mathscr{O}(s_i)$ for $i = 1, 2$ and we claim that there exists $\mathscr{M}$ a line bundle on B with $\mathscr{L}_1^\vee \otimes \mathscr{L}_2 = f^*\mathscr{M}$. This can be easily deduced by Review A.2 (see [79] Proposition 28.1.11). Let $s_0 \in H^0(C, \mathscr{L}_1)$ be the section vanishing along s_1 and $s_\infty \in H^0(C, \mathscr{L}_2)$ the section vanishing along s_2. Then we have a map of locally free sheaves on B:

$$\Psi : \mathscr{O}_B \oplus \mathscr{M}^\vee \to \mathscr{E}, (a, b) \mapsto as_0 + bs_\infty.$$

Here the section bs_∞ makes sense since by projection formula we have $\mathscr{E} = \mathscr{M}^\vee \otimes f_*\mathscr{L}_2$. On an open cover of B which trivializes $\mathscr{E}$, it is easy to check that Ψ is an isomorphism. This open cover also trivializes the line bundle $\mathscr{M}$ and then the sections s_0, s_∞ restrict to a basis of the sections of $\mathscr{L}$ on the fibres of f (since s_1, s_2 are disjoint);

(iii)Now we have the third s_3, then by (ii) we have

$$\mathscr{M} = s_3^* f^* \mathscr{M} = s_3^* \mathscr{O}_C(-s_1 + s_2) = \mathscr{O}_B,$$

hence again by (ii) we have $\mathscr{E} = \mathscr{O}_B^{\oplus 2}$. $\square$

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