NOTES ON THE GEOMETRY OF HYPERTORIC VARIETIES

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Abstract. In this note we will introduce the basic theory of hypertoric varieties.

Contents

1. Basic definitions and resolutions of hypertoric varieties

1.1. **About Poisson and symplectic structures and symplectic resolutions.** Here we give an introduction of these and we refer [Bea00] and [Fu06] for more details. See also [Fu03] for more examples and results. We work over $\mathbb{C}$.

Definition 1.1. *We consider complex algebraic schemes.*

- *We say a scheme X carries a* Poisson structure *if there is a $\mathbb{C}$-bilinear operation*
$$\{-,-\} : \mathscr{O}_X \times \mathscr{O}_X \to \mathscr{O}_X$$
which is a Lie bracket.
- *Let $f : X \to Y$ be a morphism of Poisson schemes, we say it is a* Poisson morphism *if it induce a homomorphism of Lie algebras.*

Remark 1.2. *Any Poisson structure can be induced by the $\mathscr{O}_X$-linear homomorphism $H : \Omega^1_X \to T_X = \mathrm{Der}(\mathscr{O}_X, \mathscr{O}_X)$ such that $\{f, g\} = H(df)(g)$. In particular, any symplectic variety has a canonical Poisson structure.*

We also have the relative version of Poisson schemes and we omit them here.

Definition 1.3. *Let Y_0 be a normal variety.*

- *A pair (Y_0, ω_0) of the normal algebraic variety Y_0 and a 2-form ω_0 on the smooth locus $(Y_0)_{\mathsf{sm}}$ is called a* symplectic variety *if ω_0 is symplectic and there exists (or equivalently, for any) a resolution $\pi : Y \to Y_0$ such that the pull-back of ω_0 by π extends to a holomorphic 2-form ω on Y.*
- *The resolution $\pi : Y \to Y_0$ is called* symplectic *if ω is also symplectic.*

Some basic properties:

Date: December 2, 2024.
2020 *Mathematics Subject Classification.* 14M25, 14J42 .
Key words and phrases. hypertoric varieties, GIT quotient.

Proposition 1.4 (Namikawa, 2001)**.** *A normal variety is symplectic if and only if it has only rational Gorenstein singularities and its smooth part admits a holomorphic symplectic form.*

Proof. See also Theorem 1.2 in [Fu06]. □

Proposition 1.5 (Prop.1.6 in [Fu06])**.** *Let W be a symplectic variety with a resolution $\pi : Z \to W$, then the following statements are equivalent:*

(1) π is crepant;
(2) π is symplectic;
(3) K_Z is trivial.

Next, we now care about the following special case:

Definition 1.6. *An affine symplectic variety $(Y_0 = \operatorname{Spec} R, \omega_0)$ with $\mathbb{C}^*$-action (called conical $\mathbb{C}^*$-action) is called a conical symplectic variety if it satisfies:*

- *The grading induced from the $\mathbb{C}^*$-action to the coordinate ring R is positive, i.e., $R = \bigoplus_{i\geqslant 0} R_i$ and $R_0 = \mathbb{C}$.*
- *ω_0 is homogeneous with respect to the $\mathbb{C}^*$-action, i.e., there exists $\ell \in \mathbb{Z}$ (the weight of ω_0) such that $t^*\omega_0 = t^\ell \omega_0$ ($t \in \mathbb{C}^*$).*

Remark 1.7. *We can show that the weight ℓ is always positive.*

1.2. **Algebraic symplectic quotients and hypertoric varieties.** Note that hypertoric varieties are examples of symplectic varieties.

Consider the exact sequence

$$0 \to \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \to 0$$

where $A = [\boldsymbol{a}_1, ..., \boldsymbol{a}_n] \in M_{d\times n}(\mathbb{Z})$ and $B^T = [\boldsymbol{b}_1, ..., \boldsymbol{b}_n] \in M_{(n-d)\times n}(\mathbb{Z})$ (the Gale duality of $\{\boldsymbol{a}_1, ..., \boldsymbol{a}_n\}$). Acting $\operatorname{Hom}(-, \mathbb{C}^*)$ we get

$$1 \to \mathbb{T}^d \xrightarrow{A^T} \mathbb{T}^n \xrightarrow{B^T} \mathbb{T}^{n-d} \to 1$$

an exact sequence of algebraic tori.

Via the natural action of $\mathbb{T}^n$ on $T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$ as

$$(t_1, ..., t_n) \cdot (z_1, ..., z_n, w_1, ..., w_n) = (t_1 z_1, ..., t_n z_n, t_1^{-1} w_1, ..., t_n^{-1} w_n),$$

we have the action of $\mathbb{T}^d$ on $T^*\mathbb{C}^n \cong \mathbb{C}^{2n}$ as

$$\boldsymbol{t} \cdot (z_1, ..., z_n, w_1, ..., w_n) = (\boldsymbol{t}^{\boldsymbol{a}_1} z_1, ..., \boldsymbol{t}^{\boldsymbol{a}_n} z_n, \boldsymbol{t}^{-\boldsymbol{a}_1} w_1, ..., \boldsymbol{t}^{-\boldsymbol{a}_n} w_n)$$

where $\boldsymbol{t}^{\boldsymbol{a}_i} := t_1^{a_{1,i}} \cdots t_d^{a_{d,i}}$. The moment map of this given by

$$\mu : T^*C^n \to \mathfrak{t}_d^* = \mathbb{C}^d, \quad (z_1, ..., z_n, w_1, ..., w_n) \mapsto \sum_{i=1}^n \boldsymbol{a}_i z_i w_i.$$

Definition 1.8. *Fix a character $\alpha \in \mathbb{Z}^d = \operatorname{Hom}(\mathbb{T}^d, \mathbb{C}^*)$ and a point $\xi \in \mathbb{C}^d$.*

- *We define the Lawrence toric variety as*

$$X(A, \alpha) := (\mathbb{C}^{2n})^{\alpha\text{-}ss} /\!/ \mathbb{T}^d = \operatorname{Proj}\left(\bigoplus_{k\geqslant 0} \mathbb{C}[z_i, w_j]^{\mathbb{T}^d, k\alpha}\right)$$

where $(\mathbb{C}^{2n})^{\alpha\text{-}ss} = \{u \in \mathbb{C}^{2n}$: there exists $f \in \mathbb{C}[z_i, w_j]$ such that $f(u) \neq 0$ and $\sigma(f) = \alpha^*(t)^k \otimes f$ for $k > 0\}$ where $\mathbb{C}^* = \operatorname{Spec} \mathbb{C}[t, 1/t]$ and coaction morphism $\sigma : \mathbb{C}[z_i, w_j] \to \Gamma(\mathscr{O}_{\mathbb{T}^d}) \otimes \mathbb{C}[z_i, w_j]$. Note that $\mathbb{C}[z_i, w_j]^{\mathbb{T}^d, k\alpha} = \{f \in \mathbb{C}[z_i, w_j] : \sigma(f) = \alpha^*(t)^k \otimes f\}$.

- *We define the hypertoric variety (or toric hyperkähler variety) as*

$$Y(A, \alpha, \xi) := \mu^{-1}(\xi)^{\alpha\text{-}ss} /\!/ \mathbb{T}^d = \operatorname{Proj}\left(\bigoplus_{k\geqslant 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k\alpha}\right)$$

similar as above.

Remark 1.9. *The Poisson structures coming from the usual symplectic structure $\omega_{\mathbb{C}} = \sum_{j=1}^n dz_j \wedge dw_j$ on $T^*\mathbb{C}^n = \mathbb{C}^n \oplus \mathbb{C}^{n*}$.*

Remark 1.10. *We can write the semistable locus as follows:*

$$(\mathbb{C}^{2n})^{\alpha\text{-}ss} = \left\{ (z_i, w_j) \in \mathbb{C}^{2n} : \alpha \in \sum_{i: z_i \neq 0} \mathbb{Q}_{\geqslant 0} \boldsymbol{a}_i + \sum_{j: w_j \neq 0} \mathbb{Q}_{\geqslant 0} (-\boldsymbol{a}_j) \right\}$$

and $\mu^{-1}(\xi)^{\alpha\text{-}ss} = \mu^{-1}(\xi) \cap (\mathbb{C}^{2n})^{\alpha\text{-}ss}$.

Remark 1.11. *Note that we have a natural projective morphism* $\Pi : X(A, \alpha) \to X(A, 0)$ *and* $\pi : Y(A, \alpha, \xi) \to Y(A, 0, \xi)$ *with the same reason. Indeed, we consider the case of hypertoric varieties. Note that*

$$Y(A, 0, \xi) = \operatorname{Proj}\left(\bigoplus_{k \geqslant 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k \cdot 0} \right) = \operatorname{Spec} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d}.$$

Then inclusion $\mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d} \subset \bigoplus_{k \geqslant 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k\alpha}$ *induce* $\operatorname{Spec} \bigoplus_{k \geqslant 0} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d, k\alpha} \to \operatorname{Spec} \mathbb{C}[\mu^{-1}(\xi)]^{\mathbb{T}^d}$. *Since the grade induced by* $\mathbb{C}^*$*-action and this morphism is* $\mathbb{C}^*$*-invariant, then we get* $\pi : Y(A, \alpha, \xi) \to Y(A, 0, \xi)$. *Note moreover that* $\mu^{-1}(\xi)^{\alpha\text{-}ss} \subset \mu^{-1}(\xi) = \mu^{-1}(\xi)^{0\text{-}ss}$.

Remark 1.12. *As* $X(A, 0)$ *and* $Y(A, 0)$ *are affine, we can find their coordinate rings. We refer this in Lemma 2.17 in* [Nag21].

Remark 1.13. *The hypertoric varieties are the special case of the following general contruction.*

Consider a reductive group G *and a representation* V. *Then we form* $T^*V = V \oplus V^*$ *which comes with a moment map* $\Phi : T^*V \to \mathfrak{g}^*$ *given by cup of* $T_x V^* \to \mathfrak{g}^*$ *as* $T_e G \to T_x(Gx) \subset T_x V$. *We fix a character* $\chi : G \to \mathbb{C}^\times$ *and form the GIT quotient*

$$\Phi^{-1}(\xi) /\!\!/_\chi G := \Phi^{-1}(\xi)^{\chi\text{-}ss} /\!\!/ G = \operatorname{Proj}\left(\bigoplus_{n \geqslant 0} \mathbb{C}[\Phi^{-1}(\xi)]^{G, n\chi} \right).$$

We have a natural projective morphism as before

$$\pi : Y := \Phi^{-1}(\xi) /\!\!/_\chi G \to X := \Phi^{-1}(\xi) /\!\!/_0 G = \operatorname{Spec} \mathbb{C}[\Phi^{-1}(0)]^G$$

carry Poisson structures coming from the usual symplectic structure on T^*V. *This construction will not usually give a symplectic resolution; for example,* Y *may not be smooth and* $Y \to X$ *might not be birational. Here in the physics literature,* Y *is called the* Higgs branch of the 3d supersymmetric gauge theory defined by G, V. G *is called the* gauge group *and* N *is called the* matter. *Similarly, there is a conical* $\mathbb{C}^\times$ *action on* Y *coming from its scaling action of* T^*V.

As another special case, Nakajima defined the quiver varieties and we omit it here.

Proposition 1.14 (Trivial)**.** *If* $A \in \operatorname{Mat}_{d \times n}(\mathbb{Z})$ *of form* $A = A_1 \oplus \cdots \oplus A_s$ *for* $A_i \in \operatorname{Mat}_{k_i \times l_i}(\mathbb{Z})$, *then we have the natural* $\mathbb{C}^* \times \mathbb{T}^{n-d}$*-equivariant isomorphism of Poisson varieties*

$$X(A, \alpha) \cong \prod_j X(A_j, \alpha_j), \quad Y(A, \alpha) \cong \prod_j Y(A_j, \alpha_j)$$

where $\alpha_j := p_j(\alpha)$ *for* $p_j : \mathbb{C}^d \to \mathbb{C}^{k_j}$.

1.3. **Symplectic resolutions of hypertoric varieties.** We will consider when $\pi : Y(A, \alpha, \xi) \to Y(A, 0, \xi)$ will be a symplectic resolution. So we need to consider the condition that $\mu^{-1}(\xi)^{\alpha\text{-ss}} = \mu^{-1}(\xi)^{\alpha\text{-st}}$. First we will compute their stabilizer group.

Let $(\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{C}^{2n}$ and set $J_{\boldsymbol{z},\boldsymbol{w}} := \{ j \in \{1, ..., n\} : z_j \neq 0 \text{ or } w_j \neq 0 \}$, then we have

$$\operatorname{Stab}_{\boldsymbol{z},\boldsymbol{w}} \mathbb{T}^d = \ker(\mathbb{T}^d \xrightarrow{A^T_{J_{\boldsymbol{z},\boldsymbol{w}}}} \mathbb{T}^{|J_{\boldsymbol{z},\boldsymbol{w}}|}).$$

Hence by some linear algebra, that is, $A^T_{J_{\boldsymbol{z},\boldsymbol{w}}} : \mathbb{T}^d \to \mathbb{T}^{|J_{\boldsymbol{z},\boldsymbol{w}}|}$ is injective iff $A_{J_{\boldsymbol{z},\boldsymbol{w}}} : \mathbb{Z}^{|J_{\boldsymbol{z},\boldsymbol{w}}|} \to \mathbb{Z}^d$ is surjective and $\operatorname{Stab}_{\boldsymbol{z},\boldsymbol{w}} \mathbb{T}^d$ finite iff $A_{J_{\boldsymbol{z},\boldsymbol{w}}} \otimes_{\mathbb{Z}} \mathbb{Q} : \mathbb{Q}^{|J_{\boldsymbol{z},\boldsymbol{w}}|} \to \mathbb{Q}^d$ is surjective, then we have

Corollary 1.15 (Coro.2.7 in [Nag21])**.** *We have:*

(1) $\operatorname{Stab}_{\boldsymbol{z},\boldsymbol{w}} \mathbb{T}^d$ *is finite if and only if* $\sum_{j \in J_{\boldsymbol{z},\boldsymbol{w}}} \mathbb{Q} \boldsymbol{a}_j = \mathbb{Q}^d$;
(2) $\operatorname{Stab}_{\boldsymbol{z},\boldsymbol{w}} \mathbb{T}^d = 1$ *if and only if* $\sum_{j \in J_{\boldsymbol{z},\boldsymbol{w}}} \mathbb{Z} \boldsymbol{a}_j = \mathbb{Z}^d$.

Definition 1.16. *In this setting, we call* A *is* unimodular *if all* $d \times d$*-minors of* A *are* 0 *or* ± 1.

Remark 1.17. *Note that* A *is unimodular if and only if* B *is. Note also that for a unimodular* A, *we have* $\sum_{j \in J} \mathbb{Q} \boldsymbol{a}_j = \mathbb{Q}^d$ *iff* $\sum_{j \in J} \mathbb{Z} \boldsymbol{a}_j = \mathbb{Z}^d$ *for* $J \subset \{1, ..., n\}$.

Let A is a unimodular matrix and we define

$$\mathcal{H}_A := \{H \subset \mathbb{R}^d : H \text{ is generated by some of the } \boldsymbol{a}_j \text{ and of codimension} = 1\}.$$

We say α generic if $\alpha \notin \bigcup_{H\in\mathcal{H}_A} H$.

Lemma 1.18 (Lem.2.10 and Coro.2.11 in [Nag21])**.** *In the case, for any $\alpha \in \mathbb{Z}^d$ and $\xi \in \mathbb{C}^d$, we have $(\mu^{-1}(\xi))^{\alpha\text{-}ss} \neq \varnothing$. If α generic, then $(\mu^{-1}(\xi))^{\alpha\text{-}ss} = (\mu^{-1}(\xi))^{\alpha\text{-}st}$ with free action by $\mathbb{T}^d$. In particular, if α generic then $X(A,\alpha)$ is $2n-d$-dimensional smooth Poisson variety and for any ξ, $Y(A,\alpha,\xi)$ is a $2n-2d$-dimensional smooth symplectic variety.*

Proof. The fact $(\mu^{-1}(\xi))^{\alpha\text{-ss}} \neq \varnothing$ follows from linear algebra. Now for generic α, to show $(\mu^{-1}(\xi))^{\alpha\text{-ss}} = (\mu^{-1}(\xi))^{\alpha\text{-st}}$ we just need to show $\mathbb{C}^{\alpha\text{-ss}} = \mathbb{C}^{\alpha\text{-st}}$. We only have to show that the stabilizer group at each point in $\mathbb{C}^{\alpha\text{-ss}}$ is finite by the closed orbit lemma.

To show that the stabilizer group of $\mathbb{T}^d_{\mathbb{C}}$ at each point in $(\mathbb{C}^{2n})^{\alpha\text{-ss}}$ is finite, we note that $\alpha \in \sum_{j\in J_{\boldsymbol{z},\boldsymbol{w}}} \mathbb{Q}\boldsymbol{a}_j$ holds for $(\boldsymbol{z},\boldsymbol{w}) \in (\mathbb{C}^{2n})^{\alpha\text{-ss}}$ by the fact that

$$(\mathbb{C}^{2n})^{\alpha\text{-ss}} = \left\{ (z_i, w_j) \in \mathbb{C}^{2n} : \alpha \in \sum_{i:z_i\neq 0} \mathbb{Q}_{\geqslant 0}\boldsymbol{a}_i + \sum_{j:w_j\neq 0} \mathbb{Q}_{\geqslant 0}(-\boldsymbol{a}_j) \right\}$$

On the other hand, α is generic and hence α cannot be in any hyperplanes in $\mathcal{H}_A$. Since this implies that $\sum_{j\in J_{\boldsymbol{z},\boldsymbol{w}}} \mathbb{Q}\boldsymbol{a}_j = \mathbb{Q}^d$, the stabilizer group at each point in $(\mathbb{C}^{2n})^{\alpha\text{-ss}}$ is finite by Corollary 1.15(1). Moreover, A is unimodular, we have $\sum_{j\in J_{\boldsymbol{z},\boldsymbol{w}}} \mathbb{Z}\boldsymbol{a}_j = \mathbb{Z}^d$. Thus the stabilizer group at each point in $(\mathbb{C}^{2n})^{\alpha\text{-ss}}$ is trivial by Corollary 1.15(2). The final results follows from the construction of symplectic quotient by directly check. □

Theorem 1.19 (Thm.2.16 in [Nag21])**.** *For a unimodular A and generic α and any $\xi \in \mathbb{C}^d$, the morphism*

$$\pi_\xi : Y(A,\alpha,\xi) \to Y(A,0,\xi)$$

is a projective symplectic resolution and if $\xi = 0$, then it is conical. In particular, $Y(A,0,\xi)$ is a symplectic variety and $Y(A,0,0)$ is a conical symplectic variety.

Sketch. First, we have $\mu : \mathbb{C}^{2n} \xrightarrow{\Psi} \mathbb{C}^n \xrightarrow{A} \mathbb{C}^d$ with the flat morphism $\Psi : (\boldsymbol{z},\boldsymbol{w}) \mapsto \sum_j z_j w_j \boldsymbol{e}_j$. Then from dimension counting we get $\mu^{-1}(\xi)$ is of equidimension $2n-d$. As it define by d polynomials, we know that $\mu^{-1}(\xi) \in \mathbb{C}^{2d}$ is a complete intersection and hence Cohen-Macaulay. After showing that the codimension of singular locus $\geqslant 2$, that is, we can find that the singular locus contained in $\bigcup_i \left(\mu^{-1}(\xi) \cap \{z_i = w_i = 0\}\right)$ and the latter has codimension $\geqslant 2$, then $\mu^{-1}(\xi)$ is normal by Serre's condition. One can show it is connected, hence $\mu^{-1}(\xi)$ is irreducible. As $\mathbb{T}^d$ is reductive, $Y(A,0,\xi)$ is a normal variety.

Moreover, we can show that it is birational. Indeed, $z_k w_k = 0$ defines a divisor $D_k \subset \mu^{-1}(\xi)$ and we consider the $\mathbb{T}^d$-invariant open set $U := \left(\mu^{-1}(\xi)\backslash \bigcup_k D_k\right) \cap \mu^{-1}(\xi)^{\alpha\text{-st}}$. We can show that $U \subset \mu^{-1}(0)^{0\text{-st}}$ need to add. Hence π_ξ is indentity on U and hence it is birational.

Finally the result follows from Lemma 1.18, we have a projective resolution $\pi_\xi : Y(A,\alpha,\xi) \to Y(A,0,\xi)$ such that $Y(A,\alpha,\xi)$ is a smooth symplectic variety. Consider $U_0 := \{y \in Y_0 : \dim \pi_\xi^{-1}(y) = 0\}$, then by Zariski main theorem $\operatorname{codim}_Y(Y\backslash U_0) \geqslant 2$ with isomorphism $\pi_\xi|_{\pi_\xi^{-1}(U_0)}$. Then one can easy to show that the symplectic form ω on Y can descent to ω_0 on $(Y_0)_{\mathsf{sm}}$ which is also symplectic as $\operatorname{codim}_{(Y_0)_{\mathsf{sm}}}((Y_0)_{\mathsf{sm}}\backslash U_0) \geqslant 2$. So $Y(A,0,\xi)$ is symplectic. Now by Proposition 1.5 (symplectic manifold implies that it has the trivial canonical divisor). Well done.

Finally we consider the case $\xi = 0$. The natural $\mathbb{C}^*$-action on $\mathbb{C}^{2n}$ is $s \cdot (z_1,...,z_n,w_1,...,w_n) = (s^{-1}z_1,...,s^{-1}z_n,s^{-1}w_1,...,s^{-1}w_n)$. Then $s^*\omega_{\mathbb{C}} = s^2\omega_{\mathbb{C}}$ and well done. □

Remark 1.20. *The final step is in general can be write with the same proof as follows:*

Let Y be a smooth symplectic variety with a normal algebraic variety Y_0 such that we have a projective birational morphism $f : Y \to Y_0$, then f is a symplectic resolution and Y_0 is a symplectic variety.

Here we actually have more results:

Theorem 1.21 (Namikawa, need to add refs)**.** *The matrix A as above is unimodular iff $Y(A,0)$ has a symplectic resolution.*

Theorem 1.22 (Bellamy 2023, [Bel23]). *Any hypertoric variety $Y(A, \alpha, \xi)$ is a symplectic variety without assuming A is unimodular.*

1.4. **Some basic examples.**

Example 1.23 (A_m-type surface singularity). *Consider* $0 \to \mathbb{Z} \xrightarrow{B} \mathbb{Z}^{m+1} \xrightarrow{A} \mathbb{Z}^m \to 0$ *where*

$$B = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix}.$$

Hence $Y(A, 0) \cong S_{A_m} := \left\{ \det \begin{pmatrix} u_1 & x_1 \\ y_1 & u_1^m \end{pmatrix} = 0 \right\} \cong \mathbb{C}^2/\mathbb{Z}^{m+1}$.

Example 1.24 (General minimal nilpotent orbit closure). *Consider*

$$0 \to \mathbb{Z}^k \xrightarrow{B} \mathbb{Z}^{\ell_1 + \cdots + \ell_{k+1}} \xrightarrow{A} \mathbb{Z}^{\ell_1 + \cdots + \ell_{k+1} - k} \to 0$$

where A, B given by Figure 1. Then

$$B = \left(\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ \hline 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \\ \hline 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \cdots & 0 \\ \hline & & \vdots & \\ \hline 0 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ \hline -1 & -1 & \cdots & -1 \\ \vdots & \vdots & & \vdots \\ -1 & -1 & \cdots & -1 \end{array}\right) \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} k \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \ell_1 - 1 \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \ell_2 - 1 \\ \quad\vdots \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \ell_k - 1 \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \ell_{k+1} \end{array}, \qquad A = \left(\begin{array}{cccc|c|c|c|c|c} -1 & 0 & \cdots & 0 & & & & & \\ -1 & 0 & \cdots & 0 & I_{\ell_1 - 1} & O & \cdots & O & O \\ \vdots & \vdots & & \vdots & & & & & \\ -1 & 0 & \cdots & 0 & & & & & \\ \hline 0 & -1 & \cdots & 0 & & & & & \\ 0 & -1 & \cdots & 0 & O & I_{\ell_2 - 1} & \cdots & O & O \\ \vdots & \vdots & & \vdots & & & & & \\ 0 & -1 & \cdots & 0 & & & & & \\ \hline & & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \hline 0 & 0 & \cdots & -1 & & & & & \\ 0 & 0 & \cdots & -1 & O & O & \cdots & I_{\ell_k - 1} & O \\ \vdots & \vdots & & \vdots & & & & & \\ 0 & 0 & \cdots & -1 & & & & & \\ \hline 1 & 1 & \cdots & 1 & & & & & \\ 1 & 1 & \cdots & 1 & O & O & \cdots & O & I_{\ell_{k+1}} \\ \vdots & \vdots & & & & & & & \\ 1 & 1 & \cdots & 1 & & & & & \end{array}\right)$$

$$\in M_{(\ell_1 + \cdots + \ell_{k+1} - k) \times k}(\mathbb{Z}) \qquad\qquad \in M_{(\ell_1 + \cdots + \ell_{k+1} - k) \times (\ell_1 + \cdots + \ell_{k+1})}(\mathbb{Z})$$

FIGURE 1. Matrixes of general minimal nilpotent orbit closure

$$Y(A, 0) = \overline{\mathcal{O}}^{\min}(\ell_1, ..., \ell_{k+1}) := \left\{ \begin{pmatrix} u_1 & x_{12} & \cdots & x_{1,k+1} \\ x_{21} & u_2 & \cdots & x_{2,k+1} \\ \vdots & \vdots & & \vdots \\ x_{k+1,1} & x_{k+1,2} & \cdots & u_{k+1} \end{pmatrix} \in \mathfrak{sl}_{k+1} : \right.$$

$$\left. 2 \times 2\text{-minors of} \begin{pmatrix} u_1^{\ell_1} & x_{12} & \cdots & x_{1,k+1} \\ x_{21} & u_2^{\ell_2} & \cdots & x_{2,k+1} \\ \vdots & \vdots & & \vdots \\ x_{k+1,1} & x_{k+1,2} & \cdots & u_{k+1}^{\ell_{k+1}} \end{pmatrix} \text{ is } 0 \right\}$$

of dimension $2k$. When $\ell_i = 1$ for all i, we have $\overline{\mathcal{O}}^{\min}_{A_k} := \overline{\mathcal{O}}^{\min}(1, ..., 1)$ which is the usual minimal nilpotent orbit closure of A_k-type with springer resolution $T^*\mathbb{P}^k \to \overline{\mathcal{O}}^{\min}_{A_k}$.

Actually we will focus on the case $k = 2$.

2. BASIC GEOMETRY OF HYPERTORIC VARIETIES

In this section we will introduce the core of $Y(A, 0)$ and how it determine the geometry of $Y(A, 0)$.

2.1. **Hypertoric varieties with hyperplane arrangements.** Here we consider the case $\xi = 0$. Then we define $Y(A, \alpha) := Y(A, \alpha, 0)$. It defined by

$$0 \to \mathbb{Z}^{n-d} \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \to 0$$

where $A = [\boldsymbol{a}_1, ..., \boldsymbol{a}_n] \in M_{d\times n}(\mathbb{Z})$ and $B^T = [\boldsymbol{b}_1, ..., \boldsymbol{b}_n] \in M_{(n-d)\times n}(\mathbb{Z})$.

Then we can define $H_i := \{x \in \mathbb{R}^{n-d} : x \cdot \boldsymbol{b}_i + r_i = 0\}$ for $i = 1, ..., n$ where $\boldsymbol{r} = (r_1, ..., r_n) \in \mathbb{Z}^n$ be a lifting of α along A. This defines a hyperplane arrangement $\mathcal{A} := \{H_1, ..., H_n\}$. Here we can denote $Y(\mathcal{A}) := Y(A, \alpha)$.

Definition 2.1. *In this setting, for such hyperplane arrangement $\mathcal{A}$:*

- *we call $\mathcal{A}$ is* simple *if for any subset of m hyperplanes with nonempty intersections, they intersect of codimension m.*
- *we call $\mathcal{A}$ is* unimodular *if for any $n-d$ linear independent $\{\boldsymbol{b}_{i_1}, ..., \boldsymbol{b}_{i_{n-d}}\}$ spans $\mathbb{C}^{n-d}$ over $\mathbb{Z}$.*
- *we call $\mathcal{A}$ is* smooth *if it is simple and unimodular.*

Remark 2.2. *Note that $\mathcal{A}$ is unimodular if and only if B is unimodular if and only if A is unimodular.*

Proposition 2.3 (3.2/3.3 in [BD00])**.** *The hypertoric variety $Y(\mathcal{A})$ has at worst orbifold (finite quotient) singularities if and only if $\mathcal{A}$ is simple, and is smooth if and only if $\mathcal{A}$ is smooth.*

Note that $\mathcal{A} = \{H_1, ..., H_n\}$ be a central arrangement, meaning that $r_i = 0$ for all i, so that all of the hyperplanes pass through the origin. Then we have the following result:

Corollary 2.4. *For any central arrangement $\mathcal{A}$, there exists a* simplification *$\widetilde{\mathcal{A}} = \{\widetilde{H}_1, ..., \widetilde{H}_n\}$ of $\mathcal{A}$ by which we mean an arrangement defined by the same vectors $\{\boldsymbol{b}_i\}$, but with a different choice of $\alpha, \boldsymbol{r}$ such that $\widetilde{\mathcal{A}}$ is simple. This will give us an* equivariant orbifold resolution *$Y(\widetilde{\mathcal{A}}) \to Y(\mathcal{A})$. When A is unimodular, this will give us a resolution of singularities which recover the special case of Theorem 1.19.*

2.2. **The cores and some topology properties.** Consider again $\xi = 0$. Then we have an equivariant orbifold resolution

$$\pi : Y(\widetilde{\mathcal{A}}) \to Y(\mathcal{A})$$

where $\mathcal{A} = \{H_1, ..., H_n\}$ be a central arrangement with simplification $\widetilde{\mathcal{A}} = \{\widetilde{H}_1, ..., \widetilde{H}_n\}$.

Definition 2.5. *In this case, we call $\mathfrak{c}(\widetilde{\mathcal{A}}) := \pi^{-1}([0])$ the* core *of $Y(\widetilde{\mathcal{A}})$.*

Now we will give a toric interpretation of the core $\mathfrak{c}(\widetilde{\mathcal{A}})$. For any $J \subset \{1, ..., n\}$, define the polyhedron

$$P_J := \{x \in \mathbb{R}^{n-d} : x \cdot \boldsymbol{b}_i + r_i \geqslant 0 \text{ if } i \in J \text{ and } x \cdot \boldsymbol{b}_i + r_i \leqslant 0 \text{ if } i \notin J\}.$$

Define

$$\mathfrak{E}_J := \{(\boldsymbol{z}, \boldsymbol{w}) \in T^*\mathbb{C}^n : w_i = 0 \text{ if } i \in J \text{ and } z_i = 0 \text{ if } i \notin J\}$$

and define $\mathfrak{X}_J := \mathfrak{E}_J /\!\!/_\alpha \mathbb{T}^d$, which induce the inclusion

$$\mathfrak{X}_J \hookrightarrow \mu^{-1}(0) /\!\!/_\alpha \mathbb{T}^d = Y(\widetilde{\mathcal{A}}).$$

Theorem 2.6 (Section 6 in [BD00]/ section 3.2 in [Pro04])**.** *In this setting, we have:*

1. *the scheme $\mathfrak{X}_J$ is isomorphic to the toric variety correspond to the weighted polytope P_J;*
2. *we have $\mathfrak{c}(\widetilde{\mathcal{A}}) = \bigcup_{J:P_J \text{ bounded}} \mathfrak{X}_J$, hence $\mathfrak{c}(\widetilde{\mathcal{A}})$ is a union of compact toric varieties glued together along toric subvarieties as prescribed by the combinatorics of the polytopes P_J and their intersections in $\mathbb{R}^{n-d}$.*

Sketch. Note that (1) follows from the surjectivity real moment maps and some classification theorems, see Lemma 3.8 in [Pro04]. For (2), see Proposition 3.11 in [Pro04]. □

Remark 2.7. *This is right even for $\widetilde{\mathcal{A}}$ is not simple. For the basic theory of toric varieties, we refer* [Ful93], [CLS11] *and* [Tel22].

Finally we consider some topological results.

Theorem 2.8 (6.5 in [BD00] and section 6 in [HS02])**.** *In this setting, we have:*

(1) the core $\mathfrak{c}(\widetilde{\mathcal{A}})$ is a deformation retract of $Y(\widetilde{\mathcal{A}})$;
(2) the inclusion

$$Y(\widetilde{\mathcal{A}}) = \mu^{-1}(0) /\!\!/_{\alpha} \mathbb{T}^d \hookrightarrow T^*\mathbb{C}^n /\!\!/_{\alpha} \mathbb{T}^d = X(\widetilde{\mathcal{A}})$$

is a homotopy equivalence where $X(\widetilde{\mathcal{A}})$ is the corresponding Lawrence toric variety.

So

$$\textbf{Combinatorics} \longrightarrow \textbf{Geometry/topology}$$

$$\{\text{hyperplane arrangement } \widetilde{\mathcal{A}}\} \longrightarrow \{\text{hypertoric varieties } Y(\widetilde{\mathcal{A}})\}$$

3. Universal Poisson structures and its applications

In this section we will give a concrete description of universal Poisson structure of hypertoric varieties. At the begining, we consider some general results. Here we will follows [Nag21].

3.1. Basic theories.

Definition 3.1. *For a Poisson variety $(Y, \{-,-\}_0)$ and an affine scheme $(B, 0)$ with fixed point 0, we call a Poisson B-scheme $(\mathcal{Y}, \{-,-\})$ a Poisson deformation of Y if $\mathcal{Y} \to B$ is flat, each fiber is a Poisson scheme, and the central fiber is isomorphic to $(Y, \{-,-\}_0)$ as a Poisson variety.*

A Poisson deformation $(\mathcal{Y}, \{-,-\}) \to B$ is called infinitesimal if $B = \operatorname{Spec} A$ where A is an Artinian algebra with residue field $\mathbb{C}$.

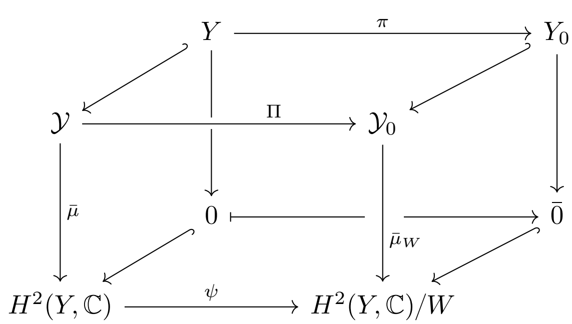
Definition 3.2. *A Poisson deformation $(\mathcal{Y}, \{-,-\}) \to B$ of a Poisson variety $(Y, \{-,-\}_0)$ is called universal at 0 if for each infinitesimal Poisson deformation $(\mathcal{X}, \{-,-\}') \to (\operatorname{Spec} A, \mathfrak{m}_A)$ there exists a unique morphism $f : \operatorname{Spec} A \to B$ such that $f(\mathfrak{m}_A) = 0$ and the diagram*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \xrightarrow{f} & B \end{array}$$

which is cartesian.

In general we have the following:

Theorem 3.3 ([Nam15])**.** *Let Y_0 be a conical symplectic variety with a projective symplectic resolution $\pi : Y \to Y_0$. Then there exists the universal Poisson deformation spaces $\mathcal{Y} \to H^2(Y, \mathbb{C})$ and $\mathcal{Y}_0 \to H^2(Y, \mathbb{C})/W$ of Y and Y_0, respectively, and they satisfy the following $\mathbb{C}^*$-commutative diagram:*



where ψ is a Galois cover with finite Galois group W acts linearly on $H^2(Y, \mathbb{C})$ which is called the Namikawa–Weyl group of Y_0.

Some comments. First, the singular locus $(Y_0)_{\text{sing}}$ can be stratified by smooth symplectic varieties. Let $\Sigma_{\text{codim} \geqslant 4}$ denote the union of strata of codimension 4 or higher, and define $\Sigma_{\text{codim2}} := (Y_0)_{\text{Sing}} \backslash \Sigma_{\text{codim} \geqslant 4}$. Then, for each component Z_k of the connected component decomposition $\Sigma_{\text{codim } 2} = \bigsqcup_{k=1}^{s} Z_k$, one can consider a transversal slice S_{ℓ_k} through a point $x \in Z_k$. Since $S_{\ell_k} = S_{\Delta_{\ell_k}}$ is a symplectic surface, i.e., the ADE type surface singularity with the corresponding Dynkin diagram Δ_{ℓ_k}, so $\pi : Y \to Y_0$ is locally (at x) isomorphic to $p \times \mathrm{id} : \tilde{S}_{\ell_k} \times \mathbb{C}^{2m-2} \to S_{\ell_k} \times \mathbb{C}^{2m-2}$, where

$2m = \dim Y_0$ and p is the minimal resolution of S_{ℓ_k}. We consider all (-2)-curves C_i $(1 \leqslant i \leqslant \ell_k)$ in $\tilde{S}_{\ell_k}$ and set

$$\Phi_{\ell_k} := \left\{ \sum_{i=1}^{\ell_k} d_i[C_i] \,\middle|\, d_i \in \mathbb{Z} \text{ s.t. } \left(\sum_{i=1}^{\ell_k} d_i[C_i]\right)^2 = -2 \right\} \subset H^2(\tilde{S}_{\ell_k}, \mathbb{R}).$$

Then, Φ_{ℓ_k} defines the corresponding ADE type root system in $H^2(\tilde{S}_{\ell_k}, \mathbb{R})$, and the associated usual Weyl group $W_{S_{\ell_k}}$ acts on $H^2(\tilde{S}_{\ell_k}, \mathbb{R})$. However this description is local at each point on Z_k, and the number of irreducible components of $\pi^{-1}(Z_k)$ may be less than ℓ_k globally. In fact, the following homomorphism is defined by the monodromy:

$$\rho_k : \pi_1(Z_k) \to \mathrm{Aut}(\Delta_{\ell_k}),$$

where Δ_{ℓ_k} is the associated Dynkin diagram and $\mathrm{Aut}(\Delta_{\ell_k})$ is its graph automorphism group. Then, we can define the subgroup of $W_{S_{\ell_k}}$ as

$$W_{Z_k} := W_{S_{\ell_k}}^{\mathrm{Im}\,\rho_k} := \{\sigma \in W_{S_{\ell_k}} : \sigma\iota = \iota\sigma, \iota \in \mathrm{Im}\,\rho_k\}.$$

Finally, taking the direct product of them, we get the Namikawa-Weyl group:

$$W := \prod_k W_{Z_k}.$$

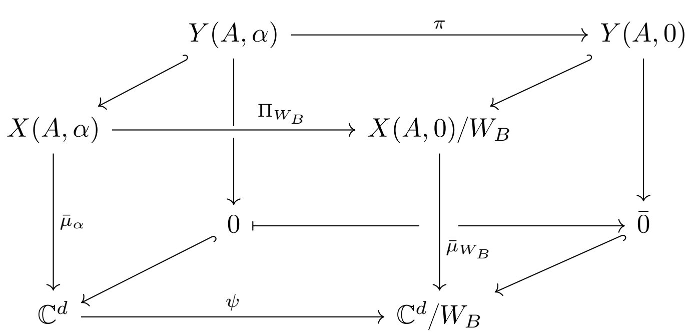
Well done. □

In our case of hypertoric varieties, we have the following results:

Theorem 3.4 (Thm 3.11 in [Nag21])**.** *Let A be a unimodular matrix and $\alpha \in \mathbb{Z}^d$ be a generic element. If for B, $\boldsymbol{b}_i \neq 0 (1 \leqslant i \leqslant n)$ and we take B as*

$$B = \begin{pmatrix} B^{(1)} \\ B^{(2)} \\ \vdots \\ B^{(s)} \end{pmatrix}, \quad B^{(k)} = \left.\begin{pmatrix} \boldsymbol{b}^{(k)} \\ \boldsymbol{b}^{(k)} \\ \vdots \\ \boldsymbol{b}^{(k)} \end{pmatrix}\right\} \ell_k$$

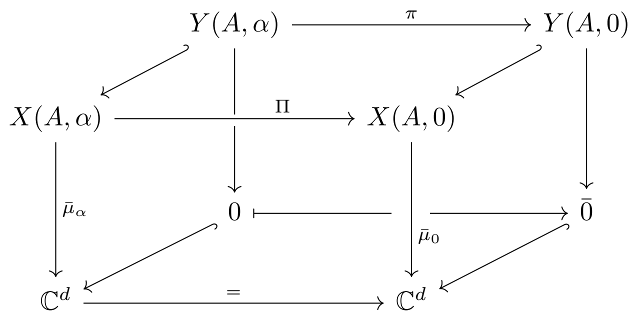
where if $k_1 \neq k_2$, then $\boldsymbol{b}^{(k_1)} \neq \pm\boldsymbol{b}^{(k_2)}$. Then the diagram of Theorem 3.3 for the affine hypertoric variety $Y(A, 0)$ is obtained as



where Π_{W_B} is the composition of $X(A, \alpha) \to X(A, 0)$ and the quotient map of $X(A, 0)$ by $W_B := \mathfrak{S}_{\ell_1} \times \cdots \times \mathfrak{S}_{\ell_s}$.

Sketch. First we need to show that $\bar{\mu}_\alpha : X(A, \alpha) \to \mathbb{C}^d$ and $\bar{\mu}_0 : X(A, 0) \to \mathbb{C}^d$ are Poisson deformations of $Y(A, \alpha)$ and $Y(A, 0)$, respectively. Note that $X(A, \alpha)$ is smooth and $X(A, 0)$ is Cohen-Macaulay by a result due to Hochster, then by miracle-flatness $\bar{\mu}_\alpha$ and $\bar{\mu}_0$ are flat. Then these are right by definition.

Next one can show that $W \subset W_B$ by analyze the singular locus carefully, using Proudfoot-Webster's result, see Theorem 3.6 and Corollary 3.7 in [Nag21]. Note that in this case we already have the following diagram:



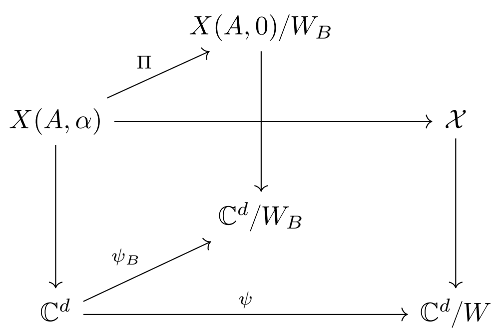
If one can construct a good W_B-action on $X(A,0)$ and $\mathbb{C}^d$, then one can show $W = W_B$ and construct the universal Poisson deformation of $Y(A,0)$ (Lemma 3.8 in [Nag21]).

Note that we have already take B as

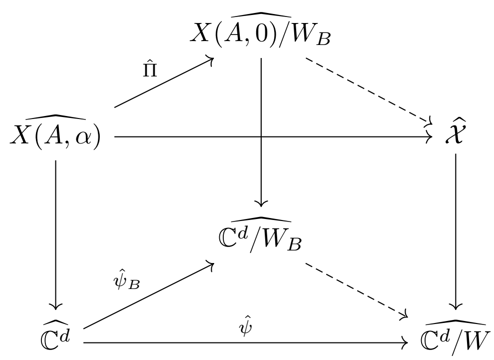
$$B = \begin{pmatrix} B^{(1)} \\ B^{(2)} \\ \vdots \\ B^{(s)} \end{pmatrix}, \quad B^{(k)} = \left.\begin{pmatrix} \boldsymbol{b}^{(k)} \\ \boldsymbol{b}^{(k)} \\ \vdots \\ \boldsymbol{b}^{(k)} \end{pmatrix}\right\} \ell_k$$

where if $k_1 \neq k_2$, then $\boldsymbol{b}^{(k_1)} \neq \pm \boldsymbol{b}^{(k_2)}$. Then we let $W_B := \mathfrak{S}_{\ell_1} \times \cdots \times \mathfrak{S}_{\ell_s}$ act $\mathbb{C}^{2n}$ as $z_i \mapsto z_{\sigma(i)}, w_i \mapsto w_{\sigma(i)}$ and act on $\mathbb{C}^n$ as $u_i \mapsto u_{\sigma(i)}$. Now one can show that W_B-action on $\mathbb{C}^{2n}$ induce an action on $X(A,0)$ and W_B-action on $\mathbb{C}^n$ induce an action on $\mathbb{C}^d$ via $A : \mathbb{C}^n \to \mathbb{C}^d$.

Now we give a sketch of the final things. Let $\mathcal{X} \to \mathbb{C}^d/W$ be the universal Poisson structure of $Y(A,0)$. Then we have



Hence after the completion at 0, we have factorization



Note that all of these are $\mathbb{C}^*$-equivariant, using this $\mathbb{C}^*$-action we can have the algebraization by Namikawa which induce $\mathbb{C}^d/W_B \to \mathbb{C}^d/W$. As $W \subset W_B$, this force $W = W_B$. Well done. □

Remark 3.5. *By definition, the W_B-action on $\mathbb{C}^{2n}$ does not commute with the $\mathbb{T}^d$-action on it in general.*

3.2. **Application 1: some classification results.** We will give an equivalent conditions of $Y(A,0) \cong Y(A',0)$ for unimodular matrixes.

Definition 3.6. *For rank d matrices $A, A' \in \mathrm{Mat}_{d\times n}(\mathbb{Z})$, we say that $A \sim A'$ if A' is obtained from A by a sequence of some elementary row operations over $\mathbb{Z}$, interchanging some column vectors, and multiplying some column vectors by -1.*

Hence $A \sim A'$ holds if and only if there exists $P \in \mathrm{GL}_d(\mathbb{Z})$ and an $n \times n$ signed permutation matrix D such that $A' = PAD$. Here, a signed permutation matrix is a product of a permutation matrix and a diagonal matrix whose all diagonal components are 1 or -1.

The following is our main theorem:

Theorem 3.7 (Arbo-Proudfoot 2016, Nagaoka 2021)**.** *For a rank d unimodular $A, A' \in \mathrm{Mat}_{d\times n}(\mathbb{Z})$, the following are equivalent:*

1. *$Y(A,0) \cong Y(A',0)$ as conical symplectic varieties, that is, $\mathbb{C}^*$-equivariant;*
2. *$Y(A,0) \cong Y(A',0)$ which is $\mathbb{C}^* \times \mathbb{T}^{n-d}$-equivariant of symplectic varieties;*
3. *$A \sim A'$;*
4. *$B^T \sim (B')^T$.*

Sketch. Note that (3) is equivalent to (4) by trivial reason. We just need to consider (1)(2)(3). As (2) implies (1) is also trivial, we will prove (1) implies (2) and the equivalence of (2)(3).

We first prove that (3) implies (2). Note that in this step we don't need the unimodular condition. Now elementary row operations don't change the image of torus embedding $A^T : \mathbb{T}^d \to \mathbb{T}^n$, then then claim is clear in the case. Next we consider interchanging some column vectors, and multiplying some column vectors by -1. This is also trivial since if we change $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$, then the new moment map is obtained from the old one with the interchange $z_i \leftrightarrow z_j$ and $w_i \leftrightarrow w_j$. This is of course $\mathbb{C}^* \times \mathbb{T}^{2n}$-equivariant on $\mathbb{C}^{2n}$ which is $\mathbb{C}^* \times \mathbb{T}^{2n-d}$-equivariant on $X(A/A',0)$ and hence $\mathbb{C}^* \times \mathbb{T}^{n-d}$-equivariant on $Y(A/A',0)$. The multiplying some column vectors by -1 is similar.

Now (2) implies (3) follows from the general results due to Arbo and Proudfoot in [AP16] by using zonotope tiling. We will finally consider (1) implies (3).

Assume we have an isomorphism $\phi : Y(A,0) \cong Y(A',0)$ as conical symplectic varieties, then the corresponding universal Poisson deformation spaces are $\mathbb{C}^*$-equivariant isomorphic to each other as Poisson varieties, moreover we have the following commutative diagram:

$$\begin{array}{ccc} X(A,0)/W_B & \xrightarrow{\Phi,\cong} & X(A',0)/W_{B'} \\ \downarrow & & \downarrow \\ \mathbb{C}^d/W_B & \xrightarrow{\cong} & \mathbb{C}^d/W_{B'} \end{array}$$

After base change to $\mathbb{C}^d$ via projection, we have

$$\begin{array}{ccc} X(A,0) & \xrightarrow{\Phi,\cong} & X(A',0) \\ & \searrow \quad \swarrow & \\ & \mathbb{C}^d & \end{array}$$

Then we just need to show that we can replace Φ (which is $\mathbb{C}^*$-equivariant) by a $\mathbb{C}^* \times \mathbb{T}^{2n-d}$ -equivariant isomorphism Ψ as Poisson varieties, which is in the form of an isomorphism induced by $\sim$. This is actually pure linear algebra where the basic idea as follows: Berchttold in [Ber03] Theorem 4.1 shows that we can replace Φ by a $\mathbb{T}^{2n-d}$-equivariant isomorphism $\Psi_\sigma : X(A,0) \cong X(A',0)$ induced from a permutation of coordinates $\tilde{\Psi}_\sigma : \mathbb{C}^{2n} \cong \mathbb{C}^{2n}$ ($\sigma \in \mathfrak{S}_{2n}$) which is equivariant with respect to an isomorphism $\psi : \mathbb{T}^d \cong \mathbb{T}^d$. This give as the following diagrams:

$$\begin{array}{ccc} \mathbb{T}^d & \xhookrightarrow{\begin{pmatrix} A^T \\ -A^T \end{pmatrix}} & \mathbb{T}^{2n} \\ {\scriptstyle \psi,\cong}\downarrow & & \downarrow{\scriptstyle \tilde{\Psi}_\sigma,\cong} \\ \mathbb{T}^d & \xhookrightarrow[\begin{pmatrix} A'^T \\ -A'^T \end{pmatrix}]{} & \mathbb{T}^{2n} \end{array} \qquad \begin{array}{ccc} \mathbb{Z}^d & \xleftarrow{\begin{pmatrix} A, -A \end{pmatrix}} & \mathbb{Z}^{2n} \\ {\scriptstyle \psi^*,\cong}\uparrow & & \uparrow{\scriptstyle \tilde{\Psi}^*_\sigma,\cong} \\ \mathbb{Z}^d & \xleftarrow[\begin{pmatrix} A', -A' \end{pmatrix}]{} & \mathbb{Z}^{2n} \end{array}$$

By the commutativity of the above diagram, note that we have

$$\psi^*\boldsymbol{a}'_i = \begin{cases} \boldsymbol{a}_{\sigma(i)} & \text{if } 1 \leqslant \sigma(i) \leqslant n, \\ -\boldsymbol{a}_{\sigma(i)-n} & \text{if } n+1 \leqslant \sigma(i) \leqslant 2n. \end{cases}$$

Now, by multiplying some of the $\boldsymbol{a}'_i$ by -1, we can assume that if $\boldsymbol{a}'_i \neq 0$ then there is no j such that $\boldsymbol{a}'_i = -\boldsymbol{a}'_j$. In this setting, we can construct a permutation $\tau \in \mathfrak{S}_n$ such that $\psi^*\boldsymbol{a}'_i = \pm\boldsymbol{a}_{\tau(i)}$ for any $1 \leqslant i \leqslant n$ as follows. First, take any bijection τ_0 between $I'_0 := \{i \in [n] \mid \boldsymbol{a}'_i = \mathbf{0}\}$ and $I_0 := \{i \in [n] \mid \boldsymbol{a}_i = \mathbf{0}\}$, then define $\tau(i) := \tau_0(i)$. Next, for $i \notin I'_0$, we define $\tau(i)$ as

$$\tau(i) := \begin{cases} \sigma(i) & \text{if } 1 \leqslant \sigma(i) \leqslant n, \\ \sigma(i)-n & \text{if } n+1 \leqslant \sigma(i) \leqslant 2n. \end{cases}$$

Then, this defines a well-defined permutation $\tau \in \mathfrak{S}_n$ since we assume that if $\boldsymbol{\alpha}'_i \neq 0$ then there is no j such that $\boldsymbol{\alpha}'_i = -\boldsymbol{\alpha}'_j$. By definition, we have

$$\psi^* a'_i = \begin{cases} \boldsymbol{a}_{\tau(i)} & \text{if } 1 \leqslant \sigma(i) \leqslant n, \\ -\boldsymbol{a}_{\tau(i)} & \text{if } n+1 \leqslant \sigma(i) \leqslant 2n. \end{cases}$$

Thus, $D := D_\tau D_\pm$ is the desired signed permutation matrix.

Then $\widetilde{\Psi}^* := (D, -D) : \mathbb{Z}^n \oplus \mathbb{Z}^n \to \mathbb{Z}^n \oplus \mathbb{Z}^n$ satisfying the above commutative diagram instead of $\widetilde{\Psi}^*_\sigma$. Hence we have $A = \psi^* A' D^{-1}$ and will done. □

Remark 3.8. *We can have the similar results for $\alpha \neq 0$.*

Remark 3.9. *Actually you can show that for the unimodular matrices A, B as above, by applying some transformations to A, B, we can get*

$$B' = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \\ & \boldsymbol{C} & \end{pmatrix}, \quad A' = \begin{pmatrix} & 1 & \cdots & 0 \\ -\boldsymbol{C} & \vdots & & \vdots \\ & 0 & \cdots & 1 \end{pmatrix}$$

where $\boldsymbol{C}$ is totally unimodular.

Corollary 3.10 (Thm 4.8 in [Nag21])**.** *Each 4-dimensional affine hypertoric variety $Y(A, 0)$ associated to a unimodular matrix A is isomorphic to exactly one of $S_{A_{\ell_1-1}} \times S_{A_{\ell_2-1}}$ and $\overline{\mathcal{O}}^{\min}(\ell_1, \ell_2, \ell_3)$.*

Sketch. This follows from Theorem 3.7 and the classifications of (the reduction) of B using the graph matoid theory and the constructions of $S_{A_{\ell_1-1}} \times S_{A_{\ell_2-1}}$ and $\overline{\mathcal{O}}^{\min}(\ell_1, \ell_2, \ell_3)$. For the classification results of these matrices, we refer Proposition 4.6 in [Nag21]. □

3.3. **Application 2: counting projective symplectic resolutions via Wall-crossings.** To count the number of projective crepent resolutions of hypertoric varieties $Y(A, 0)$ for unimodular A, we need to consider the all types of its projective crepent resolutions. Actually we have the following general result:

Theorem 3.11 (Yamagishi 2015, Braden-Proudfoot-Webster 2016)**.** *Assume a conical symplectic variety Y admits a projective crepant resolution $Y' \to Y$. Then the number of all distinct projective crepant resolutions of Y is given by*

$$\frac{\sharp\{\textit{the chambers of } \mathcal{H}_Y\}}{|W|}$$

where $\mathcal{H}_Y \subset H^2(Y', \mathbb{R})$ is the associated hyperplane arrangement whose chambers are each an ample cone of a projective crepant resolution of Y, and W is the Namikawa–Weyl group for Y.

Comments. In general, by Yamagishi in [Yam15] the number of distinct projective crepant resolutions of conical symplectic varieties (or more generally, rational Gorenstein singularities) is equal to the number of ample cones inside the movable cone. Moreover, in [BPW16], they showed that in the case of conical symplectic varieties, the movable cone is a fundamental domain with respect to the Namikawa–Weyl group action on $H^2(Y', \mathbb{R}) = \mathrm{Pic}_{\mathbb{R}}(Y')$. Then we have the result. □

Apply this into hypertoric varieties, we have the following result:

Corollary 3.12 (Nagaoya)**.** *For hypertoric variety $Y(A,0)$ associated to a unimodular matrix A, the number of all distinct projective crepant resolutions of Y is given by*

$$\frac{\sharp\{\text{the chambers of } \mathcal{H}_A\}}{|W_B|}$$

where $\mathcal{H}_A := \{H \subset \mathbb{R}^d : H$ is generated by some of the $\boldsymbol{a}_j$ and of codimension $= 1\}$ as before.

Comments. We just need to show $\mathcal{H}_{Y(A,0)} = \mathcal{H}_A$ since by Theorem 3.4 $W = W_B$. See the argumenst in Page 501 in [Nag21] which using Kirwan maps and we omitted. □

Remark 3.13. *In general, to compute the number of chambers of a hyperplane arrangement $\mathcal{H}$, we consider a more refined invariant, the characteristic polynomial $\chi_{\mathcal{H}}(t)$ defined as below.*

Recall some notations on hyperplane arrangements. Given an arrangement $\mathcal{H}$ in $\mathbb{R}^d$, let $L(\mathcal{H})$ be the set of all nonempty intersections of hyperplanes in $\mathcal{H} \subset \mathbb{R}^d$. We define the partial order $x \leqslant y$ in $L(\mathcal{H})$ if $x \supseteq y$. We call $L(\mathcal{H})$ the intersection poset of $\mathcal{H}$. For $L(\mathcal{H})$ (in general for any finite poset with the least element), we can define the Möbius function $\mu : L(\mathcal{H}) \to \mathbb{Z}$ as

$$\mu(\mathbb{R}^d) = 1 \quad \text{and} \quad \mu(x) = -\sum_{y<x} \mu(y).$$

Then we define

$$\chi_H(t) := \sum_{x \in L(\mathcal{H})} \mu(x) t^{\dim(x)},$$

where $\dim(x)$ is the dimension of x as an affine subspace of $\mathbb{R}^d$. They also showed that we have

$$\sharp\{\text{chambers of } \mathcal{H}\} = (-1)^d \chi_{\mathcal{H}}(-1).$$

Corollary 3.14 (???)**.** *For unimodular A, any projective symplectic resolutions of the hypertoric variety $Y(A,0)$ follows from $Y(A,\alpha)$ for generic α.*

Proof. I don not know this is right or not!!! □

Now we using this to deduce the number of projective crepent resolutions of $\overline{\mathcal{O}}^{\min}(\ell_1,\ell_2,\ell_3)$.

Corollary 3.15. *Let's consider $\overline{\mathcal{O}}^{\min}(\ell_1,\ell_2,\ell_3)$.*

(1) Define the hyperplane arrangement $\mathcal{H}_{\ell_1,\ell_2,\ell_3} \subset \mathbb{R}^{\ell_1+\ell_2+\ell_3}$ as the following, where we take $(x_1,\dots,x_{\ell_1},y_1,\dots,y_{\ell_2},z_1,\dots,z_{\ell_3})$ as the coordinates:

$$\mathcal{A}_{\ell_1,\ell_2,\ell_3} := \left\{ \begin{array}{lccc} H_{ijk} & : & x_i + y_j + z_k = 0 & (1 \leqslant i \leqslant \ell_1, 1 \leqslant j \leqslant \ell_2, 1 \leqslant k \leqslant \ell_3) \\ H^x_{i_1,i_2} & : & x_{i_1} - x_{i_2} = 0 & (1 \leqslant i_1 < i_2 \leqslant \ell_1) \\ H^y_{j_1,j_2} & : & y_{j_1} - y_{j_2} = 0 & (1 \leqslant j_1 < j_2 \leqslant \ell_2) \\ H^z_{k_1,k_2} & : & z_{k_1} - z_{k_2} = 0 & (1 \leqslant k_1 < k_2 \leqslant \ell_3) \end{array} \right\}.$$

Then the number of the projective symplectic resolutions of $\overline{\mathcal{O}}^{\min}(\ell_1,\ell_2,\ell_3)$ is

$$\frac{\sharp\{\text{the chambers of } \mathcal{H}_{\ell_1,\ell_2,\ell_3}\}}{\ell_1!\ell_2!\ell_3!}.$$

(2) The number of projective crepent resolutions of $\overline{\mathcal{O}}^{\min}(\ell_1,\ell_2,1)$ is

$$\binom{\ell_1+\ell_2}{\ell_1}.$$

The number of projective crepent resolutions of $\overline{\mathcal{O}}^{\min}(\ell_1,\ell_2,1)$ is

$$\frac{\binom{\ell_1+\ell_2+1}{\ell_1}\binom{\ell_1+\ell_2+1}{\ell_2}}{\ell_1+\ell_2+1}.$$

Example 3.16. *As an example, we consider the affine hypertoric variety $\overline{\mathcal{O}}^{\min}(2,1,1)$ give by matrix*

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

Then the wall-crossing structure of the parameter α and the corresponding core given by the Figure 2 as follows. Then one can see how it can connected by Mukai-flops.

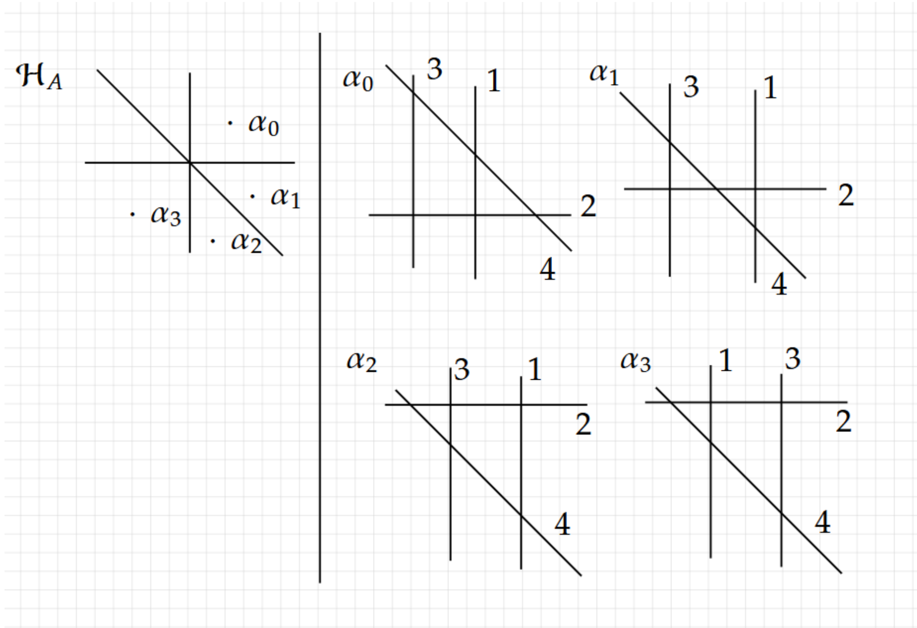


FIGURE 2. The wall-crossing of cores of $\overline{\mathcal{O}}^{\min}(2,1,1)$

Remark 3.17. *In [HD14], they shows that in general all such wall-crossing given by a family version of Mukai-flops.*

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