# LECTURES ABOUT SYMPLECTIC RESOLUTIONS, SYMPLECTIC DUALITY, AND COULOMB BRANCHES

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Abstract. In this note we will give a survey about symplectic resolutions, symplectic duality and Coulomb branches follows [Kam22].

Contents



## 1. Introduction

We will mainly follows the survey paper [Kam22]. We work over $\mathbb{C}$.

## 2. Symplectic resolutions

2.1. **Basic definitions.** Here we give some basic definitions.

**Definition 2.1.** *We consider complex algebraic schemes.*

- *We say a scheme $X$ carries a* Poisson structure *if there is a $\mathbb{C}$-bilinear operation*
$$\{-,-\} : \mathscr{O}_X \times \mathscr{O}_X \to \mathscr{O}_X$$
*which is a Lie bracket.*
- *Let $f : X \to Y$ be a morphism of Poisson schemes, we say it is a* Poisson morphism *if it induce a homomorphism of Lie algebras.*

**Remark 2.2.** *Any Poisson structure can be induced by the $\mathscr{O}_X$-linear homomorphism $H : \Omega^1_X \to T_X = \mathrm{Der}(\mathscr{O}_X, \mathscr{O}_X)$ such that $\{f, g\} = H(df)(g)$. In particular, any symplectic variety has a canonical Poisson structure.*

**Definition 2.3.** *A* symplectic resolution *is a morphism $\pi : Y \to X$ of complex algebraic varieties, where*

- *$Y$ is smooth and carries a symplectic structure.*

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- *$X$ is affine, normal, and carries a Poisson structure.*
- *$\pi$ is projective, birational, and Poisson.*

**Definition 2.4.** *Let $\pi : Y \to X$ be a symplectic resolution.*

- *$\pi$ is called conical if we are given actions of $\mathbb{C}^\times$ on $X$ and $Y$ such that $\pi$ is equivariant and $\mathbb{C}^\times$ contracts $X$ to a single point, denoted as $0$. We also assume that $\mathbb{C}^\times$ scales the symplectic form with weight $2$. The central fiber $F_0 = \pi^{-1}(0)$ is called the core of $Y$.*
- *A conical symplectic resolution is called Hamiltonian if we are given Hamiltonian actions of a torus $T$ on $X$ and $Y$, such that $\pi$ is $T$-equivariant. We also assume that the $T$ action commutes under the conical $\mathbb{C}^\times$ action, and that $Y^T$ is finite.*

Here we introduce the most basic and important example:

**Example 2.5** (Baby example)**.** *The simplest example of a symplectic resolution is*

$$\pi : Y := T^*\mathbb{P}^1 := \mathrm{Tot}(\Omega^1_{\mathbb{P}^1}) \to X := \mathcal{N}_{\mathfrak{sl}_2}$$

*where $\mathcal{N}_{\mathfrak{sl}_2}$ is the variety of nilpotent $2 \times 2$ matrices, that is,*

$$\mathcal{N}_{\mathfrak{sl}_2} = \left\{ \begin{pmatrix} w & u \\ v & -w \end{pmatrix} \in M_2(\mathbb{C}) : w^2 + uv = 0 \right\} \cong \mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$$

*which is the singular quadric cone in $\mathfrak{sl}_2(\mathbb{C}) \cong \mathbb{C}^3$ where the last isomorphism $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}) \cong \mathcal{N}_{\mathfrak{sl}_2}$ given by $(a, b) \mapsto \begin{pmatrix} ab & a^2 \\ b^2 & -ab \end{pmatrix}$. Here we have several descriptions of this morphism.*

- *Now via canonical exact sequence of tangent bundle of projective space, we have canonical description $T_{[L]}\mathbb{P}^1 = \{w \in \mathbb{C}^2 : w \cdot L = 0\}$. Hence $T^*_{[L]}\mathbb{P}^1$ is a space of linear functions over $T_{[L]}\mathbb{P}^1$. So this correspond to a matrix $\mathbf{A} \in M_2(\mathbb{C})$ such that $\mathbf{A}L = 0$ and $\mathbf{A}(w) \in L$ for $w \in T_{[L]}\mathbb{P}^1$. So*

  $$T^*\mathbb{P}^1 \cong \left\{([L], \mathbf{A}) \in \mathbb{P}^1 \times \mathcal{N}_{\mathfrak{sl}_2} : \mathbf{A}(\mathbb{C}^2) \subset L, \mathbf{A}L = 0\right\}$$

  *with canonical forgetful morphism $\pi : T^*\mathbb{P}^1 \to \mathcal{N}_{\mathfrak{sl}_2}$.*
- *Regard $\mathcal{N}_{\mathfrak{sl}_2} \subset \mathbb{C}^3$ as a cone of over a quadric plane curve $C \subset \mathbb{C}^2$. Then consider its projective completion $\overline{C} \subset \mathbb{P}^2$, then $\mathcal{N}_{\mathfrak{sl}_2} = \overline{\mathcal{N}} \backslash H_\infty$ where $\overline{\mathcal{N}} \subset \mathbb{P}^3$ be the cone of $\overline{C}$. Now*

  $$\begin{aligned} Y &\cong \mathsf{Bl}_v \mathcal{N}_{\mathfrak{sl}_2} = (\mathsf{Bl}_v \overline{\mathcal{N}}) \backslash H_\infty = \mathbb{P}_{\overline{C}}(\mathscr{O}_{\overline{C}} \oplus \mathscr{O}_{\overline{C}}(-1)) \backslash \mathbb{P}_{\overline{C}}(\mathscr{O}_{\overline{C}}) \\ &= \mathrm{Tot}(\mathscr{O}_{\overline{C}}(-1)) = \mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(-2)), \end{aligned}$$

  *well done. More precisely, we have the following:*

  $$\mathrm{Tot}(\mathscr{O}_{\mathbb{P}^1}(-2)) \to \mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}), \quad ([z_0 : z_1], \lambda^2(z_0, z_1)^2) \mapsto \lambda(z_0, z_1).$$

*Note that $\mathbb{C}^2$ admit a natural Poisson structure, then so is $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z}) \cong \mathcal{N}_{\mathfrak{sl}_2}$. It is also affine and normal. Moreover $T^*\mathbb{P}^1$ is a smooth and carries a symplectic structure and $\pi$ is projective, birational, and Poisson. So $\pi$ is a symplectic resolution.*

*Actually, we have a conical $\mathbb{C}^\times$-action which scales the matrix and a Hamiltonian action of $T := (\mathbb{C}^\times)^2$, inherited from its action on $\mathbb{C}^2$. So $\pi$ is a Hamiltonian conical symplectic resolution.*

2.2. **Example: cotangent bundles of flag varieties.** The first way to generalize the Example 2.5, we consider any semi-simple group $G$ and its parabolic subgroup $P$. Then we consider $T^*(G/P)$ which will be seen as a resolution of a nilpotent orbit closure.

**Definition 2.6.** *Consider a semi-simple Lie algebra $\mathfrak{g}$ with a adjoint algebraic group $G$.*

- *For a nilpotent element $v \in \mathfrak{g}$, its nilpotent orbit $\mathcal{O}_v$ is the orbit of $v$ under the adjoint action of $G$. Its closure $\overline{\mathcal{O}_v}$ is a nilpotent orbit closure.*
- *Consider the standard $\mathfrak{sl}_2$-action of $\mathfrak{g}$ due to Jacobson-Morozov: for a nilpotent element $v \in \mathfrak{g}$, there exist two elements $H, u \in \mathfrak{g}$ such that $[H, v] = 2v, [H, u] = -2u, [v, u] = H$. Thus $\mathfrak{g}$ is decomposed as $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, where $\mathfrak{g}_i := \{x \in \mathfrak{g} : [H, x] = ix\}$. Let $\mathfrak{p} = \bigoplus_{i \geqslant 0} \mathfrak{g}_i$ and $P$ a correspond connected subgroup of $G$ which is a parabolic subgroup. Let $\mathfrak{n} := \bigoplus_{i \geqslant 2} \mathfrak{g}_i$ and $\mathfrak{u} := \bigoplus_{i \geqslant 1} \mathfrak{g}_i$.*
- *The nilpotent orbit $\mathcal{O}_v$ is called even if $\mathfrak{g}_1 = 0$ or equivalently if $\mathfrak{g}_{2k+1} = 0$ for all $k \in \mathbb{Z}$.*

**Proposition 2.7** (Prop 2.8 and Cor 2.9 in [Fu06])**.** *For a nilpotent orbit $\mathcal{O}_v$, there exists a symplectic form $\omega$ (hence Poisson) on $\mathcal{O}_v$ which is called the Kostant-Kirillov-Souriau form. Moreover, there exists a $G$-equivariant proper resolution*

$$\pi : G \times^P \mathfrak{n} \to \overline{\mathcal{O}_v},$$

*which is called the Springer resolution such that $\pi$ maps the orbit $G \cdot (1, v)$ isomorphically to $\mathcal{O}_v$ and the symplectic form $\pi^*\omega$ on $G \cdot (1, v)$ can be extended to a global 2-form $\Omega$ on $G \times^P \mathfrak{n}$. Moreover, $\Omega$ is symplectic if and only if $\mathcal{O}_v$ is even.*

As resolution $\pi : G \times^P \mathfrak{n} \to \overline{\mathcal{O}_v}$ factor-through the normalization $\widetilde{\mathcal{O}} \xrightarrow{\nu} \overline{\mathcal{O}_v}$, we have the following directly:

**Corollary 2.8.** *The normalization $\widetilde{\mathcal{O}}$ of a nilpotent orbit closure $\overline{\mathcal{O}_v}$ is a symplectic variety. The Springer resolution $G \times^P \mathfrak{n} \to \widetilde{\mathcal{O}}$ is symplectic if and only if $\mathcal{O}_v$ is an even nilpotent orbit.*

But not that for an even nilpotent orbit closure, there can exist some symplectic resolutions not of the above form as we will talking about.

Note that not all nilpotent orbit closure admits a symplectic resolution. But this is true for $\mathfrak{sl}_n$ which we will be particularly interested.

**Example 2.9.** *Every nilpotent orbit closure in $\mathfrak{sl}_{n+1}$ admits a symplectic resolution. Indeed, note that we have bijction between the set of nilpotent orbit closures and the set of some special partitions, see Proposition 2.2.1 in* [Nam10]*. In the case of $\mathfrak{sl}_{n+1}$, the set of nilpotent orbit closures correspond to the all partitions of $n + 1$.*

*Let $\mathcal{O}$ be the nilpotent orbit corresponding to the partition $[d_1, ..., d_k]$. The dual partition is defined by $s_j = |\{i | d_i \geqslant j\}|$. The closure is $\overline{\mathcal{O}} = \{A \in \mathfrak{sl}_{n+1} : \dim \ker A^j \geqslant \sum_{i=1}^j s_j\}$ which is normal, see* [KP79]*. Consider the partial flag variety (which is of course $\mathrm{SL}_{n+1}/P$) $\mathsf{Fl} := \{(V_1, ..., V_l) : \dim V_j = \sum_{i=1}^j s_j, V_j \subset V_{j+1}\}$ with the similar description as baby example:*

$$\pi : T^*\mathsf{Fl} \cong \{(\mathbf{A}, V_\bullet) \in \mathfrak{sl}_{n+1} \times \mathsf{Fl} : \mathbf{A}V_i \subset V_{i-1}\} \to \overline{\mathcal{O}}$$

*which is of course a symplectic resolution.*

But note that not every nilpotent orbit closure admits a symplectic resolution, see Proposition 5.2 in [Fu06]. But conversely we have the following nice result:

**Theorem 2.10** (Thm 0.1 in [Fu03])**.** *Suppose that we have a symplectic resolution $\pi : Z \to \widetilde{\mathcal{O}}$ where $\widetilde{\mathcal{O}}$ is a normalization of a nilpotent orbit closure $\overline{\mathcal{O}_v}$, then there exists a parabolic subgroup $P$ of $G$ such that $Z \cong T^*(G/P)$.*

Such orbits are called the Richardson orbits.

**Corollary 2.11.** *The normalization $\widetilde{\mathcal{O}}$ of a nilpotent orbit closure in a semisimple Lie algebra admits a symplectic resolution if and only if*

- *$\mathcal{O}$ is a Richardson nilpotent orbit.*
- *There exists a polarization $P$ such that the moment map $T^*(G/P) \to \mathcal{O}$ is birational.*

The conical $\mathbb{C}^\times$ acts by linear scaling on $\overline{\mathcal{O}_v}$ (coming from its embedding in the vector space $\mathfrak{g}$). The maximal torus $T \subset G$ acts Hamiltonianly on $T^*(G/P)$ in the natural way. The fixed point set $(T^*(G/P))^T$ is in bijection with $W/W_P$.

### 2.3. More examples.

**Example 2.12** (Resolutions of Kleinian singularities)**.** *Generalizing $T^*\mathbb{P}^1$ in a different direction, we take $\Gamma \subset \mathrm{SL}_2$ a finite subgroup. Under the McKay correspondence, such subgroups are in bijection with simply-laced ADE Dynkin diagrams. We only consider*

$$\Gamma \cong \mathbb{Z}/n\mathbb{Z} = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} : \zeta^n = 1 \right\} \longleftrightarrow A_{n-1}.$$

*The affine GIT quotient $X := \mathbb{C}^2 // \Gamma$ carries a Poisson structure by descending the usual symplectic structure on $\mathbb{C}^2$. Then consider minial resolution. The conical $\mathbb{C}^\times$ action comes from the scaling action on $\mathbb{C}^2$. On the other hand, the Hamiltonian torus $T$ is given by the diagonal matrices in $\mathrm{SL}_2$.*

**Example 2.13** (Higgs branches of gauge theories)**.** *Consider a reductive group $G$ and a representation $V$. Then we form $T^*V = V \oplus V^*$ which comes with a moment map $\Phi : T^*V \to \mathfrak{g}^*$. We fix a character $\chi : G \to \mathbb{C}^\times$ and form the GIT quotient*

$$T^*V///_{0,\chi}G := \Phi^{-1}(0)//_\chi G := \operatorname{Proj}\left(\bigoplus_{n\geqslant 0}\mathbb{C}[\Phi^{-1}(0)]^{G,n\chi}\right)$$

*where $\mathbb{C}[\Phi^{-1}(0)]^{G,n\chi} = \{f \in \mathbb{C}[\Phi^{-1}(0)] : \sigma(f) = \chi^*(t)^n \otimes f\}$ where $\sigma : \mathbb{C}[\Phi^{-1}(0)] \to \Gamma(G, \mathscr{O}_G) \otimes \mathbb{C}[\Phi^{-1}(0)]$ is the coaction and $\chi^*$ induced by $\chi$ with $\mathbb{C}^\times \cong \operatorname{Spec}\mathbb{C}[t, t^{-1}]$.*

*We have a natural projective morphism*

$$\pi : Y := T^*V///_{0,\chi}G \to X := T^*V///_{0,0}G = \operatorname{Spec}\mathbb{C}[\Phi^{-1}(0)]^G$$

*carry Poisson structures coming from the usual symplectic structure on $T^*V$. This construction will not usually give a symplectic resolution; for example, $Y$ may not be smooth and $Y \to X$ might not be birational. Here in the physics literature, $Y$ is called the Higgs branch of the 3d supersymmetric gauge theory defined by $G, V$. $G$ is called the gauge group and $N$ is called the matter.*

*There is a conical $\mathbb{C}^\times$ action on $Y$ coming from its scaling action of $T^*V$. In order to define a Hamiltonian torus action, we need one piece of data. We choose an extension $1 \to G \to \widetilde{G} \to T \to 1$, where $T$ is the flavor torus, and an action of $\widetilde{G}$ on $V$, extending the action of $G$. Then we obtain a residual Hamiltonian action of $T$ on $Y$ and $X$. In general, this action does not have finitely many fixed points.*

**Example 2.14** (Hypertoric varieties)**.** *As above in a special case, let $G$ a torus and $\widetilde{G} = (\mathbb{C}^\times)^n$, so we have*

$$1 \to G \to (\mathbb{C}^\times)^n \to T \to 1.$$

*This gives us a linear action of $G$ on $T^*\mathbb{C}^n = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$. We have a moment map $\Phi : T^*\mathbb{C}^n \to \mathbb{C}^n \to \mathfrak{g}^*$ where the first map is given by $(z_i, w_i) \mapsto (z_i w_i)$. We fix a generic character $\chi : G \to \mathbb{C}^\times$ and consider $Y = \Phi^{-1}(0)//_\chi G$ and $X = \Phi^{-1}(0)//G$ as above.*

*The map $(\mathbb{C}^\times)^n \to T$ is equivalent to a map $\phi : \mathbb{Z}^n \to \mathbf{t}_\mathbb{Z}$ between the coweight lattices of these tori. We assume that $\phi$ is unimodular (this means if we choose a matrix representing $\phi$, then every invertible square submatrix has determinant $\pm 1$). This ensures that $Y$ is smooth. We also assume that the map $G \to (\mathbb{C}^\times)^n$ is not contained in any coordinate subtorus. This is needed to ensure that the natural $\mathbb{C}^\times$ action on $X$ is conical.*

*We have residual actions of $T$ on $Y$ and $X$ with moment map $X \to \mathfrak{t}^*$. The structure of $Y$ can be visualized by means of a hyperplane arrangement. We consider the real vector space $\mathfrak{t}^*_\mathbb{R}$ and define affine hyperplanes $H_1, ..., H_n$ by*

$$H_i := \{v \in \mathfrak{t}^*_\mathbb{R} : \langle\phi(e_i), v\rangle = \langle e_i, \tilde{\chi}\rangle\},$$

*where $\tilde{\chi} \in (\mathbb{R}^n)^*$ is a lift of $\chi \in \mathfrak{g}^*_\mathbb{R}$. These hyperplanes partition $t^*_\mathbb{R}$ into chambers, and $Y$ contains all the toric varieties associated to these chambers. In particular, the core $F_0$ is the union of the toric varieties associated to the compact chambers.*

*Note that given*

$$1 \to G \to (\mathbb{C}^\times)^n \to T \to 1$$

*as above, we can dualize to*

$$1 \to T^\vee \to (\mathbb{C}^\times)^n \to G^\vee \to 1$$

*where $T^\vee$ is the dual torus which satisfies $\operatorname{Hom}(\mathbb{C}^\times, T^\vee) = \operatorname{Hom}(T, \mathbb{C}^\times)$. The resulting pair $Y, Y^\vee$ of hypertoric varieties is called symplectic dual. See the survey [Pro08] for the details.*

**Example 2.15** (Quiver varieties)**.** *Another special case, we introduce the Nakajima quiver varieties, first introduced by Nakajima [Nak94]. We fix a finite directed graph $Q = (I, E)$, with head and tail maps $h, t : E \to I$. Also, we fix two dimension vectors $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$. For $i \in I$,let $V_i = \mathbb{C}^{u_i}, W_i = \mathbb{C}^{w_i}$ and consider the space of representations of the quiver $Q$ on the vector space $\oplus V_i$ framed by $\oplus W_i$.*

$$N = \bigoplus_{e\in E}\operatorname{Hom}(V_{t(e)}, V_{h(e)}) \oplus \bigoplus_{i\in I}\operatorname{Hom}(V_i, W_i).$$

*This big vector space $N$ has a natural action of $G = \prod_i \operatorname{GL}(V_i)$. We form the cotangent bundle $T^*N$ and take the Hamiltonian reduction by the action of $G$. The resulting space $Y = \Phi^{-1}(0)//_\chi G$ is called a Nakajima quiver variety. Here we choose $\chi : G \to \mathbb{C}^\times$ to be given by the product of the* *determinants. On $Y$, we have a Hamiltonian action of $T = \prod_i (\mathbb{C}^\times)^{w_i}$ inherited from its action on $\bigoplus W_i$. (In other words, we take $\widetilde{G} = G \times T$.)*

*Note that the space $Y$ is always smooth but $\pi : Y \to X$ is not always birational. Also, the Hamiltonian torus action does not always have finitely many fixed points.*

*Here we give two examples of Nakajima quiver varieties.*

- *Consider a linearly oriented type $A_{n-1}$-quiver with $\mathbf{v} = (1, ..., n-1), \mathbf{w} = (0, ..., 0, n)$:*

$$\bullet(V_1) \longrightarrow \bullet(V_2) \longrightarrow \cdots \longrightarrow \bullet(V_{n-1}) \longrightarrow \blacksquare(\mathbb{C}^n)$$

  *Then $N = \bigoplus_{i=1}^{n-1} \mathrm{Hom}(\mathbb{C}^i, \mathbb{C}^{i+1})$ with $G = \prod_{i=1}^{n-1} \mathrm{CL}_i$. Then $Y \cong T^*\mathsf{Fl}_n$ with $X = \mathcal{N}_{\mathfrak{sl}_n}$.*
- *Another important example is a quiver with one vertex and one self-loop with $V = \mathbb{C}^n$ and $W = \mathbb{C}^r$.*

$$\begin{tikzcd} \bullet(\mathbb{C}^n) \arrow[loop above] \arrow[r] & \blacksquare(\mathbb{C}^r) \end{tikzcd}$$

  *In this case, $Y$ is the moduli space of rank $r$, torsion-free sheaves on $\mathbb{P}^2$, framed at $\infty$ with second Chern class $n$.*

## 3. Topologies of symplectic resolutions

3.1. **Symplectic leaves and the Springer sheaf.** Before consider the symplectic resolution, we review some basic things.

**Definition 3.1.** *Consider a morphism of varieties $f : X \to Y$. Let $Y_t := \{y \in Y : f^{-1}(Y) = t\}$. Then $f$ is said to be* semismall *if for each stratum $Y_t$ and each point $y \in Y_t \cap f(X)$, we have $\dim f^{-1}(y) \leqslant \frac{1}{2}(\dim X - \dim Y_t)$.*

**Remark 3.2.** *This is equivalent to that for any irreducible subvariety $Z \subset X$ we have $2\dim Z - \dim f(Z) \leqslant \dim X$. It is generically finite. If it is of same dimension, this is $2\mathrm{codim}(Z) \geqslant \mathrm{codim}(f(Z))$.*

**Theorem 3.3.** *Perverse; BBDG*

For a symplectic resolution, we have the following topological properties:

**Theorem 3.4.** *Let $\pi : Y \to X$ be a symplectic resolution.*

- *$X$ has a finite partition $X = \bigsqcup_j X_j$ where each $X_j$ is locally closed, smooth, and symplectic. (These are the* symplectic leaves *of $X$.)*
- *The map $\pi$ is semismall.*
- *The pushforward $\pi_* \mathbb{C}_Y$ is constructible with respect to the stratification by the symplectic leaves.*

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3.2. **Hyperbolic decomposition.**

3.3. **Transversal slices to symplectic leaves.**

3.4. **Quantum cohomology.**

## 4. Deformations and quantizations

## 5. Symplectic duality

## 6. Geometrization and categorification of representations

## 7. Coulomb branches of 3d gauge theory

## 8. Affine Grassmannian slices as Coulomb branches

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